Characterization and properties of matrices with $k$-involutory symmetries

William F. Trench*

Trinity University, San Antonio, Texas 78212-7200, USA
Mailing address: 659 Hopkinton Road, Hopkinton, NH 03229 USA


Abstract

We say that a matrix $R \in \mathbb{C}^{n \times n}$ is $k$-involutory if its minimal polynomial is $x^k - 1$ for some $k \geq 2$, so $R^{k-1} = R^{-1}$ and the eigenvalues of $R$ are $1, \zeta, \zeta^2, \ldots, \zeta^{k-1}$, where $\zeta = e^{2\pi i/k}$. Let $\mu \in \{0, 1, \ldots, k-1\}$. If $R \in \mathbb{C}^{m \times m}$, $A \in \mathbb{C}^{m \times n}$, $S \in \mathbb{C}^{n \times n}$ and $R$ and $S$ are $k$-involutory, we say that $A$ is $(R, S, \mu)$-symmetric if $RAS^{-1} = \zeta^\mu A$. If $R, A \in \mathbb{C}^{n \times n}$, we say that $A$ is $(R, \mu)$-symmetric if $RAR^{-1} = \zeta^\mu A$. We show that an $(R, S, \mu)$-symmetric matrix $A$ can be represented in terms of matrices $F_s \in \mathbb{C}^{c_s \times d_s}$, $0 \leq s \leq k-1$, where $c_s$ and $d_s$ are respectively the dimensions of the $\zeta^s$-eigenspaces of $R$ and $S$ and $+$ denotes addition modulo $k$. The system $Az = w$ can be solved by solving $k$ independent systems with the matrices $F_0, F_1, \ldots, F_{k-1}$. If $A$ is invertible then $A^{-1}$ is can be expressed in terms of $F_0^{-1}, F_1^{-1}, \ldots, F_{k-1}^{-1}$. We do not assume in general that $R$ and $S$ are unitary; however, if they are then the Moore-Penrose inverse $A^\dagger$ of $A$ can be written in terms of $F_0^\dagger, F_1^\dagger, \ldots, F_{k-1}^\dagger$, and a singular value decomposition of $A$ can be written simply in terms of singular value decompositions of $F_0, F_1, \ldots, F_{k-1}$. If $A$ is $(R, 0)$-symmetric then solving the eigenvalue problem for $A$ reduces to solving the eigenvalue problems for $F_0, F_1, \ldots, F_{k-1}$. We also solve the eigenvalue problem for the more complicated case where $A$ is $(R, \mu)$-symmetric with $\mu \in \{1, \ldots, k-1\}$.

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*--e-mail:wtrench@trinity.edu
1 Introduction

Throughout this paper \( k \geq 2 \) is an integer and \( \zeta = e^{2\pi i/k} \). We say that a matrix \( R \) is \( k \)-involutory if its minimal polynomial is \( x^k - 1 \) for some \( k \geq 2 \), so \( R^{k-1} = R^{-1} \) and the eigenvalues of \( R \) are \( 1, \zeta, \zeta^2, \ldots, \zeta^{k-1} \), where \( \zeta = e^{2\pi i/k} \). It is to be understood that all arithmetic operations in subscripts are modulo \( k \).

Suppose \( R \in \mathbb{C}^{m \times m} \) and \( S \in \mathbb{C}^{n \times n} \) are \( k \)-involutory. We say that \( A \in \mathbb{C}^{m \times n} \) is \((R, S, \mu)\)-symmetric if

\[
RA = \zeta^\mu AS \quad \text{or, equivalently,} \quad RAS^{-1} = \zeta^\mu A.
\]

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\]

All of our results for \((R, S, \mu)\)-symmetric matrices can be applied to \((R, \mu)\)-symmetric matrices by letting \( R = S \). We consider \((R, \mu)\)-symmetric matrices separately in connection with the eigenvalue problem.

We show that an \((R, S, \mu)\)-symmetric matrix \( A \) can be represented in terms of matrices \( F_s \in \mathbb{C}^{c_s \times d_s} \), \( 0 \leq s \leq k - 1 \), where \( c_s \) and \( d_s \) are respectively the dimensions of the \( \zeta^s \)-eigenspaces of \( R \) and \( S \). The solution of \( Az = w \) can be obtained by solving \( k \) systems with the matrices \( F_0, F_1, \ldots, F_{k-1} \). If \( A \) is invertible then \( A^{-1} \) is given in terms of \( F_0^{-1}, F_1^{-1}, \ldots, F_{k-1}^{-1} \). We do not assume in general that \( R \) and \( S \) are unitary; however, if they are then the Moore-Penrose inverse \( A^\dagger \) of \( A \) can be written in terms of \( F_0^\dagger, F_1^\dagger, \ldots, F_{k-1}^\dagger \), and a singular value decomposition of \( A \) can be written simply in terms of singular value decompositions of \( F_0, F_1, \ldots, F_{k-1} \). If \( A \) is \((R, 0)\)-symmetric then solving the eigenvalue problem for \( A \) reduces to solving the eigenvalue problems for \( F_0, F_1, \ldots, F_{k-1} \). We also solve the eigenvalue problem for the more complicated case where \( A \) is \((R, \mu)\)-symmetric with \( \mu \in \{1, \ldots, k-1\} \).

