# Simplification and Strengthening of Weyl's Definition of Asymptotic Equal Distribution of Two Families of Finite Sets

William F. Trench

Trinity University, San Antonio, Texas, USA; wtrench@trinity.edu
Mailing address: 95 Pine Lane, Woodland Park, CO 80863 USA

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**Abstract**

Suppose that $-\infty < a < b < \infty$, $a \le u_{1n} \le u_{2n} \le \cdots \le u_{nn} \le b$, and $a \le v_{1n} \le v_{2n} \le \cdots \le v_{nn} \le b$, $n \ge 1$. We simplify and strenghthen Weyl's definition of asymptotic equal distribution of $\mathbf{U} = \{\{u_{in}\}_{i=1}^n\}_{n\ge1}$ and $\mathbf{V} = \{\{v_{in}\}_{i=1}^n\}_{n\ge1}$ by showing that the following statements are equivalent:

(i) $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n (F(u_{in}) - F(v_{in})) = 0$ for all $F \in C[a,b]$.

(ii) $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n |u_{in} - v_{in}| = 0$.

(iii) $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n |F(u_{in}) - F(v_{in})| = 0$ for all $F \in C[a,b]$.

## 1 Introduction

The following definition is due to H. Weyl [1, p. 62].

Definition 1.1 Suppose that $-\infty < a < b < \infty$,

$$\{u_{in}\}_{i=1}^n \subset [a,b], \quad \text{and} \quad \{v_{in}\}_{i=1}^n \subset [a,b], \quad n \ge 1.$$

Then $\mathbf{U} = \{\{u_{in}\}_{i=1}^n\}_{n\ge1}$ and $\mathbf{V} = \{\{v_{in}\}_{i=1}^n\}_{n\ge1}$ are *asymptotically equally distributed* if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n (F(u_{in}) - F(v_{in})) = 0, \quad F \in C[a,b].$$

We present a simple necessary and sufficient condition for asymptotic equal distribution and point out that a stronger conclusion is implicit in Definition 1.1.

Without loss of generality, we may assume that

$$a \le u_{1n} \le u_{2n} \le \cdots \le u_{nn} \le b, \quad a \le v_{1n} \le v_{2n} \le \cdots \le v_{nn} \le b, \quad n \ge 1. \tag{1}$$

Theorem 1.2 *If* (1) *holds then the following assertions are equivalent*:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (F(u_{in}) - F(v_{in})) = 0, \quad F \in C[a,b]; \tag{2}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |u_{in} - v_{in}| = 0; \tag{3}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |F(u_{in}) - F(v_{in})| = 0, \quad F \in C[a,b]. \tag{4}$$

Obviously, (4) implies (2). The proof that (3) implies (4) (Section 2) is straightforward. Our main effort is devoted to showing that (2) implies (3).

Theorem 1.2 is a special case of more general results in [4] concerning asymptotic relationships between the eigenvalues or singular values of two infinite sequences of matrices $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ related in some way that it is not necessary to specify here. However, [4] is quite technical and of interest mainly to the linear algebra community. We think it is worthwhile to present Theorem 1.2 in this expository article addressed to a larger audience.

Given Theorem 1.2, we suggest replacing Definition 1.1 by the following simpler definition while bearing in mind that (3) implies (4).

Definition 1.3 $\mathbf{U} = \{\{u_{in}\}_{i=1}^{n}\}_{n\ge1}$ and $\mathbf{V} = \{\{v_{in}\}_{i=1}^{n}\}_{n\ge1}$ are *asymptotically equally distributed* if (1) holds and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |u_{in} - v_{in}| = 0.$$

## 2 Proof that (3) implies (4)

Suppose that $F \in C[a,b]$ and $\epsilon > 0$. By the Weierstrass approximation theorem, there is a polynomial $P$ such that

$$|F(x) - P(x)| < \epsilon/2, \quad a \le x \le b.$$

By the triangle inequality,

$$\begin{aligned} |F(u_{in}) - F(v_{in})| &\le |F(u_{in}) - P(u_{in})| + |P(u_{in}) - P(v_{in})| \\ &\quad + |P(v_{in}) - F(v_{in})| \\ &< |P(u_{in}) - P(v_{in})| + \epsilon. \end{aligned} \tag{5}$$

Let $M = \max_{a \le x \le b} |P'(x)|$. By the mean value theorem,

$$|P(u_{in}) - P(v_{in})| \le M|u_{in} - v_{in}|.$$

This and (5) imply that

$$\frac{1}{n}\sum_{i=1}^{n} |F(u_{in}) - F(v_{in})| < \epsilon + \frac{M}{n}\sum_{i=1}^{n} |u_{in} - v_{in}|.$$

From this and (3),

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} |F(u_{in}) - F(v_{in})| \le \epsilon.$$

Since $\epsilon$ is arbitrary, this implies (4).

