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Spectral Distribution of Hermitian Toeplitz Matrices Formally Generated by Rational Functions

William F. Trench

ABSTRACT. We consider the asymptotic spectral distribution of Hermitian Toeplitz matrices $\{T_n\}_{n=1}^{\infty}$ formally generated by a rational function $h(z) = (f(z)f^*(1/z))/(g(z)g^*(1/z))$, where the numerator and denominator have no common zeros, $\deg(f) < \deg(g)$, and the zeros of g are in the open punctured disk $0 < |z| < 1$. From Szegö's theorem, the eigenvalues of $\{T_n\}$ are distributed like the values of $h(e^{i\theta})$ as $n \to \infty$ if $T_n = (t_{r-s})_{r,s=1}^n$, where $\{t_\ell\}_{\ell=-\infty}^{\infty}$ are the coefficients in the Laurent series for h that converges in an annulus containing the unit circle. We show that if $\{t_\ell\}_{\ell=-\infty}^{\infty}$ are the coefficients in certain other formal Laurent series for h, then there is an integer p such that all but the p smallest and p largest eigenvalues of T_n are distributed like the values of $h(e^{i\theta})$ as $n \to \infty$.

1. Introduction

If $P(z) = a_0 + a_1 z + \cdots + a_k z^k$, then $P^*(z) = \overline{a}_0 + \overline{a}_1 z + \cdots + \overline{a}_k z^k$. We consider the spectral distribution of families of Hermitian Toeplitz matrices $T_n = \{t_{r-s}\}_{r,s=1}^n$, $n \geq 1$, where $\{t_\ell\}_{\ell=-\infty}^{\infty}$ are the coefficients in a formal Laurent expansion of a rational function

$$h(z) = \frac{f(z)f^*(1/z)}{g(z)g^*(1/z)},$$

where

$$g(z) = \prod_{j=1}^{k} (z - \zeta_j)^{d_j},$$

$\zeta_1, \ldots, \zeta_k$ are distinct, $0 < |\zeta_r| < 1$ $(1 \leq r \leq k)$, $d_1, \ldots, d_k$ are positive integers, f is a polynomial of degree less than $d_1 + \cdots + d_k$, and $f(\zeta_j)f^*(1/\zeta_j) \neq 0$ $(1 \leq r \leq k)$. Then h has a unique convergent Laurent expansion

$$h(z) = \sum_{\ell=-\infty}^{\infty} \widetilde{t}_\ell \, z^\ell, \quad \max_{1 \leq j \leq k} |\zeta_j| < |z| < \min_{1 \leq j \leq k} 1/|\zeta_j|. \tag{1}$$

If α and β are respectively the minimum and maximum of $w(\theta) = h(e^{i\theta})$, then Szegö's distribution theorem [**1**, pp. 64-5] implies that eigenvalues of the matrices $\widetilde{T}_n = (\widetilde{t}_{r-s})_{r,s=1}^n$, $n \geq 1$, are all in $[\alpha, \beta]$, and are distributed like the values of w as $n \to \infty$; that is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\lambda_i(\widetilde{T}_n)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(w(\theta))\, d\theta \quad \text{if} \quad F \in C[\alpha, \beta]. \tag{2}$$

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We are interested in the asymptotic spectral distribution of 2^k-1 other families of Hermitian Toeplitz matrices formally generated by h, to which Szegö's theorem does not apply. To be specific, a partial fraction expansion yields $h = h_1 + \cdots + h_k$, with

$$h_j(z) = \sum_{m=0}^{d_j-1}\left(\frac{a_{mj}}{(1-\overline{\zeta}_j z)^{m+1}} + \frac{(-1)^m b_{mj}\zeta_j^{m+1}}{(z-\zeta_j)^{m+1}}\right), \tag{3}$$

where $\{a_{mj}\}$ and $\{b_{mj}\}$ are constants and

$$a_{d_j-1,j} \neq 0, \quad b_{d_j-1,j} \neq 0, \quad 1 \le j \le k. \tag{4}$$

Using the expansions

$$\frac{1}{(1-\overline{\zeta}_j z)^{m+1}} = \sum_{\ell=0}^{\infty}\binom{m+\ell}{m}\overline{\zeta}_j^{\ell} z^{\ell}, \quad |z| < 1/|\zeta_j|, \tag{5}$$

