Characterization and properties of matrices with k-involutory symmetries

William F. Trench*

Trinity University, San Antonio, Texas 78212-7200, USA
Mailing address: 659 Hopkinton Road, Hopkinton, NH 03229 USA

Linear Algebra and Its Applications 429, Issues 8-9 (2008) 2278-2290

Abstract

We say that a matrix $R \in \mathbb{C}^{n\times n}$ is k-involutory if its minimal polynomial is $x^k - 1$ for some $k \geq 2$, so $R^{k-1} = R^{-1}$ and the eigenvalues of R are 1, ζ, ζ^2,..., ζ^{k-1}, where $\zeta = e^{2\pi i/k}$. Let $\mu \in \{0, 1, \dots, k-1\}$. If $R \in \mathbb{C}^{m\times m}$, $A \in \mathbb{C}^{m\times n}$, $S \in \mathbb{C}^{n\times n}$ and R and S are k-involutory, we say that A is (R, S, μ)-symmetric if $RAS^{-1} = \zeta^\mu A$. If $R, A \in \mathbb{C}^{n\times n}$, we say that A is (R, μ)-symmetric if $RAR^{-1} = \zeta^\mu A$. We show that an (R, S, μ)-symmetric matrix A can be represented in terms of matrices $F_s \in \mathbb{C}^{c_{s+\mu}\times d_s}$, $0 \leq s \leq k-1$, where c_s and d_s are respectively the dimensions of the ζ^s- eigenspaces of R and S and $+$ denotes addition modulo k. The system $Az = w$ can be solved by solving k independent systems with the matrices F_0, F_1, ..., F_{k-1}. If A is invertible then A^{-1} is can be expressed in terms of F_0^{-1}, F_1^{-1}, ..., F_{k-1}^{-1}. We do not assume in general that R and S are unitary; however, if they are then the Moore-Penrose inverse $A^\dagger$ of A can be written in terms of $F_0^\dagger$, $F_1^\dagger$, ..., $F_{k-1}^\dagger$, and a singular value decomposition of A can be written simply in terms of singular value decompositions of F_0, F_1, ..., F_{k-1}. If A is $(R, 0)$-symmetric then solving the eigenvalue problem for A reduces to solving the eigenvalue problems for F_0, F_1, ..., F_{k-1}. We also solve the eigenvalue problem for the more complicated case where A is (R, μ)-symmetric with $\mu \in \{1, \dots, k-1\}$.

MSC: 15A18; 15A57

Keywords: k-involution; (R, μ)-symmetric; (R, S, μ)-symmetric; Eigenvalue problem; Inverse; Moore–Penrose inverse

---

*e-mail:wtrench@trinity.edu

1 Introduction

Throughout this paper $k \geq 2$ is an integer and $\zeta = e^{2\pi i/k}$. We say that a matrix R is k-involutory if its minimal polynomial is $x^k - 1$ for some $k \geq 2$, so $R^{k-1} = R^{-1}$ and the eigenvalues of R are 1, ζ, ζ^2, …, ζ^{k-1}, where $\zeta = e^{2\pi i/k}$. It is to be understood that all arithmetic operations in subscripts are modulo k.

Suppose $R \in \mathbb{C}^{m\times m}$ and $S \in \mathbb{C}^{n\times n}$ are k-involutory. We say that $A \in \mathbb{C}^{m\times n}$ is (R, S, μ)-symmetric if

$$RA = \zeta^\mu AS \quad \text{or, equivalently,} \quad RAS^{-1} = \zeta^\mu A.$$

We say that $A \in \mathbb{C}^{n\times n}$ is (R, μ)-symmetric if

$$RA = \zeta^\mu AR \quad \text{or, equivalently,} \quad RAR^{-1} = \zeta^\mu A.$$

All of our results for (R, S, μ)-symmetric matrices can be applied to (R, μ)-symmetric matrices by letting $R = S$. We consider (R, μ)-symmetric matrices separately in connection with the eigenvalue problem.

We show that an (R, S, μ)-symmetric matrix A can be represented in terms of matrices $F_s \in \mathbb{C}^{c_{s+\mu}\times d_s}$, $0 \leq s \leq k-1$, where c_s and d_s are respectively the dimensions of the ζ^s- eigenspaces of R and S. The solution of $Az = w$ can be obtained by solving k systems with the matrices F_0, F_1, …, F_{k-1}. If A is invertible then A^{-1} is given in terms of F_0^{-1}, F_1^{-1}, …, F_{k-1}^{-1}. We do not assume in general that R and S are unitary; however, if they are then the Moore-Penrose inverse $A^\dagger$ of A can be written in terms of $F_0^\dagger$, $F_1^\dagger$, …, $F_{k-1}^\dagger$, and a singular value decomposition of A can be written simply in terms of singular value decompositions of F_0, F_1, …, F_{k-1}. If A is $(R, 0)$-symmetric then solving the eigenvalue problem for A reduces to solving the eigenvalue problems for F_0, F_1, …, F_{k-1}. We also solve the eigenvalue problem for the more complicated case where A is (R, μ)-symmetric with $\mu \in \{1, \dots, k-1\}$.

Our results are natural extensions of results obtained in [15, 19] and applied in [16]–[18] for the case where $k = 2$. That work was motivated by results of several investigators including [2]–[14], [20], and [21]. We are particularly influenced by Andrew [2], who introduced the notions of symmetry and skew-symmetry of vectors by defining z to be symmetric (skew-symmetric) if $Jz = z$ ($Jz = -z$), where J is the flip matrix with ones on the secondary diagonal and zeros elsewhere. These definitions are useful in the study of centrosymmetric matrices ($JAJ = A$) and centroskew matrices ($JAJ = -A$). In [15]-[19] we studied matrices such that $RAR = A$ and $RAR = -A$, where R is an arbitrary nontrivial involution; i.e., $R^2 = I$, $R \neq \pm I$. In connection with this work it was useful to define a vector z to be R-symmetric (R-skew symmetric) if $Rz = z$ ($Rz = -z$). (It should also be noted that Yasuda made this definition in [20].) Here we say that z is (R, s)-symmetric if $Rz = \zeta^s z$; thus, the set of all (R, s)-symmetric vectors is the ζ^s-eigenspace of R.

