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Asymptotic relationships between singular values of structured matrices similarly generated by different formal expansions of a rational function

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SUMMARY

We define a class of formal expansions $\sum_{\ell=-\infty}^{\infty}\alpha_\ell z^\ell$ of a rational function with at least one nonzero pole. To distinct formal expansions $\sum_{\ell=-\infty}^{\infty}\alpha_\ell z^\ell$ and $\sum_{\ell=-\infty}^{\infty}\beta_\ell z^\ell$ in this class we associate structured arrays $\mathbf{A}=(a_{ij})_{i,j=1}^{\infty}$ and $\mathbf{B}=(b_{ij})_{i,j=1}^{\infty}$, defined by $a_{ij}=\sum_{\nu=1}^{k}a_\nu\alpha_{p_\nu i+qj+\tau_\nu}$ and $b_{ij}=\sum_{\nu=1}^{k}a_\nu\beta_{p_\nu i+qj+\tau_\nu}$, where $q\,(\neq 0)$, $p_1,\ \dots,\ p_k$, and $\tau_1,\ \dots,\ \tau_k$ are integers and $a_1,\ \dots,\ a_k$ are nonzero complex constants. We study the asymptotic relationship between the singular values of the matrices $(a_{ij})_{1\le i\le h_n,1\le j\le k_n}$ and $(b_{ij})_{1\le i\le h_n,1\le j\le k_n}$ as $\min(h_n,k_n)\to\infty$.

KEY WORDS: Singular value; rational function; structured matrix

1. INTRODUCTION

We begin by discussing the problem that motivated this paper. Problems of this kind arise in connection with preconditioning of structured linear systems.

Every rational function with at least one nonzero pole can be written as

$$R(z)=z^kP(z)+\sum_{s=1}^{m}\sum_{r=0}^{d_s-1}\frac{c_{rs}}{(z-z_s)^{r+1}},\tag{1}$$

where k is an integer, P is a polynomial, $m\ge 1$, $z_1,\ \dots,\ z_m$ are nonzero and distinct, $d_1,\ \dots,$ d_m are positive integers, and $c_{d_s-1,s}\neq 0,\quad 1\le s\le m$. We call

$$d=d_1+\cdots+d_m\tag{2}$$

the *rank index* of R. For completeness, the rank index of a rational function with no nonzero poles is zero.

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If

$$|z_s| \neq 1, \quad 1 \leq s \leq k, \tag{3}$$

then $R(z)$ has a unique expansion

$$R(z) = \sum_{\ell=-\infty}^{\infty} t_\ell z^\ell$$

that converges on an open annulus containing the unit circle $|z| = 1$. Let

$$h(\theta) = R(e^{i\theta}) \tag{4}$$

and

$$M = \max_{-\pi \leq \theta \leq \pi} |h(\theta)|. \tag{5}$$

If $C \in \mathbb{C}^{h \times k}$ where $n = \min(h, k)$, let

$$\sigma_1(C) \leq \sigma_2(C) \leq \cdots \leq \sigma_n(C)$$

denote the singular values of C. According to the Avram–Parter theorem [1, 2], the singular values of the Toeplitz matrices $T_n = (t_{r-s})_{r,s=1}^n$, $n \geq 1$, are all in the interval $[0, M]$ and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(\sigma_i(T_n)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(|h(\theta)|)\, d\theta \tag{6}$$

for every F in $C[0, M]$. Because of this it is usually said that the singular values of $\{T_n\}_{n=1}^{\infty}$ are asymptotically distributed like the values of $|h|$.

For a given (r, s),

$$\frac{1}{(z - z_s)^{r+1}} = (-1)^{r+1} \sum_{\ell=0}^{\infty} \binom{r+\ell}{r} z_s^{-r-\ell-1} z^\ell, \quad |z| < |z_s|, \tag{7}$$

and

$$\frac{1}{(z - z_s)^{r+1}} = (-1)^{r} \sum_{\ell=-\infty}^{-1} \binom{r+\ell}{r} z_s^{-r-\ell-1} z^\ell, \quad |z| > |z_s|. \tag{8}$$

(Note that the binomial coefficients on the right vanish for $-r \leq \ell \leq -1$.) Let

$$\mathcal{S} = \{(r, s) \,|\, c_{rs} \neq 0\}$$

and let $C(\mathcal{S})$ be the cardinality of $\mathcal{S}$. By choosing between the expansions (7) and (8) for each (r, s) in $\mathcal{S}$, we obtain $2^{C(\mathcal{S})}$ formal expansions for $R(z)$. We denote this class of expansions by $\mathcal{E}(R)$.

The question we were first interested in was the following: If $\sum_{\ell=-\infty}^{\infty} \alpha_\ell z^\ell$ is arbitrary member of $\mathcal{E}(R)$, what can be said about the asymptotic distribution of the singular values of the Toeplitz matrices $S_n = (\alpha_{r-s})_{r,s=1}^n$, $n \geq 1$? (We considered the analogous question for the eigenvalues of rationally generated Hermitian Toeplitz matrices in [3].) In trying to answer this question we were led to a more general problem, which is the main subject of this paper.