and

$$\frac{(-1)^m\zeta_j^{m+1}}{(z-\zeta_j)^{m+1}} = \sum_{\ell=-\infty}^{-1}\binom{m+\ell}{m}\frac{z^{\ell}}{\zeta_j^{\ell}}, \quad |z| > |\zeta_j|, \tag{6}$$

for $0 \le m \le d_j-1$ produces a Laurent series that converges to $h_j(z)$ for $|\zeta_j| < |z| < 1/|\zeta_j|$. We will call this the *convergent expansion of* h_j. However, using the expansions

$$\frac{1}{(1-\overline{\zeta}_j z)^{m+1}} = -\sum_{\ell=-\infty}^{-1}\binom{m+\ell}{m}\overline{\zeta}_j^{\ell} z^{\ell}, \quad |z| > 1/|\zeta_j|, \tag{7}$$

and

$$\frac{(-1)^m\zeta_j^{m+1}}{(z-\zeta_j)^{m+1}} = -\sum_{\ell=0}^{\infty}\binom{m+\ell}{m}\frac{z^{\ell}}{\zeta_j^{\ell}}, \quad |z| < |\zeta_j|, \tag{8}$$

for $0 \le m \le d_j-1$ produces a formal Laurent series for h_j that converges nowhere. We will call this the *formal expansion of* h_j.

Henceforth eigenvalues are numbered in nondecreasing order. We will prove the following theorem.

Theorem 1. *Let $\{\mathcal{S}_0, \mathcal{S}_1\}$ be a partition of $\{1, \dots, k\}$, with $\mathcal{S}_1 \neq \emptyset$. For $1 \le j \le k$, let $\sum_{\ell=-\infty}^{\infty} t_\ell^{(j)} z^\ell$ be the convergent expansion of h_j if $j \in \mathcal{S}_0$, or the formal expansion of h_j if $j \in \mathcal{S}_1$. Let $T_n = (t_{r-s})_{r,s=1}^n$, where $t_\ell = \sum_{j=1}^k t_\ell^{(j)}$, and let*

$$p = \sum_{j\in\mathcal{S}_1} d_j. \tag{9}$$

Then

$$\{\lambda_i(T_n)\}_{i=p+1}^{n-p} \subset [\alpha, \beta], \quad n > 2p,$$

and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=p+1}^{n-p} F(\lambda_i(T_n)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(w(\theta))\,d\theta \quad \textit{if} \quad F \in C[\alpha, \beta]. \tag{10}$$

In fact,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=p+1}^{n-p}|F(\lambda_i(T_n))-F(\lambda_i(\widetilde{T}_n))|=0 \quad \textit{if} \quad F\in C[\alpha,\beta]. \tag{11}$$

We proved a similar theorem in [**2**], concerning the asymptotic spectral distribution of Hermitian Toeplitz matrices of the form

$$T_n=\sum_j c_jK_n(\zeta_j;P_j) \quad \text{(finite sum)}$$

where $c_1,\ldots, c_k$ are real, $\zeta_1,\ \ldots,\ \zeta_k$ are distinct and nonzero, $P_1,\ \ldots,\ P_k$ are monic polynomials with real coefficients, and

$$K_n(\zeta;P)=\Big(P(|r-s|)\rho^{|r-s|}e^{i(r-s)\phi}\Big)_{r,s=1}^n.$$

2. Proof of Theorem 1

We need the following lemmas from [**2**].

Lemma 1. *Let*

$$\gamma_\ell=\sum_{j=1}^m F_j(\ell)z_j^\ell,$$

where $z_1,\ z_2,\ \ldots,\ z_m$ are distinct nonzero complex numbers and $F_1,\ F_2,\ \ldots,\ F_m$ are polynomials with complex coefficients. Define

$$\mu=\sum_{j=1}^m(1+\deg(F_j)).$$

Let $\Gamma_n=(\gamma_{r-s})_{r,s=1}^n$. Then $\operatorname{rank}(\Gamma_n)=\mu$ if $n\ge\mu$.