## 3 Four Required Lemmas

We need the following lemmas to show that (2) implies (3).

LEMMA 3.1 (HELLY'S FIRST THEOREM) *Let $\{\phi_m\}_{m=1}^{\infty}$ be an infinite sequence of functions on $[a, b]$ and suppose that there is a finite number $K$ such that*

$$|\phi_m(x)| \le K, \quad a \le x \le b, \quad \text{and} \quad V_a^b(\phi_m) \le K, \quad m \ge 1.$$

*Then there is a subsequence of $\{\phi_m\}_{m=1}^{\infty}$ that converges at every point of $[a, b]$ to a function of bounded variation on $[a, b]$.*

LEMMA 3.2 (HELLY'S SECOND THEOREM) *Let $\{\phi_m\}_{m=1}^{\infty}$ be an infinite sequence of functions on $[a, b]$ such that $V_a^b(\phi_m) \le K < \infty$, $m \ge 1$, and*

$$\lim_{m\to\infty} \phi_m(x) = \phi(x), \quad a \le x \le b.$$

*Then $V_a^b(\phi) \le K$ and*

$$\lim_{m\to\infty} \int_a^b F(x)\, d\phi_m(x) = \int_a^b F(x)\, d\phi(x), \quad F \in C[a, b].$$

LEMMA 3.3 *Suppose that $\phi(a) = \phi(b) = 0$, $\phi$ is of bounded variation on $[a, b]$, and*

$$\int_a^b F(x)\, d\phi(x) = 0, \quad F \in C[a, b].$$

*Then $\phi(x) = 0$ at all points of continuity of $\phi$. Thus, $\phi(x) \ne 0$ for at most countably many values of $x$.*

For proofs of Lemmas 3.1–3.3, see [2, p. 222], [2, p. 233], and [3, p. 111].

The following lemma is also known [5, p. 108], but we include its short proof for convenience.

LEMMA 3.4 *Suppose that* $x_1 \leq x_2 \leq \cdots \leq x_n$ *and* $y_1 \leq y_2 \leq \cdots \leq y_n$*. Let* $\{\ell_1, \ell_2, \dots \ell_n\}$ *be a permutation of* $\{1, 2, \dots, n\}$ *and define*

$$Q(\ell_1, \ell_2, \dots, \ell_n) = \sum_{i=1}^{n} (x_i - y_{\ell_i})^2.$$

*Then*

$$Q(\ell_1, \ell_2, \dots, \ell_n) \geq Q(1, 2, \dots, n). \tag{6}$$

PROOF The proof is by induction. Let $P_n$ be the stated proposition. $P_1$ is trivial. Suppose that $n > 1$ and $P_{n-1}$ is true. If $\ell_n = n$, $P_{n-1}$ implies $P_n$. If $\ell_n = s < n$, choose $r$ so that $\ell_r = n$, and define

$$\ell_i' = \begin{cases} \ell_i & \text{if } i \neq r \text{ and } i \neq n, \\ s & \text{if } i = r, \\ n & \text{if } i = n. \end{cases}$$

Then

$$\begin{aligned} Q(\ell_1, \ell_2, \dots, \ell_n) - Q(\ell_1', \ell_2', \dots, \ell_n') &= (x_n - y_s)^2 + (x_r - y_n)^2 \\ &\quad - (x_n - y_n)^2 - (x_r - y_s)^2 \\ &= 2(x_n - x_r)(y_n - y_s) \geq 0. \end{aligned} \tag{7}$$

Since $\ell_n' = n$, $P_{n-1}$ implies that

$$Q(\ell_1', \ell_2', \dots, \ell_n') \geq Q(1, 2, \dots, n).$$

Therefore (7) implies (6), which completes the induction.

# 4 Proof that (2) implies (3)

We will show that if (2) holds then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (u_{in} - v_{in})^2 = 0. \tag{8}$$

From Schwarz's inequality,

$$\frac{1}{n} \sum_{i=1}^{n} |u_{in} - v_{in}| \leq \left( \frac{1}{n} \sum_{i=1}^{n} (u_{in} - v_{in})^2 \right)^{1/2},$$

so (8) implies (3).

