FUNCTIONAL PERTURBATIONS OF NONOSCILLATORY SECOND ORDER DIFFERENCE EQUATIONS

WILLIAM F. TRENCH

Trinity University, San Antonio, Texas, 78212 USA

Dedicated to John Baxley on the occasion of his retirement from Wake Forest University

Dynamic Systems and Applications 15 (2006) 155–171

ABSTRACT. We consider functional perturbations of the nonoscillatory equation

$$\Delta(r_{n-1}\Delta x_{n-1}) + p_n x_n = 0.$$

Let $\mathcal{S}_k$ be the set of all real sequences of the form $Y = \{y_n\}_{n=k}^{\infty}$. For each $n > k$, let $f_n(Y)$ denote a real-valued functional of $Y \in \mathcal{S}_k$. We give sufficient conditions on $\{f_n(Y)\}_{n=k+1}^{\infty}$ for the equation

$$\Delta(r_{n-1}\Delta y_{n-1}) + p_n y_n = f_n(Y), \quad n > k,$$

to have a solution $\widehat{Y} \in \mathcal{S}_k$ that behaves in a precisely defined way like a given solution $\widehat{X}$ of the unperturbed equation as $n \to \infty$. A specific family of nonlinear functional perturbations is considered in detail.

AMS (MOS) Subject Classification. 39A11.

1. INTRODUCTION

Throughout this paper all quantities are real. We consider functional perturbations of the equation

$$\Delta(r_{n-1}\Delta x_{n-1}) + p_n x_n = 0 \tag{1.1}$$

under the standing assumption that $\{r_n\}^{\infty}$ and $\{p_n\}^{\infty}$ are infinite sequences of real numbers with $r_n > 0$, and (1.1) is nonoscillatory. In this case there is an integer k and sequences $X_1 = \{x_{1n}\}_{n=k}^{\infty}$ and $X_2 = \{x_{2n}\}_{n=k}^{\infty}$ of positive numbers that satisfy (1.1) for $n > k$, such that

$$r_n(x_{1n}x_{2,n+1} - x_{1,n+1}x_{2n}) = 1, \quad n \geq k, \tag{1.2}$$

and

$$\lim_{n\to\infty} \frac{x_{2n}}{x_{1n}} = \infty. \tag{1.3}$$

The solutions X_1 and X_2 are said to be *recessive* (or *principal*) and *dominant* (or *nonprincipal*), respectively, and X_1 is unique up to a positive constant multiplier [3]. Let

$$\rho_n = \frac{x_{2n}}{x_{1n}}.$$

From (1.2) and (1.3),

$$\Delta\rho_n = \frac{1}{r_n x_{1n} x_{1,n+1}} > 0 \quad \text{and} \quad \lim_{n\to\infty} \rho_n = \infty. \tag{1.4}$$

Let $\mathcal{S}_k$ be the set of real sequences $Y = \{y_n\}_{n=k}^{\infty}$. For each $n > k$, let $f_n(Y)$ denote a real-valued functional of Y. We give sufficient conditions on $\{f_n(Y)\}_{n=k+1}^{\infty}$ for the functional equation

$$\Delta(r_{n-1}\Delta y_{n-1}) + p_n y_n = f_n(Y), \quad n > k, \tag{1.5}$$

to have a solution $\widehat{Y} \in \mathcal{S}_k$ that behaves in a precisely defined way like a given solution $\widehat{X}$ of (1.1) as $n \to \infty$. The case where $f_n(Y) = f_n y_n$ has previously been considered in [1, 4, 5]. Some nonlinear and nonhomogeneous perturbations were also considered in [1].

2. PRELIMINARY CONSIDERATIONS

We impose the metric

$$d(X, Y) = \sum_{n=k}^{\infty} \frac{1}{2^{n-k}} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

on $\mathcal{S}_k$. If $\{Y_\nu\}$ is a sequence in $\mathcal{S}_k$, then $\lim_{\nu\to\infty} Y_\nu = Y$ with respect to this metric if and only if $\lim_{\nu\to\infty} y_{\nu n} = y_n$ for every $n \geq k$. Moreover, $\mathcal{S}_k$ is complete; i.e., $(\mathcal{S}_k, d)$ is a Frechét space.

We use the following lemmas to obtain our results. The first is a special case of the Schauder–Tychonoff theorem [2, Corollary 0.1, p 405].

Lemma 2.1. *If $\mathcal{U}$ is a closed, convex, and compact subset of a Frechét space and T is a continuous mapping of $\mathcal{U}$ into itself, then T has a fixed point in $T(\mathcal{U})$.*

Lemma 2.2. *Suppose that $\sum_{n=k}^{\infty} u_m x_{2m}$ converges, and let*

$$\psi_n = \sup_{\ell \geq n} \left| \sum_{m=\ell}^{\infty} u_m x_{2m} \right|, \quad n \geq k. \tag{2.1}$$

Then $\sum_{m=k}^{\infty} u_m x_{1m}$ also converges,

$$\left| \sum_{m=n}^{\infty} u_m x_{1m} \right| \leq \frac{2\psi_n}{\rho_n}, \quad n \geq k, \tag{2.2}$$

and

$$\left| \sum_{m=n+1}^{\infty} (x_{2m} - x_{1m}\rho_n) u_m \right| \leq \psi_{n+1}, \quad n \geq k. \tag{2.3}$$

Proof. Define $g_n = \sum_{m=n}^{\infty} u_m x_{2m}$. If $k \le n < N$, summation by parts yields

$$\sum_{m=n}^{N} u_m x_{1m} = \sum_{m=n}^{N} \frac{u_m x_{2m}}{\rho_m} = \sum_{m=n}^{N} \frac{g_m - g_{m+1}}{\rho_m}$$

$$= \frac{g_n}{\rho_n} - \frac{g_{N+1}}{\rho_N} + \sum_{m=n+1}^{N} g_m \left(\frac{1}{\rho_m} - \frac{1}{\rho_{m-1}} \right). \tag{2.4}$$

From (1.4) and (2.1),

$$\sum_{m=n+1}^{\infty} |g_m| \left| \frac{1}{\rho_m} - \frac{1}{\rho_{m-1}} \right| \le \frac{\psi_{n+1}}{\rho_n}, \quad n \ge k, \tag{2.5}$$

and $\lim_{N\to\infty} g_{N+1}/\rho_N = 0$. Therefore (2.4) implies that $\sum_{m=n}^{\infty} u_m x_{1m}$ converges and satisfies (2.2).