Our results are natural extensions of results obtained in [15, 19] and applied in [16]–[18] for the case where \( k = 2 \). That work was motivated by results of several investigators including [2]–[14], [20], and [21]. We are particularly influenced by Andrew [2], who introduced the notions of symmetry and skew-symmetry of vectors by defining \( z \) to be symmetric (skew-symmetric) if \( Jz = z \) (\( Jz = -z \)), where \( J \) is the flip matrix with ones on the secondary diagonal and zeros elsewhere. These definitions are useful in the study of centrosymmetric matrices \((JAJ = A)\) and centroskew matrices \((JAJ = -A)\). In [15]–[19] we studied matrices such that \( RAR = A \) and \( RAR = -A \), where \( R \) is an arbitrary nontrivial involution; i.e., \( R^2 = I, R \neq \pm I \). In connection with this work it was useful to define a vector \( z \) to be \( R \)-symmetric \((R\text{-skew symmetric})\) if \( Rz = z \) \((Rz = -z)\). (It should also be noted that Yasuda made this definition in [20].) Here we say that \( z \) is \((R, s)\)-symmetric if \( Rz = \zeta^s z \); thus, the set of all \((R, s)\)-symmetric vectors is the \( \zeta^s \)-eigenspace of \( R \).
2 Preliminaries

It is to be understood throughout that $\mu \in \{0, 1, \ldots, k-1\}$. If $\lambda$ is an eigenvalue of $B$, let $\mathcal{E}_B(\lambda)$ be the $\lambda$-eigenspace of $B$; i.e.,

$$\mathcal{E}_B(\lambda) = \{ z \mid Bz = \lambda z \}.$$ 

We are particularly interested in the eigenspaces of a $k$-involution $R$:

$$\mathcal{E}_R(\zeta^s) = \{ z \mid Rz = \zeta^s z \}, \quad 0 \leq s \leq k-1,$$

and their union,

$$\mathcal{S}_R = \bigcup_{s=0}^{k-1} \mathcal{E}_R(\zeta^s).$$

As mentioned above, we will say that $z$ is $(R, s)$-symmetric if $z \in \mathcal{E}_R(\zeta^s)$.

If $w \in \mathbb{C}$ and

$$g(w) = \sum_{\ell=0}^{P} a_\ell w^\ell, \quad \text{let} \quad \overline{g}(w) = \sum_{\ell=0}^{P} \overline{a_\ell} w^\ell,$$

Lemma 1 Let $R \in \mathbb{C}^{m \times m}$ and $S \in \mathbb{C}^{n \times n}$ be $k$- involutions, and let $c_s$ and $d_s$ be the dimensions of $\mathcal{E}_R(\zeta^s)$ and $\mathcal{E}_S(\zeta^s)$. Then $\sum_{s=0}^{k-1} c_s = m$, $\sum_{s=0}^{k-1} d_s = n$, and there are matrices $P_s \in \mathbb{C}^{m \times c_s}$ and $Q_s \in \mathbb{C}^{n \times d_s}$ such that

$$RP_s = \zeta^s P_s, \quad SQ_s = \zeta^s Q_s, \quad P_s^* P_s = I_{c_s}, \quad \text{and} \quad Q_s^* Q_s = I_{d_s}, \quad (1)$$

$0 \leq s \leq k-1$. Let

$$f(w) = \prod_{s=0}^{k-1} (w - \zeta^s), \quad f_r(w) = \frac{f(w)}{f'(\zeta^r)(w - \zeta^r)}.$$

Then

$$f_r(R)P_s = \delta_{rs} P_s \quad \text{and} \quad f_r(S)Q_s = \delta_{rs} Q_s, \quad 0 \leq r, s \leq k-1.$$

From this and (1), if we define

$$\widehat{P}_r = \mathcal{T}_r(R^*)P_r \quad \text{and} \quad \widehat{Q}_r = \mathcal{T}_r(S^*)Q_r, \quad 0 \leq r \leq k-1,$$

then

$$\widehat{P}_r^* P_s = 0 \quad \text{and} \quad \widehat{Q}_r^* Q_s = 0 \quad \text{if} \quad r \neq s, \quad \text{while} \quad P_s^* P_s = I_{c_s} \quad \text{and} \quad Q_s^* Q_s = I_{d_s}, \quad 0 \leq r, s \leq k-1. \quad \text{Therefore, if}$$

$$P = \begin{bmatrix} P_0 & P_1 & \cdots & P_{k-1} \end{bmatrix} \quad \text{and} \quad \widehat{P} = \begin{bmatrix} \widehat{P}_0 & \widehat{P}_1 & \cdots & \widehat{P}_{n-1} \end{bmatrix},$$
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then $\hat{P}^* = P^{-1}$, and if
\[
Q = \begin{bmatrix} Q_0 & Q_1 & \cdots & Q_{k-1} \end{bmatrix} \quad \text{and} \quad \hat{Q} = \begin{bmatrix} \hat{Q}_0 & \hat{Q}_1 & \cdots & \hat{Q}_{n-1} \end{bmatrix},
\]
then $\hat{Q}^* = Q^{-1}$. Hence,
\[
R = PD_R\hat{P}^* \quad \text{with} \quad D_R = \bigoplus_{s=0}^{k-1} \zeta^s I_{c_s}
\]
and
\[
S = QD_S\hat{Q}^* \quad \text{with} \quad D_S = \bigoplus_{s=0}^{k-1} \zeta^s I_{d_s}.
\]

Moreover, $\hat{P} = P$ if and only if $R$ is unitary and $\hat{Q} = Q$ if and only if $S$ is unitary. In any case,
\[
\hat{Q}^* S^{-1} = (SQ)^{-1} = (QD_S)^{-1} = D_S^* \hat{Q}^*.
\]

We also define
\[
V_\mu = \begin{bmatrix} P_\mu & P_{\mu+1} & \cdots & P_{\mu+(k-1)} \end{bmatrix}, \quad \hat{V}_\mu = \begin{bmatrix} \hat{P}_\mu & \hat{P}_{\mu+1} & \cdots & \hat{P}_{\mu+(k-1)} \end{bmatrix},
\]
\[
W_\mu = \begin{bmatrix} Q_\mu & Q_{\mu+1} & \cdots & Q_{\mu+(k-1)} \end{bmatrix} \quad \text{and} \quad \hat{W}_\mu = \begin{bmatrix} \hat{Q}_\mu & \hat{Q}_{\mu+1} & \cdots & \hat{Q}_{\mu+(k-1)} \end{bmatrix}.
\]

Then $\hat{V}_\mu^* = V_\mu^{-1}$, which reduces to $V_\mu^* = V_\mu^{-1}$ if and only if $R$ is unitary, and $\hat{W}_\mu^* = W_\mu^{-1}$, which reduces to $W_\mu^* = W_\mu^{-1}$ if and only if $S$ is unitary.