2 Preliminaries

It is to be understood throughout that $\mu \in \{0, 1, \dots, k-1\}$. If λ is an eigenvalue of B, let $\mathcal{E}_B(\lambda)$ be the λ-eigenspace of B; i.e.,

$$\mathcal{E}_B(\lambda) = \left\{ z \,\middle|\, Bz = \lambda z \right\}.$$

We are particularly interested in the eigenspaces of a k-involution R:

$$\mathcal{E}_R(\zeta^s) = \left\{ z \,\middle|\, Rz = \zeta^s z \right\}, \quad 0 \le s \le k-1,$$

and their union,

$$\mathcal{S}_R = \bigcup_{s=0}^{k-1} \mathcal{E}_R(\zeta^s).$$

As mentioned above, we will say that z is (R, s)-symmetric if $z \in \mathcal{E}_R(\zeta^s)$.

If $w \in \mathbb{C}$ and

$$g(w) = \sum_{\ell=0}^{p} a_\ell w^\ell, \quad \text{let} \quad \overline{g}(w) = \sum_{\ell=0}^{p} \overline{a}_\ell w^\ell.$$

Lemma 1 *Let $R \in \mathbb{C}^{m\times m}$ and $S \in \mathbb{C}^{n\times n}$ be k-involutions, and let c_s and d_s be the dimensions of $\mathcal{E}_R(\zeta^s)$ and $\mathcal{E}_S(\zeta^s)$. Then $\sum_{s=0}^{k-1} c_s = m$, $\sum_{s=0}^{k-1} d_s = n$, and there are matrices $P_s \in \mathbb{C}^{m\times c_s}$ and $Q_s \in \mathbb{C}^{n\times d_s}$ such that*

$$RP_s = \zeta^s P_s, \quad SQ_s = \zeta^s Q_s, \quad P_s^* P_s = I_{c_s}, \quad \textit{and} \quad Q_s^* Q_s = I_{d_s}, \tag{1}$$

$0 \le s \le k-1$. *Let*

$$f(w) = \prod_{s=0}^{k-1} (w - \zeta^s), \quad f_r(w) = \frac{f(w)}{f'(\zeta^r)(w - \zeta^r)}.$$

Then

$$f_r(R) P_s = \delta_{rs} P_s \quad \textit{and} \quad f_r(S) Q_s = \delta_{rs} Q_s, \quad 0 \le r, s \le k-1.$$

From this and (1), *if we define*

$$\widehat{P}_r = \overline{f}_r(R^*) P_r \quad \textit{and} \quad \widehat{Q}_r = \overline{f}_r(S^*) Q_r, \quad 0 \le r \le k-1,$$

then

$$\widehat{P}_r^* P_s = 0 \quad \textit{and} \quad \widehat{Q}_r^* Q_s = 0 \quad \textit{if} \quad r \ne s, \quad \textit{while} \ P_s^* P_s = I_{c_s} \quad \textit{and} \quad Q_s^* Q_s = I_{d_s},$$

$0 \le r, s \le k-1$. *Therefore, if*

$$P = \left[\begin{array}{ccc} P_0 & P_1 \cdots & P_{k-1} \end{array} \right] \quad \textit{and} \quad \widehat{P} = \left[\begin{array}{cccc} \widehat{P}_0 & \widehat{P}_1 & \cdots & \widehat{P}_{n-1} \end{array} \right],$$

then $\widehat{P}^* = P^{-1}$, *and if*

$$Q = \left[\begin{array}{cccc} Q_0 & Q_1 \cdots & Q_{k-1} \end{array}\right] \quad \textit{and} \quad \widehat{Q} = \left[\begin{array}{cccc} \widehat{Q}_0 & \widehat{Q}_1 & \cdots & \widehat{Q}_{n-1} \end{array}\right],$$

then $\widehat{Q}^* = Q^{-1}$. *Hence,*

$$R = PD_R\widehat{P}^* \quad \textit{with} \quad D_R = \bigoplus_{s=0}^{k-1} \zeta^s I_{c_s}$$

and

$$S = QD_S\widehat{Q}^* \quad \textit{with} \quad D_S = \bigoplus_{s=0}^{k-1} \zeta^s I_{d_s}.$$

Moreover, $\widehat{P} = P$ *if and only if* R *is unitary and* $\widehat{Q} = Q$ *if and only if* S *is unitary. In any case,*

$$\widehat{Q}^* S^{-1} = (SQ)^{-1} = (QD_S)^{-1} = D_S^*\widehat{Q}^*. \tag{2}$$

We also define

$$V_\mu = \left[\begin{array}{cccc} P_\mu & P_{\mu+1} & \cdots & P_{\mu+(k-1)} \end{array}\right], \quad \widehat{V}_\mu = \left[\begin{array}{cccc} \widehat{P}_\mu & \widehat{P}_{\mu+1} & \cdots & \widehat{P}_{\mu+(k-1)} \end{array}\right],$$

$$W_\mu = \left[\begin{array}{cccc} Q_\mu & Q_{\mu+1} & \cdots & Q_{\mu+(k-1)} \end{array}\right] \quad \text{and} \quad \widehat{W}_\mu = \left[\begin{array}{cccc} \widehat{Q}_\mu & \widehat{Q}_{\mu+1} & \cdots & \widehat{Q}_{\mu+(k-1)} \end{array}\right].$$

Then $\widehat{V}_\mu^* = V_\mu^{-1}$, which reduces to $V_\mu^* = V_\mu^{-1}$ if and only if R is unitary, and $\widehat{W}_\mu^* = W_\mu^{-1}$, which reduces to $W_\mu^* = W_\mu^{-1}$ if and only if S is unitary.