If $k \le n \le N$, summation by parts yields

$$\sum_{m=n+1}^{N} (x_{2m} - x_{1m}\rho_n) u_m = \sum_{m=n+1}^{N} \left(1 - \frac{\rho_n}{\rho_m}\right)(g_m - g_{m+1})$$

$$= -\left(1 - \frac{\rho_n}{\rho_N}\right) g_{N+1} + \rho_n \sum_{m=n+1}^{N} \left(\frac{1}{\rho_{m-1}} - \frac{1}{\rho_m} \right) g_m.$$

Letting $N \to \infty$ and invoking (2.5) yields (2.3). □

The following elementary lemma motivates the choices of transformations of $\mathcal{S}_k$ whose fixed points are solutions of (1.5).

Lemma 2.3. *If $\sum_{n=k+1}^{\infty} q_n x_{1n}$ converges, then*

$$z_n = -x_{1n} \sum_{\ell=k}^{n-1} (\Delta\rho_\ell) \sum_{m=\ell+1}^{\infty} q_m x_{1m}, \quad n \ge k,$$

is a solution of

$$\Delta(r_{n-1}\Delta z_{n-1}) + p_n z_n = q_n, \quad n > k. \tag{2.6}$$

If $\sum_{n=k+1}^{\infty} q_n x_{2n}$ converges, then

$$z_n = \sum_{m=n+1}^{\infty} (x_{2m} x_{1n} - x_{1m} x_{2n}) q_m, \quad n \ge k,$$

is a solution of (2.6).

3. GENERAL RESULTS

Theorem 3.1. *Let* $\widehat{X}$ *be a given solution of* (1.1) *for* $n \geq k$, *and let* $\{\sigma_n\}_{n=k+1}^{\infty}$ *be a sequence of positive numbers. Let* $\mathcal{U}_1$ *be the set of all* $Y \in \mathcal{S}_k$ *such that*

$$\left|\frac{y_n - \widehat{x}_n}{x_{1n}}\right| \leq \sigma_{n+1} \quad \text{and} \quad \left|\Delta\left(\frac{y_n - \widehat{x}_n}{x_{1n}}\right)\right| \leq 2\sigma_{n+1}\frac{\Delta\rho_n}{\rho_{n+1}}, \quad n \geq k. \tag{3.1}$$

Suppose that

(i) $\{f_n(Y)\}_{n=k+1}^{\infty}$ *is defined if* $Y \in \mathcal{U}_1$;

(ii) *If* $\{Y_\nu\}$ *is a sequence in* $\mathcal{U}_1$ *such that* $\lim_{\nu\to\infty} Y_\nu = Y$, *then* $\lim_{\nu\to\infty} f_n(Y_\nu) = f_n(Y)$, $n > k$;

(iii) $\sum_{m=k+1}^{\infty} x_{2m} f_m(Y)$ *converges for every* $Y \in \mathcal{U}_1$, *and there is a sequence* $\{\phi_n\}_{n=k+1}^{\infty}$ *such that*

$$\sup_{\ell \geq n}\left|\sum_{m=\ell}^{\infty} x_{2m} f_m(Y)\right| \leq \phi_n, \quad n > k, \quad Y \in \mathcal{U}_1, \tag{3.2}$$

$$\phi_n \leq \sigma_n, \quad n > k, \tag{3.3}$$

and $\lim_{n\to\infty} \phi_n = 0$. *Then* (1.5) *has a solution* $\widehat{Y} \in \mathcal{S}_k$ *such that*

$$\left|\frac{\widehat{y}_n - \widehat{x}_n}{x_{1n}}\right| \leq \phi_{n+1} \quad \text{and} \quad \left|\Delta\left(\frac{\widehat{y}_n - \widehat{x}_n}{x_{1n}}\right)\right| \leq 2\phi_{n+1}\frac{\Delta\rho_n}{\rho_{n+1}}, \quad n \geq k. \tag{3.4}$$

Proof. $\mathcal{U}_1$ is closed, convex, and compact. From (iii) and Lemma 2.2 with $u_m = f_m(Y)$, we can define T_1 on $\mathcal{U}_1$ by

$$(T_1(Y))_n = \widehat{x}_n + \sum_{m=n+1}^{\infty} (x_{2m}x_{1n} - x_{1m}x_{2n}) f_m(Y), \quad n \geq k.$$

Then

$$\frac{(T_1(Y))_n - \widehat{x}_n}{x_{1n}} = \sum_{m=n+1}^{\infty} (x_{2m} - x_{1m}\rho_n) f_m(Y), \tag{3.5}$$

and

$$\Delta\left(\frac{(T_1(Y))_n - \widehat{x}_n}{x_{1n}}\right) = -(\Delta\rho_n)\sum_{m=n+1}^{\infty} x_{1m} f_m(Y),$$

so (3.2) and Lemma 2.2, with $u_m = f_m(Y)$, imply that

$$\left|\frac{(T_1(Y))_n - \widehat{x}_n}{x_{1n}}\right| \leq \phi_{n+1}, \quad n \geq k, \tag{3.6}$$

and

$$\left|\Delta\left(\frac{(T_1(Y))_n - \widehat{x}_n}{x_{1n}}\right)\right| \leq 2\phi_{n+1}\frac{\Delta\rho_n}{\rho_{n+1}}, \quad n \geq k. \tag{3.7}$$

Hence, (3.1) and (3.3) imply that $T_1(\mathcal{U}_1) \subset \mathcal{U}_1$.

Now suppose that $\{Y_\nu\}_{\nu=1}^\infty$ is a sequence in $\mathcal{U}_1$ such that $\lim_{\nu\to\infty} Y_\nu = Y$. We want to show that $\lim_{\nu\to\infty} T_1(Y_\nu) = T_1(Y)$; i.e., that

$$\lim_{\nu\to\infty}(T_1(Y_\nu))_n = (T_1(Y))_n, \quad n \ge k. \tag{3.8}$$

If $\epsilon > 0$, choose $N > k+1$ so that $\phi_n < \epsilon/4$ if $n > N$. Then (3.2) implies that

$$\left|\sum_{m=n}^\infty (f_m(Y_\nu) - f_m(Y))x_{2m}\right| \le \epsilon/2, \quad \nu \ge 1, \quad n > N. \tag{3.9}$$

With N now fixed, (ii) implies that there is a ν_0 such that

$$\sum_{m=k+1}^{N} |f_m(Y_\nu) - f_m(Y)|x_{2m} < \epsilon/2, \quad \nu \ge \nu_0.$$

This and (3.9) imply that

$$\left|\sum_{m=n}^\infty (f_m(Y_\nu) - f_m(Y))x_{2m}\right| < \epsilon, \quad n > k, \quad \nu \ge \nu_0. \tag{3.10}$$

Therefore, (3.5) and Lemma 2.2 with $u_m = f_m(Y_\nu) - f_m(Y)$ imply that

$$|(T_1(Y_\nu))_n - (T_1(Y))_n| = \left|\sum_{m=n+1}^\infty (x_{2m} - x_{1m}\rho_n)(f_m(Y_\nu) - f_m(Y))\right| \le \epsilon x_{1n},$$