Henceforth we do not assume (3) except in Section 3. Let $q\,(\neq 0)$, $p_1, \ldots, p_k$, and $\tau_1, \ldots, \tau_k$ be integers, let $a_1, \ldots, a_k$ be nonzero complex constants, and define the array $\mathbf{A} = \{a_{ij}\}_{i,j=1}^{\infty}$ by

$$a_{ij} = \sum_{\nu=1}^{k} a_\nu \alpha_{p_\nu i + qj + \tau_\nu}, \quad 1 \le i, j < \infty, \tag{9}$$

where $\sum_{\ell=-\infty}^{\infty} \alpha_\ell z^\ell \in \mathcal{E}(R)$. We say that $\mathbf{A}$ is *generated by* $R(z)$ *and the parameters* q, $p_1, \ldots,$ p_k, $\tau_1, \ldots, \tau_k$, *and* $a_1, \ldots, a_k$. If $\mathbf{B} = (b_{ij})_{i,j=1}^{\infty}$ where

$$b_{ij} = \sum_{\nu=1}^{k} a_\nu \beta_{p_\nu i + qj + \tau_\nu}, \quad 1 \le i, j < \infty, \tag{10}$$

and $\sum_{\ell=-\infty}^{\infty} \beta_\ell z^\ell \in \mathcal{E}(R)$, we say that $\mathbf{A}$ and $\mathbf{B}$ are *similarly generated* by $R(z)$. If h and k are positive integers, the matrices

$$A = (a_{ij})_{1 \le i \le h, 1 \le j \le k} \quad \text{and} \quad B = (b_{ij})_{1 \le i \le h, 1 \le j \le k}$$

are the $h \times k$ *sections* of $\mathbf{A}$ and $\mathbf{B}$, respectively.

Let $\mathcal{F}$ be the set of real-valued continuous functions F on $[0, \infty)$ such that $\lim_{x \to \infty} F(x)$ exists (finite). We will prove the following theorem.

Theorem 1. *Suppose that* $\epsilon > 0$, $F \in \mathcal{F}$, *and the arrays* $\mathbf{A} = (a_{ij})_{i,j=1}^{\infty}$ *and* $\mathbf{B} = (b_{ij})_{i,j=1}^{\infty}$ *are similarly generated by a rational function* $R(z)$ *with rank index* $d > 0$. *Then there is an integer* N *that depends only on* ϵ, F, *and* d *such that if* A *and* B *are* $h \times k$ *sections of* $\mathbf{A}$ *and* $\mathbf{B}$ *and* $n := min(h, k) > N$, *then*

$$\sum_{i=1}^{n} |F(\sigma_i(A)) - F(\sigma_i(B))| < n\epsilon. \tag{11}$$

We emphasize that N is independent of the particular choice of $R(z)$ and the parameters in (9) and (10).

2. PROOF OF THEOREM 1

From (1), (7), and (8),

$$\beta_\ell - \alpha_\ell = \sum_{s=1}^{m} \sum_{r=0}^{d_s - 1} \gamma_{rs} \binom{r + \ell}{r} z_s^{-r-\ell-1},$$

where $\gamma_{rs} = 0$ if the same choice of (7) or (8) occurs in both expansions, while $\gamma_{rs} = \pm c_{rs}$ if the choices differ. Therefore, from (9) and (10),

$$b_{ij} - a_{ij} = \sum_{\nu=1}^{k} a_\nu \sum_{s=1}^{m} \sum_{r=0}^{d_s - 1} \gamma_{rs} \binom{r + p_\nu i + qj + \tau_\nu}{r} z_s^{-r - p_\nu i - qj - \tau_\nu - 1}.$$

Now suppose that $n > d$ (see (2)). If

$$\prod_{s=1}^{m} (1 - z_s^q z)^{d_s} = \sum_{\mu=0}^{d} \phi_\mu z^\mu$$

then the $k-d$ linearly independent k-vectors

$$[\phi_0\,\phi_1\,\cdots\,\phi_d\,0\,\cdots\,0]^T,\ [0\,\phi_0\,\phi_1\,\cdots\,\phi_d\,0\,\cdots\,0]^T,\ \dots,\ [0\,\cdots\,0\cdots\phi_0\,\phi_1\,\cdots\,\phi_d]^T$$

are in the null space of $B-A$. Therefore $\operatorname{rank}(B-A)\le d$. Since

$$B^*B - A^*A = (B^*-A^*)B + A^*(B-A),$$

it follows that

$$\operatorname{rank}(B^*B - A^*A) \le 2d. \tag{12}$$

Henceforth we assume that $n > 2d$. By a standard theorem (see, e.g., [5, pp. 94–97]) first applied by Tyrtyshnikov [4] to singular value distribution problems, (12) implies that there are nonnegative integers p and q such that

$$p+q \le 2d \tag{13}$$

and

$$\sigma_{i-q}(A) \le \sigma_i(B) \le \sigma_{i+p}(A), \quad q+1 \le i \le n-p. \tag{14}$$