The proof of (8) is by contradiction. If (8) is false, there is an $\epsilon_0 > 0$ and an increasing sequence $\{\ell_k\}_{k=1}^{\infty}$ of positive integers such that

$$\frac{1}{\ell_k} \sum_{i=1}^{\ell_k} (u_{i\ell_k} - v_{i\ell_k})^2 \geq \epsilon_0, \quad k \geq 1. \tag{9}$$

However, we will show that if (2) holds, then any increasing sequence $\{\ell_k\}_{k=1}^{\infty}$ of positive integers has a subsequence $\{n_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} \frac{1}{n_k} \sum_{i=1}^{n_k} (u_{in_k} - v_{in_k})^2 = 0, \tag{10}$$

contradicting (9).

If $S$ is a set, let card $S$ be the cardinality of $S$. For $a \le x \le b$, let

$$\nu_n(x;\mathbf{U}) = \operatorname{card}\left\{i \,\middle|\, u_{in} < x\right\} \quad \text{and} \quad \nu_n(x;\mathbf{V}) = \operatorname{card}\left\{i \,\middle|\, v_{in} < x\right\}. \tag{11}$$

Define

$$\rho_n(x;\mathbf{U}) = \begin{cases} \nu_n(x;\mathbf{U})/n, & a \le x < b, \\ 1, & x = b, \end{cases} \tag{12}$$

and

$$\rho_n(x;\mathbf{V}) = \begin{cases} \nu_n(x;\mathbf{V})/n, & a \le x < b, \\ 1, & x = b. \end{cases} \tag{13}$$

If $F \in C[a,b]$, then

$$\frac{1}{n} \sum_{i=1}^{n} F(u_{in}) = \int_a^b F(x)\, d\rho_n(x;\mathbf{U}) \tag{14}$$

and

$$\frac{1}{n} \sum_{i=1}^{n} F(v_{in}) = \int_a^b F(x)\, d\rho_n(x;\mathbf{V}) \tag{15}$$

[2, p. 231]. The sequences $\{\rho_n(\cdot;\mathbf{U})\}_{n=1}^{\infty}$ and $\{\rho_n(\cdot;\mathbf{V})\}_{n=1}^{\infty}$ both satisfy the hypotheses of Lemma 3.1. Therefore, there is a subsequence $\{m_k\}_{k=1}^{\infty}$ of $\{\ell_k\}_{k=1}^{\infty}$ such that

$$\gamma(x;\mathbf{U}) := \lim_{k\to\infty} \rho_{m_k}(x;\mathbf{U}) \tag{16}$$

exists for $a \le x \le b$, and there is a subsequence $\{n_k\}_{k=1}^{\infty}$ of $\{m_k\}_{k=1}^{\infty}$ such that

$$\gamma(x;\mathbf{V}) := \lim_{k\to\infty} \rho_{n_k}(x;\mathbf{V}) \tag{17}$$

exists for $a \le x \le b$. Clearly, (16) implies that

$$\gamma(x;\mathbf{U}) = \lim_{k\to\infty} \rho_{n_k}(x;\mathbf{U}), \quad a \le x \le b. \tag{18}$$

From (11)–(13), $\gamma(\cdot;\mathbf{U})$ and $\gamma(\cdot;\mathbf{V})$ are nondecreasing,

$$\gamma(a;\mathbf{U}) = \gamma(a;\mathbf{V}) = 0, \quad \text{and} \quad \gamma(b;\mathbf{U}) = \gamma(b;\mathbf{V}) = 1. \tag{19}$$

Therefore, (17), (18), and Lemma 3.2 imply that

$$\lim_{k\to\infty} \int_a^b F(x)\, d\rho_{n_k}(x;\mathbf{U}) = \int_a^b F(x)\, d\gamma(x;\mathbf{U}), \quad F \in C[a,b], \tag{20}$$

and

$$\lim_{k\to\infty}\int_a^b F(x)\,d\rho_{n_k}(x;\mathbf{V})=\int_a^b F(x)\,d\gamma(x;\mathbf{V}),\quad F\in C[a,b]. \tag{21}$$

Now (2), (14), (15) (20), and (21) imply that

$$\int_a^b F(x)\,d\gamma(x;\mathbf{U})=\int_a^b F(x)\,d\gamma(x;\mathbf{V}),\quad F\in C[a,b].$$

This, (19), and Lemma 3.3 with $\phi=\gamma(\cdot;\mathbf{U})-\gamma(\cdot;\mathbf{V})$ imply that

$$\gamma(x;\mathbf{U})=\gamma(x;\mathbf{V})$$

except for at most countably many values of $x$ in $[a,b]$.