Eqn.(1) does not determine $P_0, P_1, \ldots, P_{k-1}$ and $Q_0, Q_1, \ldots, Q_{k-1}$ uniquely. We offer one way to choose them: since
\[
(R - \zeta^s I)f_s(R) = 0 \quad \text{and} \quad (S - \zeta^s I)f_s(S) = 0,
\]
$P_s$ and $Q_s$ can be obtained by applying the Gram-Schmidt procedure to the columns of $f_s(R)$ and $f_s(S)$ respectively.

3 Characterization of $(R, S, \mu)$-symmetric matrices

**Theorem 1** $A \in \mathbb{C}^{m \times n}$ is $(R, S, \mu)$-symmetric if and only if
\[
A = PC\hat{Q}^* \quad \text{with} \quad C = [C_{rs}]_{r,s=0}^{k-1},
\]
where $C_{rs} \in \mathbb{C}^{c_r \times d_s}$,
\[
C_{rs} = 0 \quad \text{if} \quad r \not\equiv s + \mu \pmod{k},
\]
and
\[
C_{s+\mu,s} = P_{s+\mu}^*AQ_s \in \mathbb{C}^{c_{s+\mu} \times d_s}.
\]
Proof. We can write an arbitrary \( A \in \mathbb{C}^{m \times n} \) as in (3) with \( C = \hat{P}^* A Q \), and we can partition \( C \) as in (3). Then (2) implies that

\[
RAS^{-1} = (RP)(Q^* S^{-1}) = (PD_R)(C(D_S^* Q^*)) = P(D_R C D_S^*) Q^*.
\]

From this and (3), \( RAS^{-1} = \zeta^\mu A \) if and only if \( \zeta^\mu C = D_R C D_S^* \), i.e., if and only if

\[
[[\zeta^\mu C]_{r,s=0}]^{k-1} = [[\zeta^{-s} C]_{r,s=0}]^{k-1}.
\]

This is equivalent to (4).

For (5), (3) implies that \( AQ = PC \); i.e.,

\[
A \begin{bmatrix} Q_0 & Q_1 & \cdots & Q_{k-1} \end{bmatrix} = \begin{bmatrix} P_0 & P_1 & \cdots & P_{k-1} \end{bmatrix} C.
\]

Now (4) implies that

\[
AQ_s = P_{s+\mu} C_{s+\mu,s}, \quad 0 \leq s \leq k - 1.
\]

Since \( P_{s+\mu} P_{s+\mu} = I_{c+s,\mu} \), this implies (5). \( \square \)

Remark 1 There is no point in considering \((R, S, \mu)\) symmetry with

\[
\mu \notin \{0, 1, \ldots, k - 1\},
\]

since in this case (4) implies that \( RAS^{-1} = \zeta^\mu A \) if and only if \( A = 0 \).

Remark 2 We consider the problem discussed in [7, 9], modifying the terminology and notation to be consistent with ours. A matrix \( R \in \mathbb{C}^{n \times n} \) is a circulation matrix if \( R \) is unitary and \( R^k = I \) for some positive integer \( k \), and a matrix \( A \) is \((R, \alpha)\)-circulative if \( A \neq 0 \) and \( RAR^* = \zeta^\alpha A \). We will characterize the class \( C_\alpha \) of \((R, \alpha)\)-circulative matrices.

Let \( \zeta^{\nu_0}, \zeta^{\nu_1}, \ldots, \zeta^{\nu_{k-1}} \) be the distinct eigenvalues of \( R \), with multiplicities \( m_0, m_1, \ldots, m_{k-1} \). Then there are matrices \( P_0, P_1, \ldots, P_{k-1} \) such that \( P_s \in \mathbb{C}^{n \times m_s} \), \( RP_s = \zeta^{\nu_s} P_s \), \( P_s^* P_r = 0 \) if \( r \neq s \), and \( P_s^* P_r = I_{m_r} \), \( 0 \leq s, r \leq k - 1 \). Let

\[
P = \begin{bmatrix} P_0 & P_1 & \cdots & P_{k-1} \end{bmatrix}.
\]

We can write \( A = PCP^* \) with \( C = P^* AP = [C_{pq}]_{p,q=0}^{\ell-1} \) and \( C_{pq} \in \mathbb{C}^{m_p \times m_q} \), \( 0 \leq p, q \leq \ell - 1 \). Since

\[
R P = \begin{bmatrix} \zeta^{\nu_0} P_0 & \zeta^{\nu_1} P_1 & \cdots & \zeta^{\nu_{k-1}} P_{k-1} \end{bmatrix},
\]

it follows that

\[
RAR^* = (RP)C(RP)^* = P[\zeta^{(\nu_r - \nu_q)} C_{pq}]_{p,q=0}^{\ell-1} P^*.
\]

Since

\[
A = P[C_{pq}]_{p,q=0}^{\ell-1} P^*,
\]