Lemma 2. *Let*

$$\gamma_r=P(r)\zeta^r+P^*(-r)\overline{\zeta}^{-r},$$

where P is a polynomial of degee d and $|\zeta|\ne 0,\ 1$. Then the Hermitian matrix $\Gamma_n=(\gamma_{r-s})_{r,s=1}^n$ has inertia $[d+1,n-2d-2,d+1]$ if $n\ge 2d+2$.

(In [**2**] we considered only the case where P has real coefficients; however the same argument yields the more general result stated here.)

Lemma 3. *Suppose that H_n is Hermitian and*

$$-\infty<\alpha\le\lambda_i(H_n)\le\beta<\infty, \quad 1\le i\le n, \quad n\ge 1.$$

Let k be a positive integer and let p and q be nonnegative integers such that $p+q=k$. For $n\ge k$ let $T_n=H_n+B_n$, where B_n is Hermitian and of rank k, with p positive and q negative eigenvalues. Then

$$\{\lambda_i(T_n)\}_{i=q+1}^{n-p}\subset[\alpha,\beta], \quad n>k,$$

and

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=q+1}^{n-p}|F(\lambda_i(T_n))-F(\lambda_i(H_n))|=0 \quad \textit{if} \quad F\in C[\alpha,\beta].$$

Moreover,

$$\lambda_i(T_n)-\lambda_i(B_n)=O(1), \quad 1\le i\le q, \tag{12}$$

and

$$\lambda_{n-p+j}(T_n) - \lambda_{n-p+j}(B_n) = O(1), \quad 1 \le j \le p, \tag{13}$$

as $n \to \infty$.

In Lemma 3, $\{H_n\}$ and $\{B_n\}$ need not be Toeplitz matrices. Our proof of Lemma 3 was motivated in part by an observation of Tyrtyshnikov [**3**], who, to our knowledge, was the first to apply the idea of low rank perturbations to spectral distribution problems.

Let

$$u_j(z) = \sum_{m=0}^{d_j-1} a_{mj}\binom{m+z}{m} \quad \text{and} \quad v_j(z) = \sum_{m=0}^{d_j-1} b_{mj}\binom{m+z}{m}. \tag{14}$$

From (4), $\deg(u_j) = \deg(v_j) = d_j - 1$. We first show that

$$v_j(-z) = u_j^*(z), \quad 1 \le j \le k. \tag{15}$$

The proof is by contradiction. Suppose that (15) is false. Let $\Gamma_n = (\gamma_{r-s})_{r,s=1}^n$, where

$$\gamma_\ell = \sum_{j=1}^{k}(v_j(-\ell) - u_j^*(\ell))\zeta_j^\ell. \tag{16}$$

From Lemma 1, there is a positive integer ν such that

$$\operatorname{rank}(\Gamma_n) = \nu, \quad n \ge \nu. \tag{17}$$

From (3), (5), (6), and (14), the convergent expansion of h_j is

$$h_j(z) = \sum_{\ell=0}^{\infty} u_j(\ell)\overline{\zeta}_j^\ell z^\ell + \sum_{\ell=-\infty}^{-1} v_j(\ell)\zeta_j^{-\ell} z^\ell, \quad |\zeta_j| < |z| < 1/|\zeta_j|.$$

Therefore, the coefficients $\{\widetilde{t}_\ell\}$ in (1) are given by

$$\widetilde{t}_\ell = \begin{cases} \displaystyle\sum_{j=1}^{k} u_j(\ell)\overline{\zeta}_j^\ell, & \ell \ge 0, \\ \displaystyle\sum_{j=1}^{k} v_j(\ell)\zeta_j^{-\ell}, & \ell < 0. \end{cases} \tag{18}$$

Since $\widetilde{t}_{-\ell} = \overline{\widetilde{t}_\ell}$, this and (16) imply that $\gamma_\ell = 0$ if $\ell > 0$. From this and (17), there is a largest nonpositive integer ℓ_0 such that $\gamma_{\ell_0} \ne 0$. But then $\operatorname{rank}(\Gamma_n) = n - |\ell_0|$ if $n > |\ell_0| + 1$, which contradicts (17). Therefore, (15) is true.