Now suppose that $0<\rho<\infty$. Since

$$F(x) - F(\min(x,\rho)) = \begin{cases} 0, & 0 \le x \le \rho, \\ F(x) - F(\rho), & x > \rho, \end{cases}$$

and $\lim_{x\to\infty} F(x)$ exists (finite), there is a ρ such that

$$|F(x) - F(\min(x,\rho))| < \epsilon/8, \quad 0 \le x < \infty.$$

By the Weierstrass approximation theorem, there is a polynomial P such that

$$|F(x) - P(x)| < \epsilon/8, \quad 0 \le x \le \rho.$$

Then

$$|F(\sigma_i(A)) - F(\sigma_i(B))| < |P(\min(\sigma_i(A),\rho)) - P(\min(\sigma_i(B),\rho))| + \epsilon/2,$$

so

$$\sum_{i=1}^{n} |F(\sigma_i(A)) - F(\sigma_i(B))| < n\epsilon/2 + KS_n, \tag{15}$$

where $K = \max_{0\le x\le\rho} |P'(x)|$ and

$$S_n = \sum_{i=1}^{n} |\min(\sigma_i(A),\rho) - \min(\sigma_i(B),\rho)|.$$

For convenience, denote

$$s_i = \min(\sigma_i(A),\rho), \quad t_i = \min(\sigma_i(B),\rho). \tag{16}$$

Then

$$S_n = \sum_{i=1}^{n} |s_i - t_i| = T_1 + T_2 + T_3, \tag{17}$$

where

$$T_1 = \sum_{i=1}^{q} |s_i - t_i| \le q\rho, \quad T_3 = \sum_{i=n-p+1}^{n} |s_i - t_i| \le p\rho, \tag{18}$$

and

$$T_2 = \sum_{i=q+1}^{n-p} |s_i - t_i|.$$

Since $\min(x, \rho) \le \min(y, \rho)$ if $0 \le x \le y$, (14) and (16) imply that

$$s_{i-q} \le t_i \le s_{i+p}, \quad q+1 \le i \le n-p. \tag{19}$$

Since

$$s_{i-q} \le s_i \le s_{i+p}, \quad q+1 \le i \le n-p,$$

(19) implies that

$$|s_i - t_i| \le s_{i+p} - s_{i-q}, \quad q+1 \le i \le n-p.$$

Hence,

$$T_2 \le \sum_{i=q+1}^{n-p} (s_{i+p} - s_{i-q}) \le (p+q)\rho.$$

This, (17) and (18) imply that $S_n \le 2(p+q)\rho$. Hence, (13) and (15) imply that

$$\sum_{i=1}^{n} |F(\sigma_i(A)) - F(\sigma_i(B))| < n\epsilon/2 + 4d\rho K.$$

Now choose N so that $N\epsilon \ge 8d\rho K$; then (11) holds if $n > N$.

3. BACK TO THE AVRAM–PARTER THEOREM

We can now answer the question raised at the beginning of this paper. We let $k = a_1 = p_1 = -q = 1$ and and $\tau_1 = 0$ in (9) and (10).

Theorem 2. *Let $R(z)$ be as in (1) with $d > 0$ and assume that (3) holds. For $n \ge 1$, let $T_n = (t_{r-s})_{r,s=1}^n$ and $S_n = (\alpha_{r-s})_{r,s=1}^n$, where $\sum_{\ell=-\infty}^{\infty} t_\ell z^\ell$ is the expansion of $R(z)$ that converges on an open annulus containing the unit circle $|z| = 1$ and $\sum_{\ell=-\infty}^{\infty} \alpha_\ell z^\ell$ is an arbitrary member of $\mathcal{E}(R)$. Let h and M be as in (4) and (5). Then*

$$\sigma_i(S_n) \le M, \quad 1 \le i \le n-2d, \quad n > 2d, \tag{20}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-2d} F(\sigma_i(S_n)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(|h(\theta)|)\, d\theta \tag{21}$$

for every F in $C[0, M]$.

Proof From (13) and (14) with A and B replaced by T_n and S_n,

$$\sigma_i(S_n) \le \sigma_{i+2d}(T_n), \quad 1 \le i \le n-2d, \quad n > 2d.$$

Since the singular values of $\{T_n\}_{n=1}^{\infty}$ are all in $[0, M]$, this implies (20).

If $F \in C[0, M]$, define

$$G(x) = F(\min(x, M)), \quad 0 \le x < \infty. \tag{22}$$

Then $G \in \mathcal{F}$, so Theorem 1 implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |G(\sigma_i(S_n)) - G(\sigma_i(T_n))| = 0.$$

This and (6) imply that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} G(\sigma_i(S_n)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(|h(\theta)|)\, d\theta. \tag{23}$$

However, from (20) and (22),

$$\sum_{i=1}^{n} G(\sigma_i(S_n)) = \sum_{i=1}^{n-2d} F(\sigma_i(S_n)) + \sum_{i=n-2d+1}^{n} G(\sigma_i(S_n)).$$

Since G is bounded on $[0, \infty)$, this and (23) imply (21).

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