$n \ge k$, $\nu \ge \nu_0$, which implies (3.8). Now Lemma 2.1 implies that there is a $\widehat{Y} \in U_1$ such $T_1(\widehat{Y}) = \widehat{Y}$; i.e.,

$$\widehat{y}_n = \widehat{x}_n + \sum_{m=n+1}^\infty (x_{2m}x_{1n} - x_{1m}x_{2n}) f_m(\widehat{Y}), \quad n \ge k,$$

and Lemma 2.3 implies that $\widehat{Y}$ satisfies (1.5). Setting $Y = \widehat{Y} = T(\widehat{Y})$ in (3.6) and (3.7) verifies (3.4). □

Theorem 3.2. *Let $\widehat{X}$ be a given solution of* (1.1) *for $n \ge k$, and let $\{\sigma_n\}_{n=k+1}^\infty$ be a positive sequence. Let $\mathcal{U}_2$ be the set of all $Y \in \mathcal{S}$ such that*

$$\left|\frac{y_n - \widehat{x}_n}{x_{2n}}\right| \le \sigma_{n+1} \quad \text{and} \quad \left|\Delta\left(\frac{y_n - \widehat{x}_n}{x_{2n}}\right)\right| \le 2\sigma_{n+1}\frac{\Delta\rho_n}{\rho_{n+1}}, \quad n \ge k. \tag{3.11}$$

Suppose that

(i) *$\{f_n(Y)\}_{n=k+1}^\infty$ is defined if $Y \in \mathcal{U}_2$;*

(ii) *If $\{Y_\nu\}$ is a sequence in $\mathcal{U}_2$ such that* $\lim_{\nu\to\infty} Y_\nu = Y$, *then* $\lim_{\nu\to\infty} f_n(Y_\nu) = f_n(Y)$, $n > k$;

(iii) *$\sum_{m=k+1}^\infty x_{1m} f_m(Y)$ converges for every $Y \in \mathcal{U}_2$, and there is a sequence $\{\phi_n\}_{n=k+1}^\infty$ such that*

$$\sup_{\ell \ge n}\left|\sum_{m=\ell}^\infty x_{1m} f_m(Y)\right| \le \phi_n, \quad n > k, \quad Y \in \mathcal{U}_2, \tag{3.12}$$

$\lim_{n\to\infty} \phi_n = 0$, *and*

$$\max(\widehat{\phi}_n, \phi_{n+1}) \le \sigma_{n+1}, \quad n \ge k, \tag{3.13}$$

where

$$\widehat{\phi}_n = \frac{1}{\rho_n} \sum_{\ell=k}^{n-1} \phi_{\ell+1} \Delta\rho_\ell. \tag{3.14}$$

Then (1.5) *has a solution* $\widehat{Y} \in \mathcal{S}_k$ *such that*

$$\left|\frac{\widehat{y}_n - \widehat{x}_n}{x_{2n}}\right| \le \widehat{\phi}_n \quad \text{and} \quad \left|\Delta\left(\frac{\widehat{y}_n - \widehat{x}_n}{x_{2n}}\right)\right| \le (\widehat{\phi}_n + \phi_{n+1})\frac{\Delta\rho_n}{\rho_{n+1}}, \quad n \ge k. \tag{3.15}$$

Proof. $\mathcal{U}_2$ is closed, convex, and compact. From (iii), we can define T_2 on $\mathcal{U}_2$ by

$$(T_2(Y))_n = \widehat{x}_n - x_{1n} \sum_{\ell=k}^{n-1} (\Delta\rho_\ell) \sum_{m=\ell+1}^{\infty} f_m(Y) x_{1m}, \quad n \ge k. \tag{3.16}$$

Then

$$\frac{(T_2(Y))_n - \widehat{x}_n}{x_{2n}} = -\frac{1}{\rho_n} \sum_{\ell=k}^{n-1} (\Delta\rho_\ell) \sum_{m=\ell+1}^{\infty} f_m(Y) x_{1m}$$

and

$$\Delta\left(\frac{(T_2(Y))_n - \widehat{x}_n}{x_{2n}}\right) = \frac{\Delta\rho_n}{\rho_n\rho_{n+1}} \sum_{\ell=k}^{n-1} (\Delta\rho_\ell) \sum_{m=\ell+1}^{\infty} f_m(Y) x_{1m} - \frac{\Delta\rho_n}{\rho_{n+1}} \sum_{m=n+1}^{\infty} f_m(Y) x_{1m}.$$

Therefore (3.12) and(3.14) imply that

$$\left|\frac{(T_2(Y))_n - \widehat{x}_n}{x_{2n}}\right| \le \widehat{\phi}_n, \quad n \ge k. \tag{3.17}$$

and

$$\left|\Delta\left(\frac{(T_2(Y))_n - \widehat{x}_n}{x_{2n}}\right)\right| \le (\widehat{\phi}_n + \phi_{n+1})\frac{\Delta\rho_n}{\rho_{n+1}}, \quad n \ge k. \tag{3.18}$$

Now (3.11) and (3.13) imply that $T_2(\mathcal{U}_2) \subset \mathcal{U}_2$.

Now suppose that $\{Y_\nu\}_{\nu=1}^{\infty}$ is a sequence in $\mathcal{U}_2$ such that $\lim_{\nu\to\infty} Y_\nu = Y$. We want to show that $\lim_{\nu\to\infty} T_2(Y_\nu) = T_2(Y)$; i.e. that,

$$\lim_{\nu\to\infty} T_2(Y_\nu)_n = T_2(Y)_n, \quad n \ge k. \tag{3.19}$$

Let $\epsilon > 0$. The argument used to obtain (3.10) shows that there is an integer ν_0 such that

$$\left|\sum_{m=n}^{\infty} (f_m(Y_\nu) - f_m(Y)) x_{1m}\right| < \epsilon, \quad n > k, \quad \nu \ge \nu_0.$$

Therefore, from (3.16),

$$|(T_2Y_\nu)_n - (T_2Y)_n| \le \epsilon x_{1n} \sum_{\ell=k}^{n-1} \Delta\rho_\ell = \epsilon x_{1n}(\rho_n - \rho_k), \quad n \ge k, \quad \nu \ge \nu_0.$$

This implies (3.19). Now Lemma 2.1 implies that there is a $\widehat{Y} \in \mathcal{U}_2$ such that $T_2(\widehat{Y}) = \widehat{Y}$; i.e.,

$$\widehat{y}_n = \widehat{x}_n - x_{1n} \sum_{\ell=k}^{n-1} (\Delta\rho_\ell) \sum_{m=\ell+1}^{\infty} f_m(\widehat{Y}) x_{1m}, \quad n \ge k,$$

and Lemma 2.3 implies that $\widehat{Y}$ satisfies (1.5). Setting $Y = \widehat{Y} = T_2(\widehat{Y})$ in (3.17) and (3.18) verifies (3.15). □

Note that $\lim_{n\to\infty} \widehat{\phi}_n = 0$. To see this, suppose $\epsilon > 0$. Since $\lim_{n\to\infty} \phi_n = 0$, there is an integer N such that $\phi_m < \epsilon$ if $m > N$. Therefore, (3.14) implies that

$$\widehat{\phi}_n < \frac{1}{\rho_n} \sum_{\ell=k}^{N} (\Delta\rho_\ell)\phi_{\ell+1} + \epsilon, \quad n > N+1.$$

Hence, $\limsup_{n\to\infty} \widehat{\phi}_n \le \epsilon$. Since ϵ is arbitrary, it follows that $\lim_{n\to\infty} \widehat{\phi}_n = 0$.