We can now rewrite (18) as

$$\widetilde{t}_\ell = \begin{cases} \displaystyle\sum_{j=1}^{k} u_j(\ell)\overline{\zeta}_j^\ell, & \ell \ge 0, \\ \displaystyle\sum_{j=1}^{k} u_j^*(-\ell)\zeta_j^{-\ell}, & \ell < 0. \end{cases} \tag{19}$$

From (3), (7), (8), and (14), the formal expansion of $h_j(z)$ is

$$-\sum_{\ell=0}^{\infty} v_j(\ell)\zeta_j^{-\ell} z^\ell - \sum_{\ell=-\infty}^{-1} u_j(\ell)\overline{\zeta}_j^{\ell} z^\ell = -\sum_{\ell=0}^{\infty} u_j^*(-\ell)\zeta_j^{-\ell} z^\ell - \sum_{\ell=-\infty}^{-1} u_j(\ell)\overline{\zeta}_j^{\ell} z^\ell.$$

Therefore,

$$t_\ell = \begin{cases} \displaystyle\sum_{j\in\mathcal{S}_0} u_j(\ell)\overline{\zeta}_j^{\ell} - \sum_{j\in\mathcal{S}_1} u_j^*(-\ell)\zeta_j^{-\ell}, & \ell \ge 0, \\ \displaystyle\sum_{j\in\mathcal{S}_0} u_j^*(-\ell)\zeta_j^{-\ell} - \sum_{j\in\mathcal{S}_1} u_j(\ell)\overline{\zeta}_j^{\ell}, & \ell < 0. \end{cases}$$

(Note that $t_{-\ell} = \overline{t}_\ell$.) From this and (19),

$$t_\ell - \widetilde{t}_\ell = -\sum_{j\in\mathcal{S}_1} \left(u_j(\ell)\overline{\zeta}_j^{\ell} + u_j^*(-\ell)\zeta_j^{-\ell} \right), \quad -\infty < \ell < \infty.$$

Now let $B_n = T_n - \widetilde{T}_n$ and $\mathcal{S}_1 = \{j_1, \dots, j_k\}$. Then Lemma 1 with $m = 2k$,

$$\{z_1, z_2, \dots, z_{2k}\} = \{\overline{\zeta}_{j_1}, 1/\zeta_1, \dots, \overline{\zeta}_{j_k}, 1/\zeta_{j_k}\},$$

and

$$\{F_1(\ell), \dots, F_{2k}(\ell)\} = \{u_{j_1}(\ell), u_{j_1}^*(-\ell), \dots, u_{j_k}(\ell), u_{j_k}^*(-\ell)\}, \quad -\infty < \ell < \infty,$$

implies that $\operatorname{rank}(B_n) = 2p$ if $n \ge 2p$. (Recall (9) and that $\deg(u_j) = d_j - 1$.)

If $\Gamma_n^{(j)} = \left(\gamma_{r-s}^{(j)}\right)_{r,s=1}^{n}$ with

$$\gamma_\ell^{(j)} = u_j(\ell)\overline{\zeta}_j^{\ell} + u_j^*(-\ell)\zeta_j^{-\ell},$$

then Lemma 2 implies that $\Gamma_n^{(j)}$ has d_j positive and d_j negative eigenvalues if $n \ge 2d_j$. Therefore, the quadratic form associated with $\Gamma_n^{(j)}$ can be written as a sum of squares with d_j positive and d_j negative coefficients if $n \ge 2d_j$. It follows that B_n can be written as a sum of squares with p positive and p negative coefficients if $n \ge 2p$. Therefore, Sylvester's law of inertia implies that B_n has p positive and p negative eigenvalues if $n \ge 2p$. Now Lemma 3 with $q = p$ implies (11). Since (2) and (11) imply (10), this completes the proof of Theorem 1. ☐

From (12) with $q = p$ and (13), the asymptotic behavior of the $\lambda_i(T_n)$ for $1 \le i \le p$ and $n - p + 1 \le i \le n$ is completely determined by the asymptotic behavior of the $2p$ nonzero eigenvalues of B_n. We believe that the latter all tend to $\pm\infty$ as $n \to \infty$, but we have not been able to prove this.

References

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Trinity University, San Antonio, Texas, USA

E-mail address: wtrench@trinity.edu