\( RAR^* = \zeta^\alpha A \) if and only if \( C_{pq} = 0 \) whenever \( \nu_p \neq \nu_q + \alpha \) (mod \( k \)). Therefore \( C_\alpha = \emptyset \) if \( \alpha \notin \{0, 1, \ldots, k - 1\} \). \( C_\alpha \) may be empty even if \( \alpha \in \{0, 1, \ldots, k - 1\} \); for example, if \( \alpha \) is odd and \( \nu_0, \nu_1, \ldots, \nu_{k-1} \) are all even.
Corollary 1 Any \( A \in \mathbb{C}^{m \times n} \) can be written uniquely as \( A = \sum_{\mu=0}^{k-1} A^{(\mu)} \), where \( A^{(\mu)} \) is \((R, S, \mu)\)-symmetric, \( 0 \leq \mu \leq k - 1 \). Specifically, if \( A \) is as in (3), then \( A^{(\mu)} \) is given uniquely by
\[
A^{(\mu)} = P \left( \left[ C^{(\mu)}_{rs} \right]^{k-1}_{r,s=0} \right) \hat{Q}^*,
\]
where
\[
C^{(\mu)}_{rs} = \begin{cases} 
0 & \text{if } r \not\equiv s + \mu \pmod{k}, \\
C_{s+\mu,s} & \text{if } r \equiv s + \mu \pmod{k}.
\end{cases}
\]

The next theorem is a convenient reformulation of Theorem 1.

Theorem 2 \( A \in \mathbb{C}^{m \times n} \) is \((R, S, \mu)\)-symmetric if and only if
\[
A = V_{\mu} \left( \bigoplus_{s=0}^{k-1} F_s \right) \hat{Q}^* = \sum_{s=0}^{k-1} P_{s+\mu} F_s \hat{Q}^*_s \quad \text{where} \quad F_s = P_{s+\mu}^* A Q_s. \quad (6)
\]

Remark 3 In (6) have suppressed the dependence of \( F_s \) on \( \mu \); however, it is important to bear in mind that \( F_s \) in general depends upon \( \mu \).

It may be reassuring to verify directly that \( A \) in (6) is in fact \((R, S, \mu)\)-symmetric. To this end we note from (2) that
\[
\hat{Q}^*_s S^{-1} = \zeta^{-s} \hat{Q}^*_s, \quad 0 \leq s \leq k - 1.
\]
Hence, if \( A \) is as in the second sum in (6), then
\[
RAS^{-1} = \sum_{s=0}^{k-1} R P_{s+\mu} F_s \hat{Q}^*_s S^{-1} = \sum_{s=0}^{k-1} \zeta^{s+\mu} P_{s+\mu} F_s \zeta^{-s} \hat{Q}^*_s = \zeta^\mu A.
\]

Remark 4 We also note that if
\[
B = P \left( \bigoplus_{s=0}^{k-1} G_s \right) \hat{W}^*_\mu = \sum_{s=0}^{k-1} P_s G_s \hat{Q}^*_s + \mu
\]
with \( G_s \in \mathbb{C}_{c_s \times d_s+\mu} \), then \( B \) is \((R, S, k - \mu)\)-symmetric, since
\[
RBS^{-1} = \sum_{s=0}^{k-1} R P_s G_s \hat{Q}^*_s + \mu S^{-1} = \sum_{s=0}^{k-1} \zeta^s P_s G_s \zeta^{-s-\mu} \hat{Q}^*_s + \mu = \zeta^\mu B = \zeta^{k-\mu} B,
\]
where we have invoked (7) with \( s \) replaced by \( s + \mu \) to obtain the second equality.

Theorem 3 If \( A \) is \((R, S, \mu)\)-symmetric and \( B \) is \((S, R, \nu)\)-symmetric, then \( AB \) is \((R, \mu + \nu)\)-symmetric and \( BA \) is \((S, \mu + \nu)\)-symmetric; more specifically, if
\[
A = \sum_{s=0}^{k-1} P_{s+\mu} F_s \hat{Q}^*_s \quad \text{and} \quad B = \sum_{s=0}^{k-1} Q_{s+\nu} G_s \hat{P}^*_s, \quad (8)
\]
then
\[ AB = \sum_{s=0}^{k-1} P_{s+\mu+\nu} C_s \widehat{P}^*_s \quad \text{with} \quad C_s = F_{s+\nu} G_s \]

and
\[ BA = \sum_{s=0}^{k-1} Q_{s+\mu+\nu} D_s \widehat{Q}^*_s \quad \text{with} \quad D_s = G_{s+\mu} F_s. \]

Proof. The first statement follows immediately from the definition of \((R, S, \mu)\)-symmetry: if \(RAS^{-1} = \zeta^\mu A\) and \(SBR^{-1} = \zeta^\nu B\), then
\[ RABS^{-1} = (RAS^{-1})(SBR^{-1}) = (\zeta^\mu A)(\zeta^\nu B) = \zeta^{\mu+\nu} AB \]
and
\[ SBAR^{-1} = (SBR^{-1})(RAS^{-1}) = (\zeta^\nu B)(\zeta^\mu A) = \zeta^{\nu+\mu} BA. \]