Eqn.(1) does not determine $P_0, P_1, \dots, P_{k-1}$ and $Q_0, Q_1, \dots, Q_{k-1}$ uniquely. We offer one way to choose them: since

$$(R - \zeta^s I)f_s(R) = 0 \quad \text{and} \quad (S - \zeta^s I)f_s(S) = 0,$$

P_s and Q_s can be obtained by applying the Gram-Schmidt procedure to the columns of $f_s(R)$ and $f_s(S)$ respectively.

3 Characterization of (R, S, μ)-symmetric matrices

Theorem 1 $A \in \mathbb{C}^{m\times n}$ *is* (R, S, μ)*-symmetric if and only if*

$$A = PC\widehat{Q}^* \quad \textit{with} \quad C = [C_{rs}]_{r,s=0}^{k-1}, \tag{3}$$

where $C_{rs} \in \mathbb{C}^{c_r \times d_s}$,

$$C_{rs} = 0 \quad \textit{if} \quad r \not\equiv s + \mu \pmod{k}, \tag{4}$$

and

$$C_{s+\mu,s} = P_{s+\mu}^* A Q_s \in \mathbb{C}^{c_{s+\mu}\times d_s}. \tag{5}$$

Proof. We can write an arbitrary $A \in \mathbb{C}^{m\times n}$ as in (3) with $C = \widehat{P}^* AQ$, and we can partition C as in (3). Then (2) implies that

$$RAS^{-1} = (RP)C(\widehat{Q}^* S^{-1}) = (PD_R)C(D_S^*\widehat{Q}^*) = P(D_R C D_S^*)\widehat{Q}^*.$$

From this and (3), $RAS^{-1} = \zeta^\mu A$ if and only if $\zeta^\mu C = D_R C D_S^*$, i.e., if and only if

$$[\zeta^\mu C_{rs}]_{r,s=0}^{k-1} = [\zeta^{r-s} C_{rs}]_{r,s=0}^{k-1}.$$

This is equivalent to (4).

For (5), (3) implies that $AQ = PC$; i.e.,

$$A \left[\begin{array}{cccc} Q_0 & Q_1 & \cdots & Q_{k-1} \end{array}\right] = \left[\begin{array}{cccc} P_0 & P_1 & \cdots & P_{k-1} \end{array}\right] C.$$

Now (4) implies that

$$AQ_s = P_{s+\mu} C_{s+\mu,s}, \quad 0 \le s \le k-1.$$

Since $P_{s+\mu}^* P_{s+\mu} = I_{c_{s+\mu}}$, this implies (5). $\square$

Remark 1 There is no point in considering (R, S, μ) symmetry with

$$\mu \notin \{0, 1, \dots, k-1\},$$

since in this case (4) implies that $RAS^{-1} = \zeta^\mu A$ if and only if $A = 0$.

Remark 2 We consider the problem discussed in [7, 9], modifying the terminology and notation to be consistent with ours. A matrix $R \in \mathbb{C}^{n\times n}$ is a circulation matrix if R is unitary and $R^k = I$ for some positive integer k, and a matrix A is (R,α)-circulative if $A \ne 0$ and $RAR^* = \zeta^\alpha A$. We will characterize the class $\mathcal{C}_\alpha$ of (R,α)-circulative matrices.

Let ζ^{ν_0}, ζ^{ν_1}, …, $\zeta^{\nu_{\ell-1}}$ be the distinct eigenvalues of R, with multiplicities m_0, m_1, …, $m_{\ell-1}$. Then there are matrices P_0, P_1, …, $P_{\ell-1}$ such that $P_s \in \mathbb{C}^{n\times m_s}$, $RP_s = \zeta^{\nu_s} P_s$, $P_r^* P_s = 0$ if $r \ne s$, and $P_r^* P_r = I_{m_r}$, $0 \le r, s \le \ell - 1$. Let

$$P = \left[\begin{array}{cccc} P_0 & P_1 & \cdots & P_{\ell-1} \end{array}\right].$$

We can write $A = PCP^*$ with $C = P^*AP = [C_{pq}]_{p,q=0}^{\ell-1}$ and $C_{pq} \in \mathbb{C}^{m_p\times m_q}$, $0 \le p, q \le \ell - 1$. Since

$$RP = \left[\begin{array}{cccc} \zeta^{\nu_0} P_0 & \zeta^{\nu_1} P_1 & \cdots & \zeta^{\nu_{\ell-1}} P_{\ell-1} \end{array}\right],$$

it follows that

$$RAR^* = (RP)C(RP)^* = P[\zeta^{(\nu_p-\nu_q)} C_{pq}]_{p,q=0}^{\ell-1} P^*.$$

Since

$$A = P[C_{pq}]_{p,q=0}^{\ell-1} P^*,$$

$RAR^* = \zeta^\alpha A$ if and only if $C_{pq} = 0$ whenever $\nu_p \not\equiv \nu_q + \alpha \pmod{k}$. Therefore $\mathcal{C}_\alpha = \emptyset$ if $\alpha \notin \{0, 1, \dots, k-1\}$. $\mathcal{C}_\alpha$ may be empty even if $\alpha \in \{0, 1, \dots, k-1\}$; for example, if α is odd and ν_0, ν_1, …, ν_{k-1} are all even.

Corollary 1 *Any $A \in \mathbb{C}^{m\times n}$ can be written uniquely as $A = \sum_{\mu=0}^{k-1} A^{(\mu)}$, where $A^{(\mu)}$ is (R,S,μ)-symmetric, $0 \le \mu \le k-1$. Specifically, if A is as in (3), then $A^{(\mu)}$ is given uniquely by*

$$A^{(\mu)} = P\left(\left[C_{rs}^{(\mu)}\right]_{r,s=0}^{k-1}\right)\widehat{Q}^*,$$

where

$$C_{rs}^{(\mu)} = \begin{cases} 0 & \textit{if } r \not\equiv s+\mu \pmod{k}, \\ C_{s+\mu,s} & \textit{if } r \equiv s+\mu \pmod{k}. \end{cases}$$

The next theorem is a convenient reformulation of Theorem 1.