4. A SPECIFIC RESULT

Now we consider the equation

$$\Delta(r_{n-1}\Delta y_{n-1}) + p_n y_n = \rho_n^{-\alpha} a_n \sum_{\ell=k}^{n} b_\ell y_\ell^\gamma, \quad n > k. \tag{4.1}$$

We will need the following lemmas.

Lemma 4.1 (Abel-Dini Theorem)**.** *If $s > 0$ then*

$$\sum^{\infty} \frac{\Delta\rho_{n-1}}{\rho_n^{s+1}} < \infty.$$

Lemma 4.2. *If $\rho_{n+1} = O(\rho_n)$ and s, t are real, then*

$$\sum_{m=n}^{\infty} \rho_m^s \Delta\rho_m^t = O(\rho_n^{s+t}) \quad \text{if} \quad s+t < 0, \tag{4.2}$$

and

$$\sum_{m=k+1}^{n} \rho_m^s \Delta\rho_{m-1}^t = \begin{cases} O(\rho_n^{s+t}) & \text{if } s+t > 0, \\ O(1) & \text{if } s+t < 0. \end{cases} \tag{4.3}$$

Proof. For (4.2), it suffices to show that $\rho_m^s \Delta\rho_m^t = O\left(\Delta\rho_m^{s+t}\right)$. By the mean value theorem,

$$\rho_m^s \Delta\rho_m^t = t\rho_m^s u_m^{t-1} \Delta\rho_m \quad \text{and} \quad \Delta\rho_m^{s+t} = (s+t) v_m^{s+t-1} \Delta\rho_m,$$

where $\rho_m < u_m, v_m < \rho_{m+1}$. Therefore

$$\frac{\rho_m^s \Delta\rho_m^t}{\Delta\rho_m^{s+t}} = \frac{t}{s+t}\left(\frac{\rho_m}{v_m}\right)^s \left(\frac{u_m}{v_m}\right)^{t-1},$$

which is bounded for all m.

The proof of (4.3) is similar. □

Theorem 4.3. *Suppose that* $\gamma(\neq 0, 1)$ *is real*, $c > 0$, *and*

$$A_n = \sum_{\ell=k}^{n-1} a_\ell x_{1\ell}, \quad B_n = \sum_{\ell=k}^{n-1} b_\ell x_{1\ell}^\gamma, \quad C_n = \sum_{\ell=k}^{n-1} A_\ell b_\ell x_{1\ell}^\gamma \quad n \geq k, \tag{4.4}$$

are bounded. Suppose also that one of the following hypotheses hold:

$H_1 : i = r = 1,\ j = 2, \alpha > 1,$ *and* $\lambda = \alpha - 1$.

$H_2 : i = 1,\ j = r = 2, \alpha > \max(1, \gamma + 1), \alpha \neq 2\gamma, \lambda = \alpha - \max(\gamma + 1, 1),$

$$\rho_n^{-\lambda} = O(\Delta\rho_{n-1}), \tag{4.5}$$

and

$$\rho_{n+1} = O(\rho_n). \tag{4.6}$$

$H_3 : i = r = 2,\ j = 1, 0 < \alpha - \max(\gamma, 0) < 1, \alpha \neq 2\gamma,$ (4.6) *holds, and*

$$\lambda = \alpha - \max(\gamma, 0). \tag{4.7}$$

Then (4.1) *has a solution* $\widehat{Y} \in \mathcal{S}_k$ *such that*

$$\frac{\widehat{y}_n - c x_{rn}}{x_{in}} = O(\rho_{n+1}^{-\lambda}) \quad \textit{and} \quad \Delta\left(\frac{\widehat{y}_n - c x_{rn}}{x_{in}}\right) = O\left(\frac{\Delta\rho_n}{\rho_{n+1}^{\lambda+1}}\right)$$

if $c^{\gamma-1}$ *is sufficiently small.*

Proof. Let $0 < \theta < 1$ and let $\mathcal{U}$ be the set of all $Y \in \mathcal{S}_k$ such that

$$\left|\frac{y_n - c x_{rn}}{x_{in}}\right| \leq \theta c \frac{\rho_k^{\lambda+r-i}}{\rho_{n+1}^\lambda} \tag{4.8}$$

and

$$\left|\Delta\left(\frac{y_n - c x_{rn}}{x_{in}}\right)\right| \leq 2\theta c \rho_k^{\lambda+r-i} \frac{\Delta\rho_n}{\rho_{n+1}^{\lambda+1}}, \quad n \geq k. \tag{4.9}$$

To avoid annoying repetition, we invoke the boundedness of $\{A_n\}$, $\{B_n\}$ and $\{C_n\}$ repeatedly without stating that we are doing so. We also use summation by parts repeatedly without stating this explicitly, and without verifying in advance that the infinite series that arise from this are convergent. They are, in fact, all absolutely convergent, as our subsequent estimates will show. To avoid unnecesary and annoying subscripts, we will use the symbol M (as in $|u_n| \leq M|v_n|$) as a generic constant throughout this proof; thus, the value of M may be different in each appearance. It is to be understood throughout that M is independent of $Y \in \mathcal{U}$ and c. If g_n is a functional of Y, we write $g_n(Y) = O(c^\gamma v_n)$ to indicate that $|g_n(Y)| \le Mc^\gamma |v_n|$, $n \ge k$, for all $Y \in \mathcal{U}$ and $c > 0$. Finally, whenever we invoke Lemma 4.2 we are also invoking (4.6).

Under all three hypotheses,

$$\lambda = \begin{cases} \alpha - j + 1 - (r-1)\gamma & \text{if } \gamma > 0, \\ \alpha - j + 1 & \text{if } \gamma < 0. \end{cases} \tag{4.10}$$

It is to be understood throughout the proof that whenever we state that a quantity is $O(\rho_n^{-\lambda})$ or $O(c^\gamma \rho_n^{-\lambda})$, we are invoking (4.10).