From (8),
\[
AB = \left( \sum_{r=0}^{k-1} P_{r+\mu} F_r \widehat{Q}^*_r \right) \left( \sum_{s=0}^{k-1} Q_{s+\nu} G_s \widehat{P}^*_s \right)
\]
\[
= \left( \sum_{r=0}^{k-1} P_{r+\mu+\nu} F_{r+\nu} \widehat{Q}^*_r \right) \left( \sum_{s=0}^{k-1} Q_{s+\nu} G_s \widehat{P}^*_s \right)
\]
\[
= \sum_{s=0}^{k-1} P_{s+\mu+\nu} F_{s+\nu} G_s \widehat{P}^*_s = \sum_{s=0}^{k-1} P_{s+\mu+\nu} C_s \widehat{P}^*_s ,
\]
where the second equality implies the third because \(\widehat{Q}^*_{r+\nu} Q_{s+\nu} = 0\) if \(r \neq s\) and \(Q_{s+\nu+\nu} Q_{s+\nu} = I_{d_{s+\nu}}, 0 \leq r, s \leq k - 1\). Also from (8),
\[
BA = \left( \sum_{r=0}^{k-1} Q_{r+\mu} G_r \widehat{P}^*_r \right) \left( \sum_{s=0}^{k-1} P_{s+\mu} F_s \widehat{Q}^*_s \right)
\]
\[
= \left( \sum_{r=0}^{k-1} Q_{r+\mu+\nu} G_{r+\nu} \widehat{P}^*_r \right) \left( \sum_{s=0}^{k-1} P_{s+\mu} F_s \widehat{Q}^*_s \right)
\]
\[
= \sum_{s=0}^{k-1} Q_{s+\mu+\nu} G_{s+\mu} F_s \widehat{Q}^*_s = \sum_{s=0}^{k-1} Q_{s+\mu+\nu} D_s \widehat{Q}^*_s ,
\]
where the second equality implies the third because \(\widehat{P}^*_{r+\mu} P_{s+\mu} = 0\) if \(r \neq s\) and \(\widehat{P}^*_{s+\mu} P_{s+\mu} = I_{c_{s+\nu}}, 0 \leq r, s \leq k - 1\).

Theorem 4 Suppose
\[
A = V_\mu \left( \bigoplus_{s=0}^{k-1} F_s \right) \widehat{Q}^* \quad \text{and} \quad B = Q \left( \bigoplus_{s=0}^{k-1} F^\dagger_s \right) \widehat{V}^*_\mu .
\]
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Then $ABA = A$ and $BAB = B$. Moreover, if $R$ and $S$ are unitary then $B = A^\dagger$
i.e.,

$$A^\dagger = Q \left( \bigoplus_{s=0}^{k-1} F_s^\dagger \right) V^*_{\mu}.$$  

Hence, $A^\dagger$ is $(S, R, k - \mu)$-symmetric.

**Proof.** Since $\hat{Q}^* = Q^{-1}$ and $\hat{V}_{\mu}^* = V_{\mu}^{-1}$,

$$AB = V_{\mu} \left( \bigoplus_{s=0}^{k-1} F_s F_s^\dagger \right) \hat{V}_{\mu}^*, \quad BA = Q \left( \bigoplus_{s=0}^{k-1} F_s F_s^\dagger \right) \hat{Q}^*,$$

$$BAB = Q \left( \bigoplus_{s=0}^{k-1} F_s F_s^\dagger F_s^\dagger \right) \hat{V}_{\mu}^* = Q \left( \bigoplus_{s=0}^{k-1} F_s \right) \hat{V}_{\mu}^* = B,$$

and

$$ABA = V_{\mu} \left( \bigoplus_{s=0}^{k-1} F_s F_s^\dagger F_s^\dagger \right) \hat{Q}^* = V_{\mu} \left( \bigoplus_{s=0}^{k-1} F_s \right) \hat{Q}^* = A.$$  

If $R$ is unitary then $\hat{P} = P$ and $\hat{V}_{\mu} = V_{\mu}$. Therefore, since $F_s F_s^\dagger$ and $F_s^\dagger F_s$ are Hermitian, $AB$ and $BA$ are Hermitian. Hence, $A$ and $B$ satisfy the Penrose conditions, so $B = A^\dagger$. Remark 4 implies that $A^\dagger$ is $(S, R, k - \mu)$-symmetric.  

**Theorem 5** Suppose $A = \sum_{s=0}^{k-1} P_{s+\mu} F_s P_s^*$ is $(R, \mu)$-symmetric and $R$ is unitary. Then:

(i) $A$ is Hermitian if and only if $F_s P_s^* = F_{s+\mu} P_s^{*+2\mu}, 0 \leq s \leq k - 1.$

(ii) $A$ is normal if and only if $F_s F_s^* = F_{s+\mu}^* F_{s+\mu}, 0 \leq s \leq k - 1.$

(iii) $A$ is EP (i.e., $A^\dagger A = AA^\dagger$) if and only if $F_s F_s^\dagger = F_{s+\mu}^\dagger F_{s+\mu}, 0 \leq s \leq k - 1.$

**Proof.** Since $R$ is unitary, Theorems 2 and 4 imply that

$$A = \sum_{s=0}^{k-1} P_{s+\mu} F_s P_s^*, \quad A^\dagger = \sum_{s=0}^{k-1} P_s F_s^\dagger P_{s+\mu}^*, \quad A^\dagger = \sum_{s=0}^{k-1} P_s F_s^\dagger P_{s+\mu}^*. \quad (9)$$

Replacing $s$ by $s + \mu$ in the second sum in (9) yields

$$A^\dagger = \sum_{s=0}^{k-1} P_{s+\mu} F_{s+\mu}^* P_{s+2\mu}^*,$$

and comparing this with the first sum yields (i). From (9),

$$AA^\dagger = \sum_{s=0}^{k-1} P_{s+\mu} F_s F_s^\dagger P_{s+\mu}^* \quad (10)$$
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and

$$A^* A = \sum_{s=0}^{k-1} P_s F_s^* F_s P_s^* = \sum_{s=0}^{k-1} P_{s+\mu} F_{s+\mu}^* F_{s+\mu} P_{s+\mu}^*.$$  

Comparing the second sum here with (10) yields (ii). From (9),

$$A A^\dagger = \sum_{s=0}^{k-1} P_s F_{s, s}^\dagger P_s^*$$  

(11)

and

$$A^\dagger A = \sum_{s=0}^{k-1} P_s F_{s, s}^\dagger P_s^*$$  

Comparing the second sum here with (11) yields (iii). □

The next theorem follows from (6).