Theorem 2 *$A \in C^{m\times n}$ is (R,S,μ)-symmetric if and only if*

$$A = V_\mu\left(\bigoplus_{s=0}^{k-1} F_s\right)\widehat{Q}^* = \sum_{s=0}^{k-1} P_{s+\mu}F_s\widehat{Q}_s^* \quad \textit{where} \quad F_s = P_{s+\mu}^* A Q_s. \tag{6}$$

Remark 3 In (6) have suppressed the dependence of F_s on μ; however, it is important to bear in mind that F_s in general depends upon μ.

It may be reassuring to verify directly that A in (6) is in fact (R,S,μ)-symmetric. To this end we note from (2) that

$$\widehat{Q}_s^* S^{-1} = \zeta^{-s}\widehat{Q}_s^*, \quad 0 \le s \le k-1. \tag{7}$$

Hence, if A is as in the second sum in (6), then

$$RAS^{-1} = \sum_{s=0}^{k-1} RP_{s+\mu}F_s\widehat{Q}_s^*S^{-1} = \sum_{s=0}^{k-1}\zeta^{s+\mu}P_{s+\mu}F_s\zeta^{-s}\widehat{Q}_s^* = \zeta^\mu A.$$

Remark 4 We also note that if

$$B = P\left(\bigoplus_{s=0}^{k-1} G_s\right)\widehat{W}_\mu^* = \sum_{s=0}^{k-1} P_sG_s\widehat{Q}_{s+\mu}^*$$

with $G_s \in \mathbb{C}^{c_s\times d_{s+\mu}}$, then B is $(R,S,k-\mu)$-symmetric, since

$$RBS^{-1} = \sum_{s=0}^{k-1} RP_sG_s\widehat{Q}_{s+\mu}^*S^{-1} = \sum_{s=0}^{k-1}\zeta^sP_sG_s\zeta^{-s-\mu}\widehat{Q}_{s+\mu}^* = \zeta^{-\mu}B = \zeta^{k-\mu}B,$$

where we have invoked (7) with s replaced by $s+\mu$ to obtain the second equality.

Theorem 3 *If A is (R,S,μ)-symmetric and B is (S,R,ν)-symmetric, then AB is $(R,\mu+\nu)$-symmetric and BA is $(S,\mu+\nu)$-symmetric; more specifically, if*

$$A = \sum_{s=0}^{k-1} P_{s+\mu}F_s\widehat{Q}_s^* \;\; \textit{and} \;\; B = \sum_{s=0}^{k-1} Q_{s+\nu}G_s\widehat{P}_s^*, \tag{8}$$

then

$$AB = \sum_{s=0}^{k-1} P_{s+\mu+\nu} C_s \widehat{P}_s^* \quad \textit{with} \quad C_s = F_{s+\nu} G_s$$

and

$$BA = \sum_{s=0}^{k-1} Q_{s+\mu+\nu} D_s \widehat{Q}_s^* \quad \textit{with} \quad D_s = G_{s+\mu} F_s.$$

Proof. The first statement follows immediately from the definition of (R, S, μ)-symmetry: if $RAS^{-1} = \zeta^\mu A$ and $SBR^{-1} = \zeta^\nu B$, then

$$RABS^{-1} = (RAS^{-1})(SBR^{-1}) = (\zeta^\mu A)(\zeta^\nu B) = \zeta^{\mu+\nu} AB$$

and

$$SBAR^{-1} = (SBR^{-1})(RAS^{-1}) = (\zeta^\nu B)(\zeta^\mu A) = \zeta^{\nu+\mu} BA.$$

From (8),

$$\begin{aligned} AB &= \left(\sum_{r=0}^{k-1} P_{r+\mu} F_r \widehat{Q}_r^*\right)\left(\sum_{s=0}^{k-1} Q_{s+\nu} G_s \widehat{P}_s^*\right) \\ &= \left(\sum_{r=0}^{k-1} P_{r+\mu+\nu} F_{r+\nu} \widehat{Q}_{r+\nu}^*\right)\left(\sum_{s=0}^{k-1} Q_{s+\nu} G_s \widehat{P}_s^*\right) \\ &= \sum_{s=0}^{k-1} P_{s+\mu+\nu} F_{s+\nu} G_s \widehat{P}_s^* = \sum_{s=0}^{k-1} P_{s+\mu+\nu} C_s \widehat{P}_s^*, \end{aligned}$$

where the second equality implies the third because $\widehat{Q}_{r+\nu}^* Q_{s+\nu} = 0$ if $r \neq s$ and $Q_{s+\nu}^* Q_{s+\nu} = I_{d_{s+\nu}}$, $0 \le r, s \le k-1$. Also from (8),

$$\begin{aligned} BA &= \left(\sum_{r=0}^{k-1} Q_{r+\nu} G_r \widehat{P}_r^*\right)\left(\sum_{s=0}^{k-1} P_{s+\mu} F_s \widehat{Q}_s^*\right) \\ &= \left(\sum_{r=0}^{k-1} Q_{r+\mu+\nu} G_{r+\mu} \widehat{P}_{r+\mu}^*\right)\left(\sum_{s=0}^{k-1} P_{s+\mu} F_s \widehat{Q}_s^*\right) \\ &= \sum_{s=0}^{k-1} Q_{s+\mu+\nu} G_{s+\mu} F_s \widehat{Q}_s^* = \sum_{s=0}^{k-1} Q_{s+\mu+\nu} D_s \widehat{Q}_s^*, \end{aligned}$$

where the second equality implies the third because $\widehat{P}_{r+\mu}^* P_{s+\mu} = 0$ if $r \neq s$ and $\widehat{P}_{s+\mu}^* P_{s+\mu} = I_{c_{s+\nu}}$, $0 \le r, s \le k-1$.