It is convenient to work with the variable $z_n = y_n/x_{in}$. Then (4.8) and (4.9) become

$$|z_n - c\rho_n^{r-i}| \le \theta c \frac{\rho_k^{\lambda+r-i}}{\rho_{n+1}^{\lambda}} \tag{4.11}$$

and

$$|\Delta(z_n - c\rho_n^{r-i})| \le 2\theta c \rho_k^{\lambda+r-i} \frac{\Delta\rho_n}{\rho_{n+1}^{\lambda+1}}, \quad n \ge k. \tag{4.12}$$

Note that if $Y \in \mathcal{U}$ and

$$|\zeta_n - c\rho_n^{r-i}| \le |z_n - c\rho_n^{r-i}|, \quad n \ge k,$$

then

$$c(1-\theta)\rho_n^{r-i} \le \zeta_n \le c(1+\theta)\rho_n^{r-i}, \quad n \ge k. \tag{4.13}$$

From (1.5) and (4.1),

$$f_n(Y) = \rho_n^{-\alpha} a_n \sum_{\ell=k}^{n} b_\ell y_\ell^\gamma, \quad n > k.$$

We will show that

$$\sum_{m=n}^{\infty} x_{jm} f_m(Y) = O(c^\gamma \rho_n^{-\lambda}) \tag{4.14}$$

under all three hypotheses. Then we will apply Theorems 3.1 and 3.2 obtain the conclusions.

To establish (4.14), we write

$$\sum_{m=n}^{\infty} x_{jm} f_m(Y) = c^\gamma I_n + Z_n(Y),$$

where

$$I_n = \sum_{m=n}^{\infty} x_{jm} f_m(X_r) = \sum_{m=n}^{\infty} \rho_m^{-\alpha} a_m x_{jm} \sum_{\ell=k}^{n} b_\ell x_{r\ell}^\gamma \tag{4.15}$$

and

$$Z_n(Y) = \sum_{m=n}^{\infty} x_{jm} f_m(Y) - c^\gamma I_n = \sum_{m=n}^{\infty} \rho_m^{-\alpha} a_m x_{jm} \sum_{\ell=k}^{m} \left(y_\ell^\gamma - c^\gamma x_{r\ell}^\gamma\right) b_\ell. \tag{4.16}$$

It suffices to show that

$$I_n = O(\rho_n^{-\lambda}) \tag{4.17}$$

and

$$Z_n(Y) = O(c^\gamma \rho_n^{-\lambda}), \quad Y \in \mathcal{U}, \quad c > 0. \tag{4.18}$$

We begin with (4.17). Recall that $x_{jm} = \rho_m^{j-1} x_{1m}$ and $x_{r\ell} = \rho_\ell^{r-1} x_{1m}$. From (4.4) and (4.15),

$$I_n = \sum_{m=n}^{\infty} \rho_m^{j-\alpha-1} a_m x_{1m} \sum_{\ell=k}^{m} \rho_\ell^{(r-1)\gamma} b_\ell x_{1\ell}^\gamma = \sum_{m=n}^{\infty} \rho_m^{j-\alpha-1} h_m \Delta A_m, \tag{4.19}$$

with

$$h_m = \sum_{\ell=k}^{m} \rho_\ell^{(r-1)\gamma} \Delta B_\ell. \tag{4.20}$$

If $r = 1$ then $h_m = B_{m+1} = O(1)$. If $r = 2$ then

$$h_m = \rho_m^\gamma B_{m+1} - \sum_{\ell=k+1}^{m} B_\ell \Delta\left(\rho_{\ell-1}^\gamma\right) = \begin{cases} O(\rho_m^\gamma) & \text{if } \gamma > 0, \\ O(1) & \text{if } \gamma < 0. \end{cases}$$

In either case,

$$h_m = \begin{cases} O(\rho_m^{(r-1)\gamma}) & \text{if } \gamma > 0, \\ O(1) & \text{if } \gamma < 0. \end{cases} \tag{4.21}$$

From the second equality in (4.19),

$$I_n = -E_n - F_n - G_n,$$

where

$$E_n = \rho_n^{j-\alpha-1} h_n A_n, \tag{4.22}$$

$$F_n = \sum_{m=n+1}^{\infty} A_m \rho_m^{j-\alpha-1} \Delta h_{m-1}, \tag{4.23}$$

and

$$G_n = \sum_{m=n}^{\infty} A_{m+1} h_m \Delta\left(\rho_m^{j-\alpha-1}\right). \tag{4.24}$$

From (4.21) and (4.22), $E_n = O(\rho_n^{-\lambda})$. From (4.4), (4.20), and (4.23)

$$F_n = \sum_{m=n+1}^{\infty} A_m \rho_m^{j-\alpha-1+(r-1)\gamma} b_m x_{1m}^{\gamma} = \sum_{m=n+1}^{\infty} \rho_m^{j-\alpha-1+(r-1)\gamma} \Delta C_m$$
$$= -\rho_{n+1}^{j-\alpha-1+(r-1)\gamma} C_{n+1} - \sum_{m=n+2}^{\infty} C_m \Delta\left(\rho_{m-1}^{j-\alpha-1+(r-1)\gamma}\right) = O(\rho_n^{-\lambda}).$$

From (4.21) and (4.24), $G_n = O(\rho_n^{-\lambda})$ if $r = 1$ or $\gamma < 0$, while if $r = 2$ and $\gamma > 0$ then

$$|G_n| \leq M \sum_{m=n}^{\infty} \rho_m^{(r-1)\gamma} |\Delta \rho_m^{j-\alpha-1}| = O(\rho_n^{-\lambda})$$

by Lemma 4.2. We have now verified (4.14) under all three hypotheses.

Turning to (4.18), substituting

$$y_\ell = z_\ell x_{i\ell} = \rho_\ell^{i-1} z_\ell x_{1\ell} \quad \text{and} \quad x_{r\ell} = \rho_\ell^{(r-1)} x_{1\ell}$$

into (4.16) and invoking (4.4) yields

$$Z_n(Y) = \sum_{m=n}^{\infty} \rho_m^{j-\alpha-1} g_m \Delta A_m, \tag{4.25}$$

where

$$g_m = \sum_{\ell=k}^{m} \left(z_\ell^{\gamma} - c^{\gamma} \rho_\ell^{(r-i)\gamma}\right) \rho_\ell^{(i-1)\gamma} \Delta B_\ell = P_m - Q_m - R_m, \tag{4.26}$$

with

$$P_m = B_{m+1}(z_m^{\gamma} - c^{\gamma} \rho_m^{(r-i)\gamma}) \rho_m^{(i-1)\gamma}, \tag{4.27}$$

$$Q_m = \sum_{\ell=k+1}^{m} B_\ell \rho_\ell^{(i-1)\gamma} \Delta(z_{\ell-1}^{\gamma} - c^{\gamma} \rho_{\ell-1}^{(r-i)\gamma}), \tag{4.28}$$

and

$$R_m = \sum_{\ell=k+1}^{m} B_\ell (z_{\ell-1}^{\gamma} - c^{\gamma} \rho_{\ell-1}^{(r-i)\gamma}) \Delta \rho_{\ell-1}^{(i-1)\gamma}. \tag{4.29}$$