**Theorem 6** Suppose $A$ is $(R, S, \mu)$-symmetric and $F_s = \Omega_s \Sigma_s \Phi_s^*$ is a singular value decomposition of $F_s$, $0 \leq s \leq k - 1$. Let

$$\Omega = \begin{bmatrix} P_{\mu} \Omega_0 & P_{\mu+1} \Omega_1 & \cdots & P_{\mu+(k-1)} \Omega_{k-1} \end{bmatrix}$$

and

$$\Phi = \begin{bmatrix} \hat{Q}_0 \Phi_0 & \hat{Q}_1 \Phi_1 & \cdots & \hat{Q}_{k-1} \Phi_{k-1} \end{bmatrix}.$$  

Then

$$A = \Omega \left( \bigoplus_{s=0}^{k-1} \Sigma_s \right) \Phi^*.$$  

(12)

Moreover, if $R$ and $S$ are unitary then

$$\Phi = \begin{bmatrix} Q_0 \Phi_0 & Q_1 \Phi_1 & \cdots & Q_{k-1} \Phi_{k-1} \end{bmatrix},$$

$\Omega$ and $\Phi$ are unitary, and (12) is a singular value decomposition of $A$, except that the singular values are not necessarily ordered. Thus, each singular value of $F_s$ is a singular value of $A$ associated with an $(R, s+\mu)$-symmetric left singular vector and an $(S, s)$-symmetric right singular vector.

4 Solving $Az = w$

If $z, w \in \mathbb{C}^n$ and we are interested in solving $Az = w$, we write

$$z = Qu \text{ and } w = Pv$$  

where $u = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{k-1} \end{bmatrix}$ and $v = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{k-1} \end{bmatrix}$.  

(13)
Theorem 7 If \( A = \sum_{s=0}^{k-1} P_{s+\mu} F_s \hat{Q}_s \) is \((R, S, \mu)\)-symmetric then \( Az = w \) if and only if
\[
F_s u_s = v_{s+\mu}, \quad 0 \leq s \leq k - 1.
\] (14)

Proof. From (6) and (13),
\[
Az - w = \sum_{s=0}^{k-1} P_{s+\mu} F_s u_s - \sum_{s=0}^{k-1} P_s v_s = \sum_{s=0}^{k-1} P_{s+\mu} (F_s u_s - v_{s+\mu}),
\]
which vanishes if and only if (14) holds.

If \( A \in \mathbb{C}^{m \times n} \) let \( \mathcal{N}(A) \) denote the null space of \( A \).

Theorem 8 If \( A \) is \((R, S, \mu)\)-symmetric then \( Az = 0 \) if and only if \( z = Pu \) with
\[
F_s u_s = 0, \quad 0 \leq s \leq k - 1.
\]
Moreover, if \( Az = 0 \) has a nontrivial solution, then \( \mathcal{N}(A) \) has a basis in
\[
\mathcal{S}_S = \bigcup_{r=0}^{k-1} \{ z \mid Sz = \zeta^r z \}.
\]

Proof. The first statement is obvious from (14). For the second, let
\[
\mathcal{U} = \{ s \in \{0, 1, \ldots, k - 1\} \mid F_s u_s = 0 \text{ and } u_s \neq 0 \}.
\]
Since \( Az = 0 \) has a nontrivial solution, \( \mathcal{U} \neq \emptyset \). If \( s \in \mathcal{U} \) and \( \{u_s^{(1)}, u_s^{(2)}, \ldots, u_s^{(m_s)}\} \) is a basis for \( \mathcal{N}(F_s) \), then \( Q_s u_s^{(1)}, Q_s u_s^{(2)}, \ldots, Q_s u_s^{(m_s)} \) are linearly independent \((S, s)\)-symmetric vectors in \( \mathcal{N}(A) \), and
\[
\bigcup_{s \in \mathcal{U}} \{ Q_s u_s^{(1)}, Q_s u_s^{(2)}, \ldots, Q_s u_s^{(m_s)} \}
\]
is a basis for \( \mathcal{N}(A) \).

Theorem 9 If \( A \) is \((R, S, \mu)\)-symmetric then \( A \) is invertible if and only if
\[
c_{s+\mu} = d_s, \quad 0 \leq s \leq k - 1,
\] (15)
and \( F_0, F_1, \ldots, F_{k-1} \) are all invertible. In this case,
\[
A^{-1} = P \left( \bigoplus_{s=0}^{k-1} F_s^{-1} \right) \hat{V}_\mu
\] (16)
is \((S, R, k - \mu)\)-symmetric. If \( w = Pv \), the solution of \( Az = w \) is \( z = Pu \), where
\[
u_s = F_s^{-1} v_{s+\mu}, \quad 0 \leq s \leq k - 1.
\] (17)

Proof. From Theorem 7, \( Az = w \) has a solution for every \( z \) if and only (14) has a solution for every \( \{v_0, v_1, \ldots, v_{k-1}\} \). Since \( F_s \in \mathbb{C}^{c_{s+\mu} \times d_s} \), this is true if and only if (15) holds and \( F_0, F_1, \ldots, F_{k-1} \) are all invertible. It is easy to verify (16) and (17) from (6) and (14).
5 The eigenvalue problem for an \((R, 0)\)-symmetric matrix

Henceforth we assume that \(m = n\) and \(R = S\). Recall that a matrix \(A\) is \((R, \mu)\) symmetric if \(RAR^{-1} = \zeta^\mu A\). If \(\mu = 0\) then \(F_s \in \mathbb{C}^{d_s \times d_s}\), where \(d_s\) is the dimension of the \(\zeta^s\)-eigenspace of \(R\), \(0 \leq s \leq k - 1\). Setting \(z = Pu\) and \(w = \lambda z\) in (14) yields the following theorem, which reduces the eigenvalue problem for an \((R, 0)\)-symmetric matrix \(A\) to solving the eigenvalue problem for the smaller matrices \(F_0, F_1, \ldots, F_{k-1}\).