Theorem 4 *Suppose*

$$A = V_\mu \left(\bigoplus_{s=0}^{k-1} F_s\right) \widehat{Q}^* \quad \textit{and} \quad B = Q \left(\bigoplus_{s=0}^{k-1} F_s^\dagger\right) \widehat{V}_\mu^*.$$

Then $ABA = A$ and $BAB = B$. Moreover, if R and S are unitary then $B = A^\dagger$ i.e.,

$$A^\dagger = Q\left(\bigoplus_{s=0}^{k-1} F_s^\dagger\right) V_\mu^*.$$

Hence, $A^\dagger$ is $(S, R, k-\mu)$-symmetric.

Proof. Since $\widehat{Q}^* = Q^{-1}$ and $\widehat{V}_\mu^* = V_\mu^{-1}$,

$$AB = V_\mu\left(\bigoplus_{s=0}^{k-1} F_s F_s^\dagger\right)\widehat{V}_\mu^*, \quad BA = Q\left(\bigoplus_{s=0}^{k-1} F_s^\dagger F_s\right)\widehat{Q}^*,$$

$$BAB = Q\left(\bigoplus_{s=0}^{k-1} F_s^\dagger F_s F_s^\dagger\right)\widehat{V}_\mu^* = Q\left(\bigoplus_{s=0}^{k-1} F_s^\dagger\right)\widehat{V}_\mu^* = B,$$

and

$$ABA = V_\mu\left(\bigoplus_{s=0}^{k-1} F_s F_s^\dagger F_s\right)\widehat{Q}^* = V_\mu\left(\bigoplus_{s=0}^{k-1} F_s\right)\widehat{Q}^* = A.$$

If R is unitary then $\widehat{P} = P$ and $\widehat{V}_\mu = V_\mu$. Therefore, since $F_s F_s^\dagger$ and $F_s^\dagger F_s$ are Hermitian, AB and BA are Hermitian. Hence, A and B satisfy the Penrose conditions, so $B = A^\dagger$. Remark 4 implies that $A^\dagger$ is $(S, R, k-\mu)$-symmetric. ▯

Theorem 5 *Suppose $A = \sum_{s=0}^{k-1} P_{s+\mu} F_s P_s^*$ is (R,μ)-symmetric and R is unitary. Then:*

(i) *A is Hermitian if and only if $F_s P_s^* = F_{s+\mu}^* P_{s+2\mu}^*$, $0 \le s \le k-1$.*

(ii) *A is normal if and only if $F_s F_s^* = F_{s+\mu}^* F_{s+\mu}$, $0 \le s \le k-1$.*

(iii) *A is EP (i.e., $A^\dagger A = AA^\dagger$) if and only if $F_s F_s^\dagger = F_{s+\mu}^\dagger F_{s+\mu}$, $0 \le s \le k-1$.*

Proof. Since R is unitary, Theorems 2 and 4 imply that

$$A = \sum_{s=0}^{k-1} P_{s+\mu} F_s P_s^*, \quad A^* = \sum_{s=0}^{k-1} P_s F_s^* P_{s+\mu}^*, \quad \text{and} \quad A^\dagger = \sum_{s=0}^{k-1} P_s F_s^\dagger P_{s+\mu}^*. \tag{9}$$

Replacing s by $s+\mu$ in the second sum in (9) yields

$$A^* = \sum_{s=0}^{k-1} P_{s+\mu} F_{s+\mu}^* P_{s+2\mu}^*,$$

and comparing this with the first sum yields (i). From (9),

$$AA^* = \sum_{s=0}^{k-1} P_{s+\mu} F_s F_s^* P_{s+\mu}^* \tag{10}$$

and

$$A^*A = \sum_{s=0}^{k-1} P_s F_s^* F_s P_s^* = \sum_{s=0}^{k-1} P_{s+\mu} F_{s+\mu}^* F_{s+\mu} P_{s+\mu}^*.$$

Comparing the second sum here with (10) yields (ii). From (9),

$$AA^\dagger = \sum_{s=0}^{k-1} P_{s+\mu} F_s F_s^\dagger P_{s+\mu}^* \tag{11}$$

and

$$A^\dagger A = \sum_{s=0}^{k-1} P_s F_s^\dagger F_s P_s^* = \sum_{s=0}^{k-1} P_{s+\mu} F_{s+\mu}^\dagger F_{s+\mu} P_{s+\mu}^*$$

Comparing the second sum here with (11) yields (iii). □

The next theorem follows from (6).

Theorem 6 *Suppose A is (R, S, μ)-symmetric and $F_s = \Omega_s \Sigma_s \Phi_s^*$ is a singular value decomposition of F_s, $0 \le s \le k-1$. Let*

$$\Omega = \left[\begin{array}{cccc} P_\mu \Omega_0 & P_{\mu+1}\Omega_1 & \cdots & P_{\mu+(k-1)}\Omega_{k-1} \end{array}\right]$$

and

$$\Phi = \left[\begin{array}{cccc} \widehat{Q}_0 \Phi_0 & \widehat{Q}_1 \Phi_1 & \cdots & \widehat{Q}_{k-1}\Phi_{k-1} \end{array}\right].$$

Then

$$A = \Omega \left(\bigoplus_{s=0}^{k-1} \Sigma_s \right) \Phi^*. \tag{12}$$

Moreover, if R and S are unitary then

$$\Phi = \left[\begin{array}{cccc} Q_0 \Phi_0 & Q_1 \Phi_1 & \cdots & Q_{k-1}\Phi_{k-1} \end{array}\right],$$

Ω and Φ are unitary, and (12) is a singular value decomposition of A, except that the singular values are not neccesarily ordered. Thus, each singular value of F_s is a singular value of A associated with an $(R, s+\mu)$-symmetric left singular vector and an (S, s)-symmetric right singular vector.