If $\epsilon>0$, choose $a_0$, $a_1$, …, $a_m$ so that

$$a=a_0<a_1<\cdots<a_m=b,$$

$$a_j-a_{j-1}<\sqrt{\epsilon},\quad 1\le j\le m, \tag{22}$$

and

$$\gamma(a_j;\mathbf{U})=\gamma(a_j;\mathbf{V}),\quad 1\le j\le m. \tag{23}$$

Let

$$I_j=[a_{j-1},a_j),\quad 1\le j\le m-1,\quad I_m=[a_{m-1},a_m].$$

Define

$$U_{jk}=\begin{cases}\nu_{n_k}(a_1;\mathbf{U}), & j=1,\\ \nu_{n_k}(a_j;\mathbf{U})-\nu_{n_k}(a_{j-1};\mathbf{U}), & 2\le j\le m-1,\\ n_k-\nu_{n_k}(a_{m-1};\mathbf{U}), & j=m,\end{cases}$$

and

$$V_{jk}=\begin{cases}\nu_{n_k}(a_1;\mathbf{V}), & j=1,\\ \nu_{n_k}(a_j;\mathbf{V})-\nu_{n_k}(a_{j-1};\mathbf{V}), & 2\le j\le m-1,\\ n_k-\nu_{n_k}(a_{m-1};\mathbf{V}), & j=m.\end{cases}$$

Then

$$U_{jk}=\operatorname{card}\left\{i\,\middle|\,u_{in_k}\in I_j\right\},\quad V_{jk}=\operatorname{card}\left\{i\,\middle|\,v_{in_k}\in I_j\right\},$$

and

$$\lim_{k\to\infty}\frac{U_{jk}-V_{jk}}{n_k}=0,\quad 1\le j\le m, \tag{24}$$

from (12), (13), (17), (18), and (23). Since

$$\min(U_{jk},V_{jk})=\frac{U_{jk}+V_{jk}-|U_{jk}-V_{jk}|}{2},$$

and

$$\sum_{j=1}^m U_{jk}=\sum_{j=1}^m V_{jk}=n_k,$$

it follows that

$$\sum_{j=1}^{m} \min(U_{jk}, V_{jk}) = n_k - r_k, \tag{25}$$

where

$$r_k = \frac{1}{2}\sum_{j=1}^{m} |U_{jk} - V_{jk}|.$$

From (24),

$$\lim_{k\to\infty} \frac{r_k}{n_k} = 0. \tag{26}$$

From (22) and (25), there is a permutation $\tau_{n_k}$ of $\{1, \dots, n_k\}$ such that

$$(u_{in_k} - v_{\tau_k(i),n_k})^2 < \epsilon$$

for $n_k - r_k$ values of $i$; hence

$$\sum_{i=1}^{n_k} (u_{in_k} - v_{\tau_k(i),n_k})^2 < n_k\epsilon + r_k(b-a)^2.$$

Now Lemma 3.4 implies that

$$\sum_{i=1}^{n_k} (u_{in_k} - v_{in_k})^2 < n_k\epsilon + r_k(b-a)^2.$$

Hence, from (26),

$$\limsup_{k\to\infty} \frac{1}{n_k} \sum_{i=1}^{n_k} (u_{in_k} - v_{in_k})^2 \le \epsilon.$$

Since $\epsilon$ is arbitrary, this implies (10), which completes the proof.

## 5 Acknowledgment

I thank Professor Paolo Tilli for a suggestion that enabled me to complete the proof in Section 4.

## References

[1] U. Grenander, G. Szegö, *Toeplitz Forms and Their Applications*, Univ. of California Press, Berkeley and Los Angeles, 1958.

[2] I. P. Natanson, *Theory of Functions of a Real Variable*, Frederick Ungar Publishing Co., New York, 1955.

[3] F. Riesz and B. Sz.–Nagy, *Functional Analysis*, Frederick Ungar Publishing Co., New York, 1955.

[4] W. F. Trench, Absolute equal distribution of families of finite sets, *Linear Algebra Appl.* 367 (2003), 131-146.

[5] J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, 1965.