Theorem 10 If \(A\) is \((R, 0)\)-symmetric then \((\lambda, z)\) is an eigenpair of \(R\) if and only if \(z = Pu \neq 0\), where
\[
F_su_s = \lambda u_s, \quad 0 \leq s \leq k - 1.
\]

Theorem 11 If \(A\) is \((R, 0)\)-symmetric, then \(\lambda\) is an eigenvalue of \(A\) if and only if \(\lambda\) is an eigenvalue of one or more of the matrices \(F_0, F_1, \ldots, F_{k-1}\). Assuming this to be true, let
\[
S_A(\lambda) = \{ s \in \{0, 1, \ldots, k - 1\} \mid \lambda \text{ is an eigenvalue of } F_s \}.
\]
If \(s \in S_A(\lambda)\) and \(\{u_s^{(1)}, u_s^{(2)}, \ldots, u_s^{(m_s)}\}\) is a basis for
\[
E_{F_s}(\lambda) = \{ u_s \in \mathbb{C}^{n \times n} \mid F_s u_s = \lambda u_s \},
\]
then \(P_s u_s^{(1)}, P_s u_s^{(2)}, \ldots, P_s u_s^{(m_s)}\) are linearly independent \((R, s)\)-symmetric \(\lambda\)-eigenvectors of \(A\). Moreover,
\[
\bigcup_{s \in S_A(\lambda)} \{P_s u_s^{(1)}, P_s u_s^{(2)}, \ldots, P_s u_s^{(m_s)}\}
\]
is a basis for
\[
E_A(\lambda) = \{ z \mid Az = \lambda z \}.
\]
Finally, \(A\) is diagonalizable if and only if \(F_0, F_1, \ldots, F_{k-1}\) are all diagonalizable. In this case, \(A\) has \(d_s\) linearly independent \((R, s)\)-symmetric eigenvectors, \(0 \leq s \leq k - 1\).

It seems useful to consider the case where \(A\) is diagonalizable more explicitly.

Theorem 12 Suppose \(A\) is \((R, 0)\)-symmetric and diagonalizable and \(F_s = \Omega_s D_s \Omega_s^{-1}\) is a spectral decomposition of \(F_s\), \(0 \leq s \leq k - 1\). Let
\[
\Omega = \begin{bmatrix}
P_0 \Omega_0 & P_1 \Omega_1 & \cdots & P_{k-1} \Omega_{k-1}
\end{bmatrix}
\]
Then
\[
A = \Omega \left( \bigoplus_{s=0}^{k-1} D_s \right) \Omega^{-1}
\]
Matrices with $k$-involutory symmetries

is a spectral decomposition of $A$. Moreover, $A$ is normal if and only if $F_0$, $F_1$, 
\[ \ldots, F_{k-1} \] are all normal.

Note that Theorem 5(ii) implies the last sentence.

The original version of the following theorem, which dealt with centrosymmetric matrices, is due to Andrew [2, Theorem 6]. We generalized Andrew’s theorem to $R$-symmetric matrices ($k = 2$) in [15]. The next theorem generalizes it to $(R, 0)$-symmetric matrices $R$ is an arbitrary $k$-involutory matrix. The proof is practically identical to Andrew’s original proof.

**Theorem 13**

(i) If $A$ is $(R, 0)$-symmetric and $\lambda$ is an eigenvalue of $A$, then $E_A(\lambda)$ has a

$$
\mathcal{S}_R = \bigcup_{s=0}^{k-1} \{ z \mid Rz = \zeta^s z \}.
$$

(ii) If $A$ has $n$ linearly independent eigenvectors in $\mathcal{S}_R$, then $A$ is $(R, 0)$-
symmetric.

**Proof.** (i) See Theorem 11. (ii) We must show that $RA = AR$. If $Az = \lambda z$ and $Rz = \zeta^s z$, then

$$
RAz = \lambda Rz = \zeta^s \lambda z \quad \text{and} \quad ARz = \zeta^s Az = \lambda \zeta^s z;
$$

hence, $RAz = ARz$. Now suppose that $A$ has $n$ linearly independent eigenvectors \{ $z_1, z_2, \ldots, z_n$ \} in $\mathcal{S}_R$. Then we can write an arbitrary $z \in \mathbb{C}^n$ as $z = \sum_{i=1}^{n} a_i z_i$. Since $RAz_i = ARz_i$, $1 \leq i \leq n$, it follows that $RAz = ARz$. Therefore $AR = RA$. \[ \Box \]

6 The eigenvalue problem for an $(R, \mu)$-symmetric matrix with $\mu \neq 0$

We now consider the eigenvalue problem for an $(R, \mu)$-symmetric matrix $A$ with $\mu \neq 0$. Since Theorem 8 characterizes the null space of $A$, we confine our attention to nonzero eigenvalues. From Theorem 7, $Az = \lambda z \neq 0$ if and only if $z = Pu$ where

$$
F_s u_s = \lambda u_{s+\mu}, \quad 0 \leq s \leq k - 1,
$$

with $u_s \neq 0$ for some $s \in \{0, 1, \ldots, k - 1\}$. Suppose

$$
\mu \neq 0, \quad q = \text{gcd}(k, \mu) \quad \text{and} \quad m = k/q.
$$
We know this holds for \( j \). Proof

Theorem 14

Hence we must show that \( \sum \) and \( \ell \) with one nontrivial solution of (20), then \( \sum_{s=0}^{m-1} P_{\ell+\mu} u_{\ell+\mu} \) is a \( \lambda \)-eigenvector of \( A \).