4 Solving $Az = w$

If z, $w \in \mathbb{C}^n$ and we are interested in solving $Az = w$, we write

$$z = Qu \text{ and } w = Pv \text{ where } u = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{k-1} \end{bmatrix} \text{ and } v = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{k-1} \end{bmatrix}. \tag{13}$$

Theorem 7 *If* $A = \sum_{s=0}^{k-1} P_{s+\mu} F_s \widehat{Q}_s^*$ *is* (R, S, μ)*-symmetric then* $Az = w$ *if and only if*

$$F_s u_s = v_{s+\mu}, \quad 0 \le s \le k-1. \tag{14}$$

Proof. From (6) and (13),

$$Az - w = \sum_{s=0}^{k-1} P_{s+\mu} F_s u_s - \sum_{s=0}^{k-1} P_s v_s = \sum_{s=0}^{k-1} P_{s+\mu}(F_s u_s - v_{s+\mu}),$$

which vanishes if and only if (14) holds. □

If $A \in \mathbb{C}^{m\times n}$ let $\mathcal{N}(A)$ denote the null space of A.

Theorem 8 *If* A *is* (R, S, μ)*-symmetric then* $Az = 0$ *if and only* $z = Pu$ *with*

$$F_s u_s = 0, \quad 0 \le s \le k-1.$$

Moreover, if $Az = 0$ *has a nontrivial solution, then* $\mathcal{N}(A)$ *has a basis in*

$$\mathcal{S}_S = \bigcup_{r=0}^{k-1} \left\{ z \,\middle|\, Sz = \zeta^r z \right\}.$$

Proof. The first statement is obvious from (14). For the second, let

$$\mathcal{U} = \left\{ s \in \{0, 1, \dots, k-1\} \,\middle|\, F_s u_s = 0 \text{ and } u_s \ne 0 \right\}.$$

Since $Az = 0$ has a nontrivial solution, $\mathcal{U} \ne \emptyset$. If $s \in \mathcal{U}$ and $\{u_s^{(1)}, u_s^{(2)}, \cdots, u_s^{(m_s)}\}$ is a basis for $\mathcal{N}(F_s)$, then $Q_s u_s^{(1)}$, $Q_s u_s^{(2)}$, …, $Q_s u_s^{(m_s)}$ are linearly independent (S, s)-symmetric vectors in $\mathcal{N}(A)$, and

$$\bigcup_{s\in\mathcal{U}} \{Q_s u_s^{(1)}, Q_s u_s^{(2)}, \cdots, Q_s u_s^{(m_s)}\}$$

is a basis for $\mathcal{N}(A)$. □

Theorem 9 *If* A *is* (R, S, μ)*-symmetric then* A *is invertible if and only if*

$$c_{s+\mu} = d_s, \quad 0 \le s \le k-1, \tag{15}$$

and F_0, F_1, …, F_{k-1} *are all invertible. In this case,*

$$A^{-1} = P \left(\bigoplus_{s=0}^{k-1} F_s^{-1} \right) \widehat{V}_\mu^* \tag{16}$$

is $(S, R, k-\mu)$*-symmetric. If* $w = Pv$*, the solution of* $Az = w$ *is* $z = Pu$*, where*

$$u_s = F_s^{-1} v_{s+\mu}, \quad 0 \le s \le k-1. \tag{17}$$

Proof. From Theorem 7, $Az = w$ has a solution for every z if and only (14) has a solution for every $\{v_0, v_1, \dots, v_{k-1}\}$. Since $F_s \in \mathbb{C}^{c_{s+\mu}\times d_s}$, this is true if and only if (15) holds and F_0, F_1, …, F_{k-1} are all invertible. It is easy to verify (16) and (17) from (6) and (14). □

5 The eigenvalue problem for an $(R,0)$-symmetric matrix

Henceforth we assume that $m = n$ and $R = S$. Recall that a matrix A is (R,μ) symmetric if $RAR^{-1} = \zeta^{\mu}A$. If $\mu = 0$ then $F_s \in \mathbb{C}^{d_s \times d_s}$, where d_s is the dimension of the ζ^s-eigenspace of R, $0 \le s \le k-1$. Setting $z = Pu$ and $w = \lambda z$ in (14) yields the following theorem, which reduces the eigenvalue problem for an $(R,0)$-symmetric matrix A to solving the eigenvalue problem for the smaller matrices F_0, F_1, $\dots$, F_{k-1}.

Theorem 10 *If A is $(R,0)$-symmetric then (λ, z) is an eigenpair of R if and only if $z = Pu \neq 0$, where*

$$F_s u_s = \lambda u_s, \quad 0 \le s \le k-1.$$

Theorem 11 *If A is $(R,0)$-symmetric, then λ is an eigenvalue of A if and only if λ is an eigenvalue of one or more of the matrices F_0, F_1, $\dots$, F_{k-1}. Assuming this to be true, let*

$$S_A(\lambda) = \left\{ s \in \{0,1,\dots,k-1\} \,\middle|\, \lambda \textit{ is an eigenvalue of } F_s \right\}.$$

If $s \in S_A(\lambda)$ and $\{u_s^{(1)}, u_s^{(2)}, \cdots, u_s^{(m_s)}\}$ is a basis for

$$\mathcal{E}_{F_s}(\lambda) = \left\{ u_s \in \mathbb{C}^{n\times n} \,\middle|\, F_s u_s = \lambda u_s \right\},$$

then $P_s u_s^{(1)}$, $P_s u_s^{(2)}$, $\dots$, $P_s u_s^{(m_s)}$ are linearly independent (R,s)-symmetric λ-eigenvectors of A. Moreover,

$$\bigcup_{s\in\mathcal{S}_A(\lambda)} \{P_s u_s^{(1)}, P_s u_s^{(2)}, \cdots, P_s u_s^{(m_s)}\}$$

is a basis for

$$\mathcal{E}_A(\lambda) = \left\{ z \,\middle|\, Az = \lambda z \right\}.$$

Finally, A is diagonalizable if and only if F_0, F_1, $\dots$, F_{k-1} are all diagonalizable. In this case, A has d_s linearly independent (R,s)-symmetric eigenvectors, $0 \le s \le k-1$.