From (4.25),

$$Z_n(Y) = -S_n - T_n - U_n,$$

with

$$S_n = \rho_n^{j-\alpha-1} g_n A_n, \tag{4.30}$$

$$T_n = \sum_{m=n}^{\infty} A_{m+1} g_m \Delta \rho_m^{j-\alpha-1}, \tag{4.31}$$

and

$$\begin{aligned} U_n &= \sum_{m=n+1}^{\infty} A_m \rho_m^{j-\alpha-1} \Delta g_{m-1} \\ &= \sum_{m=n+1}^{\infty} \rho_m^{j-\alpha-1+(i-1)\gamma} (z_m^\gamma - c^\gamma \rho_m^{(r-i)\gamma}) \Delta C_m \quad \text{(from (4.4) and (4.26))} \\ &= -V_n - W_n - X_n, \end{aligned}$$

with

$$V_n = \rho_{n+1}^{j-\alpha-1+(i-1)\gamma} (z_{n+1}^\gamma - c^\gamma \rho_{n+1}^{(r-i)\gamma}) C_{n+1}, \tag{4.32}$$

$$W_n = \sum_{m=n+2}^{\infty} C_m \rho_m^{j-\alpha-1+(i-1)\gamma} \Delta (z_{m-1}^\gamma - c^\gamma \rho_{m-1}^{(r-i)\gamma}), \tag{4.33}$$

and

$$X_n = \sum_{m=n+1}^{\infty} C_{m+1} (z_m^\gamma - c^\gamma \rho_m^{(r-i)\gamma}) \Delta \rho_m^{j-\alpha-1+(i-1)\gamma}. \tag{4.34}$$

To complete the proof, we must show that S_n, T_n, V_n, W_n and X_n are $O(c^\gamma \rho_n^{-\lambda})$ under all three hypotheses. We consider the three cases separately.

Case 1: Suppose H_1 holds.

From (4.11), (4.12), (4.13), and the mean value theorem,

$$z_n^\gamma - c^\gamma = O(c^\gamma \rho_{n+1}^{-\lambda}) \tag{4.35}$$

and

$$\Delta z_n^\gamma = O\left(c^\gamma \frac{\Delta \rho_n}{\rho_{n+1}^{\lambda+1}} \right). \tag{4.36}$$

From (4.27) and (4.35), $P_m = O(c^\gamma \rho_{m+1}^{-\lambda})$. By Lemma 4.1,

$$\sum^{\infty} \frac{\Delta \rho_n}{\rho_{n+1}^{\lambda+1}} < \infty, \tag{4.37}$$

so (4.28) and (4.36) imply that $Q_m = 0(c^\gamma)$. From (4.29), $R_m = 0$. Hence, from (4.26), $g_m = O(c^\gamma)$. Therefore, from (4.30) and (4.31), $S_n = O(c^\gamma \rho_n^{-\lambda})$ and $T_n = O(c^\gamma \rho_n^{-\lambda})$. Moreover, (4.32) and (4.35) imply that $V_n = O(c^\gamma \rho_{n+1}^{-2\lambda})$. From (4.33), (4.36), and (4.37),

$$|W_n| \le M c^\gamma \sum_{m=n+2}^{\infty} \rho_m^{-\lambda} \frac{\Delta \rho_{m-1}}{\rho_m^{\lambda+1}} = O(c^\gamma \rho_{n+2}^{-\lambda}),$$

by Lemma 4.1. Finally, from (4.34) and (4.35),

$$|X_n| \le M c^\gamma \sum_{m=n+2}^{\infty} \rho_m^{-\lambda} |\Delta \rho_{m-1}^{-\lambda}| = O(c^\gamma \rho_{n+1}^{-2\lambda}).$$

This concludes the proof of (4.18) under the hypothesis H_1.

Case 2. Suppose H_2 holds.

From (4.11), (4.13), and the mean value theorem,

$$z_m^\gamma - c^\gamma \rho_m^\gamma = O(c^\gamma \rho_m^{\gamma-\lambda-1}). \tag{4.38}$$

Therefore, from (4.27),

$$P_m = O(c^\gamma \rho_m^{\gamma-\lambda-1}). \tag{4.39}$$

To deal with Q_m (see (4.28) with $i = 1$ and $r = 2$), we denote

$$\Phi_\ell = \Delta(z_{\ell-1}^\gamma - c^\gamma \rho_{\ell-1}^\gamma). \tag{4.40}$$

By the mean value theorem,

$$z_\ell^\gamma - c^\gamma \rho_\ell^\gamma = \gamma \zeta_\ell^{\gamma-1}(z_\ell - c\rho_\ell) \quad \text{and} \quad z_{\ell-1}^\gamma - c^\gamma \rho_{\ell-1}^\gamma = \gamma \zeta_{\ell-1}^{\gamma-1}(z_{\ell-1} - c\rho_{\ell-1}),$$

where

$$|\zeta_\ell - c\rho_\ell| \le |z_\ell - c\rho_\ell|, \quad |\zeta_{\ell-1} - c\rho_{\ell-1}| \le |z_{\ell-1} - c\rho_{\ell-1}|, \tag{4.41}$$

and, from (4.13),

$$c(1-\theta)\rho_\ell \le \zeta_\ell, \zeta_{\ell-1} \le c(1+\theta)\rho_\ell. \tag{4.42}$$

Now we can write

$$\Phi_\ell = \gamma \left(\zeta_\ell^{\gamma-1} - \zeta_{\ell-1}^{\gamma-1}\right)(z_\ell - c\rho_\ell) + \gamma \zeta_{\ell-1}^{\gamma-1} \Delta(z_{\ell-1} - c\rho_{\ell-1}),$$

and so, by the mean value theorem,

$$|\Phi_\ell| \le |\gamma(\gamma-1)| \widehat{u}_\ell^{\gamma-2} |\Delta \zeta_{\ell-1}| (z_\ell - c\rho_\ell) + |\gamma| \zeta_{\ell-1}^{\gamma-1} \left|\Delta(z_{\ell-1} - c\rho_{\ell-1})\right|, \tag{4.43}$$

where

$$c(1-\theta)\rho_\ell \le \widehat{u}_\ell \le c(1+\theta)\rho_\ell. \tag{4.44}$$