**Theorem 14** Suppose that \( A \) is \( (R, \mu) \)-symmetric and (19) holds. Suppose also that for some \( \ell \in \{0, 1, \ldots, q-1\} \), \( u_\ell \) is a nontrivial solution of

\[
F_{\ell+\mu} u_{\ell+\mu} = \lambda u_{\ell+(s+1)\mu}, \quad 0 \leq s \leq m - 1, \tag{20}
\]

where \( 0 \leq \ell \leq q-1 \); hence there are \( q \) independent systems here, each associated with one \( \ell \in \{0, 1, \ldots, q-1\} \). If \( \lambda \neq 0 \) and \( (u_\ell, u_{\ell+\mu}, \ldots, u_{\ell+(m-1)\mu}) \) is a nontrivial solution of (20), then \( \sum_{s=0}^{m-1} P_{\ell+\mu} u_{\ell+\mu} \) is a \( \lambda \)-eigenvector of \( A \).

Then (18) can be rewritten as

\[
F_{\ell+\mu} u_{\ell+\mu} = \lambda^{-m} F_{\ell+(m-1)\mu} \cdots F_{\ell+\mu} F_{\ell}. \tag{21}
\]

Define

\[
u_\ell = \lambda^{-m} F_{\ell+(m-1)\mu} \cdots F_{\ell+\mu} F_{\ell}. \tag{22}
\]

and

\[
z_{\ell j} = \sum_{s=0}^{m-1} \zeta^{(\ell+\mu)s} P_{\ell+\mu} u_{\ell+\mu}, \quad 0 \leq j \leq m - 1.
\]

Then

\[
A z_{\ell j} = \zeta^{-j\mu} \lambda z_{\ell j}, \quad 0 \leq j \leq m - 1.
\]

Proof. Equ. (22) implies (20) for \( 0 \leq s \leq m - 2 \), and, since \( \ell + m \mu = \ell \mod k \), (21) implies (20) for \( s = m - 1 \). Hence \( z_{\ell 0} \) is a \( \lambda \)-eigenvector of \( A \). Since

\[
RP_{\ell+\mu} = \zeta^{\ell+\mu} P_{\ell+\mu},
\]

it follows that \( z_{\ell j} = R^j z_{\ell 0} \), \( 0 \leq \ell \leq m - 1 \).

Hence must show that

\[
A (R^j z_{\ell 0}) = \zeta^{-j\mu} \lambda (R^j z_{\ell 0}), \quad 0 \leq j \leq m - 1.
\]

We know this holds for \( j = 0 \). If it holds for some \( j \in \{0, \ldots, m - 2\} \) then

\[
RA(R^j z_{\ell 0}) = \zeta^{-j\mu} \lambda (R^{j+1} z_{\ell 0}), \quad RAR^{-1}(R^{j+1} z_{\ell 0}) = \zeta^{-j\mu} \lambda (R^{j+1} z_{\ell 0}),
\]

and, since \( RAR^{-1} = \zeta^\mu A \),

\[
A(R^{j+1} z_{\ell 0}) = \zeta^{-(j+1)\mu} \lambda (R^{j+1} z_{\ell 0}),
\]

which completes the finite induction. \( \blacksquare \)

**7 Applications**

Let \( \sigma \) and \( \rho \) be permutations of \( \mathbb{Z}_k \), and let \( E \) and \( F \) be the permutation matrices

\[
E = [\delta_{i, \sigma^{-1}(j)}]_{i, j=0}^{k-1} = \begin{bmatrix} e_{\sigma^{-1}(0)} & e_{\sigma^{-1}(1)} & \cdots & e_{\sigma^{-1}(k-1)} \end{bmatrix}
\]

and

\[
F = [\delta_{i, \rho^{-1}(j)}]_{i, j=0}^{k-1} = \begin{bmatrix} e_{\rho^{-1}(0)} & e_{\rho^{-1}(1)} & \cdots & e_{\rho^{-1}(k-1)} \end{bmatrix}.
\]
If \( p \) and \( q \) are positive integers let \( R = E \otimes I_p \) and \( S = F \otimes I_q \). The methods of this paper are applicable to matrices \( [A_{rs}]_{r,s=0}^{k-1} \) with \( A_{rs} \in \mathbb{C}^{p \times q} \) such that \( EAF^* = \zeta^\mu A \). Since \( EAF^* = [A_{\sigma(r),\rho(s)}]_{r,s=0}^{k-1} \), \( EAF^* = \zeta^\mu A \) if and only if \( A_{\sigma(r),\sigma(s)} = \zeta^\mu A_{rs}, \quad 0 \leq r \leq s - 1 \). For example, \( A_{rs} = [\zeta^\mu C_{s-r}]_{r,s=0}^{k-1} \) satisfies this condition with \( \sigma(r) = \rho(r) = r + 1 \) (mod \( k \)).

Chen and Sameh [6] have studied matrices \( A \) such that \( PAP = A \), where \( P \) is a signed permutation matrix. Since \( PAP = A \) is equivalent to \( PAQ^{-1} = A \) with \( Q = P^{-1} \), our results also apply to these matrices.

References