It seems useful to consider the case where A is diagonalizable more explicitly.

Theorem 12 *Suppose A is $(R,0)$-symmetric and diagonalizable and $F_s = \Omega_s D_s \Omega_s^{-1}$ is a spectral decomposition of F_s, $0 \le s \le k-1$. Let*

$$\Omega = \left[\begin{array}{cccc} P_0\Omega_0 & P_1\Omega_1 & \cdots & P_{k-1}\Omega_{k-1} \end{array}\right]$$

Then

$$A = \Omega \left(\bigoplus_{s=0}^{k-1} D_s \right) \Omega^{-1}$$

with

$$\Omega^{-1} = \begin{bmatrix} \Omega_0^{-1}\widehat{P}_0^* \\ \Omega_1^{-1}\widehat{P}_1^* \\ \vdots \\ \Omega_{k-1}^{-1}\widehat{P}_{k-1}^* \end{bmatrix}$$

is a spectral decomposition of A. Moreover, A is normal if and only if F_0, F_1, ..., F_{k-1} are all normal.

Note that Theorem 5(ii) implies the last sentence.

The original version of the following theorem, which dealt with centrosymmetric matrices, is due to Andrew [2, Theorem 6]. We generalized Andrew's theorem to R-symmetric matrices ($k = 2$) in [15]. The next theorem generalizes it to $(R, 0)$-symmetric matrices R is an arbitrary k-involutory matrix. The proof is practically identical to Andrew's original proof.

Theorem 13

(i) *If A is $(R, 0)$-symmetric and λ is an eigenvalue of A, then $\mathcal{E}_A(\lambda)$ has a basis in*

$$\mathcal{S}_R = \bigcup_{s=0}^{k-1} \left\{ z \,\middle|\, Rz = \zeta^s z \right\}.$$

(ii) *If A has n linearly independent eigenvectors in $\mathcal{S}_R$, then A is $(R, 0)$-symmetric.*

Proof. (i) See Theorem 11. (ii) We must show that $RA = AR$. If $Az = \lambda z$ and $Rz = \zeta^s z$, then

$$RAz = \lambda Rz = \zeta^s \lambda z \quad \text{and} \quad ARz = \zeta^s Az = \lambda\zeta^s z;$$

hence, $RAz = ARz$. Now suppose that A has n linearly independent eigenvectors $\{z_1, z_2, \ldots, z_n\}$ in $\mathcal{S}_R$. Then we can write an arbitrary $z \in \mathbb{C}^n$ as $z = \sum_{i=1}^n a_i z_i$. Since $RAz_i = ARz_i$, $1 \le i \le n$, it follows that $RAz = ARz$. Therefore $AR = RA$. $\square$

6 The eigenvalue problem for an (R, μ)-symmetric matrix with $\mu \neq 0$

We now consider the eigenvalue problem for an (R, μ)-symmetric matrix A with $\mu \neq 0$. Since Theorem 8 characterizes the null space of A, we confine our attention to nonzero eigenvalues. From Theorem 7, $Az = \lambda z \neq 0$ if and only if $z = Pu$ where

$$F_s u_s = \lambda u_{s+\mu}, \quad 0 \le s \le k-1, \tag{18}$$

with $u_s \neq 0$ for some $s \in \{0, 1, \ldots, k-1\}$. Suppose

$$\mu \neq 0, \quad q = \gcd(k, \mu) \quad \text{and} \quad m = k/q. \tag{19}$$

Then (18) can be rewritten as

$$F_{\ell+s\mu}u_{\ell+s\mu} = \lambda u_{\ell+(s+1)\mu}, \quad 0 \le s \le m-1, \tag{20}$$

where $0 \le \ell \le q-1$; hence there are q independent systems here, each associated with one $\ell \in \{0,1,\dots,q-1\}$. If $\lambda \ne 0$ and $(u_\ell, u_{\ell+\mu}, \dots, u_{\ell+(m-1)\mu})$ is a nontrivial solution of (20), then $\sum_{s=0}^{m-1} P_{\ell+s\mu}u_{\ell+s\mu}$ is a λ-eigenvector of A.

Theorem 14 *Suppose that A is (R,μ)-symmetric and (19) holds. Suppose also that for some $\ell \in \{0,1,\dots,q-1\}$, u_ℓ is a nontrivial solution of*

$$u_\ell = \lambda^{-m} F_{\ell+(m-1)\mu} \cdots F_{\ell+\mu}F_\ell u_\ell. \tag{21}$$

Define

$$u_{\ell+(s+1)\mu} = \lambda^{-1}F_{\ell+s\mu}u_{\ell+s\mu}, \quad 0 \le s \le m-2, \tag{22}$$

and

$$z_{\ell j} = \sum_{s=0}^{m-1} \zeta^{j(\ell+s\mu)} P_{\ell+s\mu}u_{\ell+s\mu}, \quad 0 \le j \le m-1.$$

Then

$$Az_{\ell j} = \zeta^{-j\mu}\lambda z_{\ell j}, \quad 0 \le j \le m-1.$$

Proof. Eqn. (22) implies (20) for $0 \le s \le m-2$, and, since $\ell + m\mu \equiv \ell \pmod{k}$, (21) implies (20) for $s = m-1$. Hence $z_{\ell 0}$ is a λ-eigenvector of A. Since

$$RP_{\ell+s\mu} = \zeta^{\ell+s\mu}P_{\ell+s\mu}, \text{ it follows that } z_{\ell j} = R^j z_{\ell 0}, \quad 0 \le \ell \le m-1.$$

Hence we must show that

$$A(R^j z_{\ell 0}) = \zeta^{-j\mu}\lambda(R^j z_{\ell 0}), \quad 0 \le j \le m-1.$$