However,

$$\begin{aligned} |\Delta \zeta_{\ell-1}| &\le |\zeta_\ell - c\rho_\ell| + c\Delta\rho_{\ell-1} + |\zeta_{\ell-1} - c\rho_{\ell-1}| \\ &\le |z_\ell - c\rho_\ell| + c\Delta\rho_{\ell-1} + |z_{\ell-1} - c\rho_{\ell-1}| \quad \text{(from (4.41))} \\ &= c\Delta\rho_{\ell-1} + O(c\rho_\ell^{-\lambda}) = O(c\Delta\rho_{\ell-1}) \end{aligned}$$

from (4.5) and (4.11). Now (4.11), (4.12), (4.40), (4.42), (4.43), and (4.44) imply that

$$\Delta(z_{\ell-1}^\gamma - c^\gamma \rho_{\ell-1}^\gamma) = O\left(c^\gamma \frac{\Delta\rho_{\ell-1}}{\rho_\ell^{\lambda-\gamma+2}}\right). \tag{4.45}$$

From this and (4.28),

$$|Q_m| \le M c^\gamma \sum_{\ell=k+1}^{m} \frac{\Delta\rho_{\ell-1}}{\rho_\ell^{\lambda-\gamma+2}} = \begin{cases} O(c^\gamma) & \text{if } \gamma < \alpha/2, \\ O(c^{\gamma-\lambda-1}) & \text{if } \gamma > \alpha/2, \end{cases}$$

by Lemmas 4.1 and 4.2. (Note that if $\gamma > 0$ then $\gamma - \lambda - 1 = 2\gamma - \alpha$.) Since $R_m = 0$ (see (4.29) with $i = 1$), this, (4.26), and (4.39) imply that

$$g_m = \begin{cases} O(c^\gamma) & \text{if } \gamma < \alpha/2, \\ O(c^\gamma \rho_m^{\gamma-\lambda-1}) & \text{if } \gamma > \alpha/2. \end{cases} \tag{4.46}$$

From this and (4.30),

$$S_n = \begin{cases} O(c^\gamma \rho_n^{1-\alpha}) & \text{if } \gamma < \alpha/2, \\ O(c^\gamma \rho_n^{\gamma-\alpha-\lambda}) & \text{if } \gamma > \alpha/2; \end{cases}$$

hence, $S_n = O(c^\gamma \rho_n^{-\lambda})$. From (4.31) and (4.46),

$$|T_n| \le Mc^\gamma \sum_{m=n}^{\infty} |\Delta \rho_m^{1-\alpha}| = O(c^\gamma \rho_n^{1-\alpha})$$

if $\gamma < \alpha/2$, and

$$|T_n| \le Mc^\gamma \sum_{m=n}^{\infty} \rho_m^{\gamma-\lambda-1} |\Delta \rho_m^{1-\alpha}| = O(c^\gamma \rho_n^{\gamma-\alpha-\lambda})$$

if $\gamma > \alpha/2$, by Lemma 4.2; hence, $T_n = O(c^\gamma \rho_n^{-\lambda})$.

From (4.32) and (4.38), $V_n = O(c^\gamma \rho_{n+1}^{\gamma-\alpha-\lambda})$. From (4.33), (4.45), and Lemma 4.1,

$$|W_n| \le Mc^\gamma \sum_{m=n+2}^{\infty} \rho_m^{1-\alpha} \frac{\Delta \rho_{m-1}}{\rho_m^{\lambda-\gamma+2}} \le Mc^\gamma \rho_{n+2}^{-\lambda} \sum_{m=n+1}^{\infty} \frac{\Delta \rho_{m-1}}{\rho_m^{\alpha-\gamma+1}} = O(c^\gamma \rho_{n+2}^{-\lambda}).$$

Finally, from (4.34) and (4.38),

$$|X_n| \le Mc^\gamma \sum_{m=n+1}^{\infty} \rho_m^{\gamma-\lambda-1} \left|\Delta \rho_m^{1-\alpha}\right| = O(c^\gamma \rho_{n+1}^{\gamma-\alpha-\lambda}),$$

by Lemma 4.2.

Case 3. Suppose that H_3 holds. We first note that since $\alpha \ne 2\gamma$, $\gamma \ne \lambda$.

Now (4.35) and (4.36) hold. From (4.27) and (4.35), $P_m = O(c^\gamma \rho_m^{\gamma-\lambda})$. From (4.28) and (4.36),

$$|Q_m| \le Mc^\gamma \sum_{\ell=k+1}^{m} \rho_\ell^{\gamma-\lambda-1} \Delta \rho_{\ell-1} = \begin{cases} O(c^\gamma \rho_m^{\gamma-\lambda}) & \text{if } \gamma > \lambda, \\ O(c^\gamma) & \text{if } \gamma < \lambda, \end{cases}$$

by Lemma 4.2. From (4.29), (4.35), and Lemma 4.2,

$$|R_m| \le Mc^\gamma \sum_{\ell=k+1}^{m} \rho_\ell^{-\lambda} \Delta \rho_{\ell-1}^{\gamma} = \begin{cases} O(c^\gamma \rho_m^{\gamma-\lambda}) & \text{if } \gamma > \lambda, \\ O(c^\gamma) & \text{if } \gamma < \lambda. \end{cases}$$

Therefore, from (4.26),

$$g_m = \begin{cases} O(c^\gamma \rho_m^{\gamma-\lambda}) & \text{if } \gamma > \lambda, \\ O(c^\gamma) & \text{if } \gamma < \lambda. \end{cases} \tag{4.47}$$

From this and (4.30),

$$S_n = \begin{cases} O(c^\gamma \rho_n^{-2\lambda}) & \text{if } \gamma > \lambda, \\ O(c^\gamma \rho_n^{-\alpha}) & \text{if } \gamma < \lambda; \end{cases}$$

hence, $S_n = O(c^\gamma \rho_n^{-\lambda})$. From (4.31) and (4.47), $T_n = O(\rho_n^{-\alpha})$ if $\gamma < \lambda$, and

$$|T_n| \le Mc^\gamma \sum_{m=n}^{\infty} \rho_m^{\gamma-\lambda} \Delta\rho_m^{-\alpha} = O(c^\gamma \rho_n^{-2\lambda})$$

if $\gamma > \lambda$, by (4.7) and Lemma 4.2. In either case, $T_n = O(c^\gamma \rho_n^{-\lambda})$. From (4.32) and (4.35), $V_n = O(c^\gamma \rho_{n+1}^{-2\lambda})$. From (4.33) and (4.36),

$$|W_n| \le c^\gamma \sum_{m=n+2}^{\infty} \rho_m^{\gamma-\alpha-\lambda-1} \Delta\rho_{m-1} = O(c^\gamma \rho_{n+2}^{-2\lambda}),$$

by Lemma 4.2. Finally, from (4.34) and (4.35),

$$|X_n| \le Mc^\gamma \sum_{m=n+1}^{\infty} \rho_m^{-\lambda} \Delta\rho_m^{\gamma-\alpha} = O(c^\gamma \rho_{n+1}^{-2\lambda}),$$

by Lemma 4.2.