We know this holds for $j = 0$. If it holds for some $j \in \{0,\dots,m-2\}$ then

$$RA(R^j z_{\ell 0}) = \zeta^{-j\mu}\lambda(R^{j+1}z_{\ell 0}), \quad RAR^{-1}(R^{j+1}z_{\ell 0}) = \zeta^{-j\mu}\lambda(R^{j+1}z_{\ell 0}),$$

and, since $RAR^{-1} = \zeta^\mu A$,

$$A(R^{j+1}z_{\ell 0}) = \zeta^{-(j+1)\mu}\lambda(R^{j+1}z_{\ell 0}),$$

which completes the finite induction. □

7 Applications

Let σ and ρ be permutations of $\mathbb{Z}_k$, and let E and F be the permutation matrices

$$E = [\delta_{i,\sigma^{-1}(j)}]_{i,j=0}^{k-1} = \begin{bmatrix} e_{\sigma^{-1}(0)} & e_{\sigma^{-1}(1)} & \cdots & e_{\sigma^{-1}(k-1)} \end{bmatrix}$$

and

$$F = [\delta_{i,\rho^{-1}(j)}]_{i,j=0}^{k-1} = \begin{bmatrix} e_{\rho^{-1}(0)} & e_{\rho^{-1}(1)} & \cdots & e_{\rho^{-1}(k-1)} \end{bmatrix}.$$

If p and q are positive integers let $R = E \otimes I_p$ and $S = F \otimes I_q$. The methods of this paper are applicable to matrices $[A_{rs}]_{r,s=0}^{k-1}$ with $A_{rs} \in \mathbb{C}^{p\times q}$ such that $EAF^* = \zeta^\mu A$. Since $EAF^* = [A_{\sigma(r),\rho(s)}]_{r,s=0}^{k-1}$, $EAF^* = \zeta^\mu A$ if and only if $A_{\sigma(r),\sigma(s)} = \zeta^\mu A_{rs}$, $0 \le r \le s-1$. For example, $A_{rs} = [\zeta^{\mu r} C_{s-r}]_{r,s=0}^{k-1}$ satisfies this condition with $\sigma(r) = \rho(r) = r+1 \pmod{k}$.

Chen and Sameh [6] have studied matrices A such that $PAP = A$, where P is a signed permutation matrix. Since $PAP = A$ is equivalent to $PAQ^{-1} = A$ with $Q = P^{-1}$, our results also apply to these matrices.

References

[1] C. M. Ablow, J. L. Brenner, Roots and canonical forms for circulant matrices, Trans. Amer. Math. Soc. 107 (1963) 360–376.

[2] A. L. Andrew, Eigenvectors of certain matrices, Linear Algebra Appl. 7 (1973) 151–162.

[3] A. L. Andrew, Solution of equations involving centrosymmetric matrices, Technometrics 15 (1973) 405-407.

[4] A. L. Andrew, Centrosymmetric matrices, SIAM Review 40 (1998) 697-698.

[5] A. Cantoni, F. Butler, Eigenvalues and eigenvectors of symmetric centrosymmetric matrices, Linear Algebra Appl. 13 (1976), 275–288.

[6] H.-C. Chen, A. Sameh, A matrix decomposition method for orthotropic elasticity problems, SIAM J. Matrix Anal. Appl. 10 (1989), 39–64.

[7] H.-C. Chen, Circulative matrices of degree θ, SIAM J. Matrix Anal. Appl. 13 (1992) 1172–1188.

[8] A. R. Collar, On centrosymmetric and centroskew matrices, Quart. J. Mech. Appl. Math. 15 (1962) 265–281.

[9] D. Fasino, Circulative properties revisited: Algebraic properties of a generalization of cyclic matrices, Italian J. Pure Appl. Math 4 (1998) 33–43.

[10] I. J. Good, The inverse of a centrosymmetric matrix, Technometrics 12 (1970) 925–928.

[11] G. L. Li, Z. H. Feng, Mirrorsymmetric matrices, their basic properties, and an application on odd/even decomposition of symmetric multiconductor transmission lines, SIAM J. Matrix Anal. Appl. 24 (2002) 78–90.

[12] I. S. Pressman, Matrices with multiple symmetry properties: applications of centrohermitian and perhermitian matrices, Linear Algebra Appl. 284 (1998) 239–258.

[13] W. C. Pye, T. L. Boullion, T. A. Atchison, The pseudoinverse of a centrosymmetric matrix, Linear Algebra Appl. 6 (1973) 201–204.

[14] D. Tao, M. Yasuda, A spectral characterization of generalized real symmetric centrosymmetric and generalized real symmetric skew-centrosymmetric matrices, SIAM J. Matrix Anal. Appl. 23 (2002) 885–895.

[15] W. F. Trench, Characterization and properties of matrices with generalized symmetry or skew symmetry, Linear Algebra Appl. 377 (2004) 207-218.

[16] W. F. Trench, Inverse eigenproblems and associated approximation problems for matrices with generalized symmetry or skew symmetry Linear Algebra Appl. 380 (2004) 199–211.

[17] W. F. Trench, Hermitian, Hermitian R-symmetric, and Hermitian R-skew symmetric Procrustes problems, Linear Algebra Appl. 387 (2004) 83–98.

[18] W. F. Trench, Minimization problems for (R, S)-symmetric and (R, S)-skew symmetric matrices, Linear Algebra Appl. 389 (2004) 23–31.

[19] W. F. Trench, Characterization and properties of (R, S)-symmetric, (R, S)-skew symmetric, and (R, S)-conjugate matrices, SIAM J. Matrix Anal. Appl. 26 (2005) 748–757.

[20] M. Yasuda, A Spectral Characterization of Hermitian Centrosymmetric and Hermitian Skew-Centrosymmetric K-Matrices; SIAM Journal on Matrix Analysis and Applications, Volume 25, No. 3 (2003) pp 601-605.

[21] J. R. Weaver, Centrosymmetric (cross-symmetric) matrices, their basic properties, eigenvalues, eigenvectors, American Mathematical Monthly 92 (1985) 711-717.