To complete the proof, we apply Theorems 3.1 and 3.2 with $\sigma_n = \theta c \rho_k^{\lambda+r-i} \rho_n^{-\lambda}$. (If $i = 1$ as in H_1 and H_2, compare (4.8) and (4.9) with (3.1); if $i = 2$ as in H_3, compare (4.8) and (4.9) with (3.11).) Under all three hypotheses, we have shown that if (4.8) and (4.9) hold, then

$$\sum_{m=n}^{\infty} x_{jm} f_m(Y) = O(c^\gamma \rho_n^{-\lambda}).$$

Thus, under H_1 and H_2 (where $j = 2$), (3.2) holds with $\phi_n = Mc^\gamma \rho_n^{-\lambda}$, so $\phi_n/\sigma_n < 1$ for $n > k$ if $c^{\gamma-1}$ is sufficiently small. Hence, Theorem 3.1 implies the conclusion. Under H_3 (where $j = 1$), (3.12) holds with $\phi_n = Mc^\gamma \rho_n^{-\lambda}$. Therefore, since $\lambda < 1$ in this case, (3.14) and Lemma 4.2 imply that

$$\widehat{\phi}_n \le \frac{Mc^\gamma}{\rho_n} \sum_{\ell=k}^{n-1} \rho_{\ell+1}^{-\lambda} \Delta\rho_\ell \le M_1 c^\gamma \rho_n^{-\lambda}.$$

Hence, (3.13) holds if $c^{\gamma-1}$ is sufficiently small, so Theorem 3.2 implies the conclusion. □

5. AN EXAMPLE

It is clear that $\{A_n\}_{n=k}^\infty$, $\{B_n\}_{n=k}^\infty$, and $\{C_n\}_{n=k}^\infty$ in (4.4) are bounded if, for example, $\{A_n\}_{n=k}^\infty$ is bounded and $\sum^\infty |b_\ell| x_{1\ell}^\gamma < \infty$. However, they may be bounded even if $\sum^\infty a_\ell x_{1\ell}$, $\sum^\infty b_\ell x_{1\ell}^\gamma$, and $\sum^\infty A_\ell b_\ell x_{1\ell}^\gamma$ all diverge. For example, if

$$a_\ell = \frac{a\cos(\ell t_1 + s_1)}{x_{1\ell}} \quad \text{and} \quad b_\ell = \frac{\cos(\ell t_2 + s_2)}{x_{1\ell}^\gamma}$$

where $a, b \neq 0$, s_1 and s_2 are abitrary,

$$0 < t_1, t_2 < 2\pi, \quad t_1 \neq t_2, \qquad \text{and} \quad t_1 + t_2 \neq 2\pi,$$

then $\{A_n\}_{n=k}^{\infty}$, $\{B_n\}_{n=k}^{\infty}$, and $\{C_n\}_{n=k}^{\infty}$ are bounded but divergent.

For a specific example, consider the equation

$$\Delta\left(\frac{\Delta y_{n-1}}{n(n-1)}\right) + \frac{2y_n}{n^2(n^2-1)} = a_n n^{-\alpha} \sum_{\ell=1}^{n} b_\ell y_\ell^{\gamma}, \quad n > 1, \tag{5.1}$$

where

$$a_\ell = \frac{a\cos(\ell t_1 + s_1)}{\ell}, \quad b_\ell = \frac{\cos(\ell t_2 + s_2)}{\ell^{\gamma}}, \quad \ell \geq 1,$$

and the constants are as above. The solutions of the unperturbed equation

$$\Delta\left(\frac{\Delta x_{n-1}}{n(n-1)}\right) + \frac{2x_n}{n^2(n^2-1)} = 0, \quad n > 1,$$

that satisfy (1.2) are $x_{1n} = n$ and $x_{2n} = n^2$. Note that $\Delta x_{1n} = 1$ and $\Delta x_{2n} = 2n + 1$. Theorem 4.3 implies the following assertions for sufficiently small $c^{\gamma-1}$:

(i) If $\alpha > 1$ then (5.1) has a solution $\widehat{Y} = \{\widehat{y}_n\}_{n=1}^{\infty}$ such that

$$\frac{\widehat{y}_n - cn}{n} = O(n^{1-\alpha}) \quad \text{and} \quad \Delta\left(\frac{\widehat{y}_n - cn}{n}\right) = O(n^{-\alpha}),$$

which implies that

$$\widehat{y}_n = \left[c + O(n^{1-\alpha})\right]n \quad \text{and} \quad \Delta\widehat{y}_n = c + O(n^{1-\alpha}).$$

(ii) If $\alpha > \max(\gamma + 1, 1)$, $\alpha \neq 2\gamma$, and $\lambda = \alpha - \max(\gamma + 1, 1)$, then (5.1) has a solution $\widehat{Y} = \{\widehat{y}_n\}_{n=1}^{\infty}$ such that

$$\frac{\widehat{y}_n - cn^2}{n} = O(n^{-\lambda}) \quad \text{and} \quad \Delta\left(\frac{\widehat{y}_n - cn^2}{n}\right) = O(n^{-\lambda-1}),$$

which implies that

$$\widehat{y}_n = \left[c + O(n^{-\lambda-1})\right]n^2 \quad \text{and} \quad \Delta\widehat{y}_n = \left[c + O(n^{-\lambda-1})\right](2n+1).$$

(iii) If $0 < \alpha - \max(\gamma, 0) < 1$, $\alpha \neq 2\gamma$, and $\lambda = \alpha - \max(\gamma, 0)$, then (5.1) has a solution $\widehat{Y} = \{\widehat{y}_n\}_{n=1}^{\infty}$ such that

$$\frac{\widehat{y}_n - cn^2}{n^2} = O(n^{-\lambda}) \quad \text{and} \quad \Delta\left(\frac{\widehat{y}_n - cn^2}{n^2}\right) = O(n^{-\lambda-1}),$$

which implies that

$$\widehat{y}_n = \left[c + O(n^{-\lambda})\right]n^2 \quad \text{and} \quad \Delta\widehat{y}_n = \left[c + O(n^{-\lambda})\right](2n+1).$$

REFERENCES

[1] S. Chen, A problem of Hartman and Wintner: approximation for discrete perturbations, Compute. Math. Appl. 4:655-669, (2001).

[2] P. Hartman, *Ordinary Differential Equations*, John Wiley & Sons, Inc., New York, London, Sydney, 1964.

[3] W. T. Patula, Growth and oscillation properties of second order linear difference equations, SIAM J. Math. Anal. 10:55-61, (1979).

[4] W. F. Trench, Linear perturbations of a nonoscillatory second order difference equation, J. Math. Anal. Appl. 255:627-635, (2001).

[5] W. F. Trench, Linear perturbations of a nonoscillatory second order difference equation II, J. Math. Anal. Appl. 273:548-557 (2002).