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RESEARCH IN METHODS OF GENERATING LIAPUNOV FUNCTIONS

Submitted by

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Abstract

This report gives the results of a number of attempts to find underlying principles uniting the many methods for generating Liapunov functions for nonlinear systems of ordinary differential equations. Our conclusions indicate that if general principles exist for the generation of Liapunov functions they lie very deep, and are not to be found among the methods now in use.

Summary

The report falls naturally into two parts. Part I gives a critique of the methods now in use for the generation of Liapunov functions for nonlinear systems of ordinary differential equations. The critique is illustrated by a technique for generating Liapunov functions which we call the "Conservative Spring Concept". This technique, which we emphasize is not really a new method, is capable of solving a large number of previously published examples.

Part II of this report is a description of some partial results obtained for the stability analysis of systems of autonomous ordinary differential equations whose right hand sides are homogeneous polynomial forms.

The report is concerned exclusively with the problem of Liapunov stability or asymptotic stability of an isolated singular point, usually the origin.

I. The purpose of this research was to try to uncover any general principles that may exist for the systematic generation of Liapunov functions. After looking for quite some time at the various methods used to generate Liapunov functions for the analysis of the stability of systems of autonomous differential equations it is our opinion that, despite the great bulk of published material, little progress has been made. Our efforts to deal with the problem of the systematic generation of Liapunov functions has led us to feel that a specific, rather than a general, approach is most likely to succeed in the long run. It is, in fact, the very generality, or seeming generality, of the various methods now in use for finding Liapunov functions which is their principal weakness. The authors of the various methods claim generality, but they always end up applying their methods to basically the same classes of second, third, and a few fourth order systems.* The exceptions to this rule, which we know of, are high order systems describing certain rigid body motions, and control theory problems with high order linear parts combined with a single nonlinear element. The methods of obtaining Liapunov functions in these two areas are for the former, to consider a linear combination of first integrals, and for the latter to consider a quadratic form plus an integral for the nonlinear element.

Most of the methods studied are, as claimed, very general in principle, but they all suffer from the following defects in practice.

(i) A large amount of computation is required for high order systems. This is the least serious defect.

*It should be understood that we are here talking of practical procedures and not of the very substantial body of theoretical results.

the evolution of the method. Each new type of equation is further removed from the original idea. As the analogy grows weaker additional assumptions must be introduced until finally there are gaps to be filled in by a judicious choice of functions.

In summary, we believe at this point that effective progress in the problem of generating Liapunov functions has not been made mainly because the proposed methods are too general. We feel that one should consider each order of nonlinear system in turn with respect to classifying the various types of nonlinearities according to how they affect the stability of the system. The Conservative Spring method presented in this report is not really a new method. It serves to illustrate the pitfalls of generality and also serves to illustrate the possibilities of the above type of classification.

The Conservative Spring Concept of Generating Liapunov Functions

1. Motivation.

The differential equation governing the motion of an undamped nonlinear spring is

(1) $$\ddot{x} + g(x) = 0 \; ; \; g(0) = 0 \, .$$

The total energy of the system at any time is given by the function

$$V = \frac{\dot{x}^2}{2} + G(x) \quad ,$$

where $G(x) = \int_0^x g(s)\,ds$. Differentiating V with respect to time along a trajectory of (1) yields

$$\dot{V} = \dot{x}\ \ddot{x} + g(x)\ \dot{x} = \dot{x}\ (-g(x)) + \dot{x}\ g(x) \equiv 0 \, .$$

Hence if we require that

$$G(x) > 0 \text{ for } x \neq 0 \quad ,$$

the function V becomes a Liapunov function for (1) and we may conclude that (1) is stable.

2. Liénard's equation.

We may consider Liénard's equation

(2) $$\ddot{x} + f(x)\ \dot{x} + g(x) = 0$$

as representing the motion of a non-conservative spring. We attempt to cast equation (2) into a form which resembles (1). Let

$$F(x) = \int_0^x f(s)\,ds$$

$$G(x) = \int_0^x g(s)\,ds \quad ,$$

and rewrite (2) as

$$\frac{d}{dt}\left[\dot{x} + F(x)\right] + g(x) = 0$$

By analogy with (1) we choose the function

$$V = \frac{1}{2}\left[\dot{x} + F(x)\right]^2 + G(x) \tag{3}$$

as a candidate for a Liapunov function. Note that now V no longer represents the total energy. Differentiating V with respect to t along a trajectory of (2) yields

$$\dot{V} = -g(x)\,F(x)\ .$$

We may conclude that (2) is asymptotically stable in a region Ω defined by the conditions

$$g(x)\,F(x) > 0 \quad \text{for } |x| < a,\ x \neq 0$$

$$G(x) < \ell \implies |x| < a$$

$$\Omega = \{\,(x, \dot{x}) \ / \ V < \ell\,\}\ .$$

Complete details of the analysis may be found in [5].

We now compare, by means of several examples, the proposed Liapunov function given in equation (3) with the Liapunov function obtained by other methods.*

Example A. Van-der Pol's equation

$$\ddot{x} + \varepsilon\,(1-x^2)\,\dot{x} + x = 0$$

$$F(x) = \varepsilon \int_0^x (1-s^2)\,ds = \varepsilon\,(x - \frac{x^3}{3})$$

$$G(x) = \int_0^x s\,ds = \frac{x^2}{2} .$$

$$V = \frac{1}{2}\left[\dot{x} + \varepsilon\,(x - \frac{x^3}{3})\right]^2 + \frac{x^2}{2}$$

$$\dot{V} = -\,g(x)\,F(x) = -\,\varepsilon x^2\,(1 - \frac{x^2}{3}) .$$

The Liapunov function given by this analysis is equal to 1/2 of the Liapunov function given by Infante {2;p 74}.

Example B. Infante {2;p 78} considers a non-symmetrical oscillator with equation

$$\ddot{x} + a\dot{x} + bx + X^2 = 0 \text{ ; } a, b \text{ constants} > 0.$$

Here

$$F(x) = ax, \quad G(x) = \frac{b}{2}\,x^2 + \frac{1}{3}\,x^3$$

In accordance with (3) we take

* The method with which we are comparing will be denoted by the name of its author; references are to section and page of the report [6].

$$V = \frac{1}{2}\left[\dot{x} + ax\right]^2 + \frac{b}{2}x^2 + \frac{1}{3}x^3$$

$$\dot{V} = -(bx + x^2)\,ax = -ax^2(b + x) .$$

This is the same Liapunov function as obtained by Infante.

Example C. Infante {2;p 81} considers a nonlinear damped pendulum with equation

$$\ddot{x} + (\varepsilon \cos x)\,\dot{x} + \sin x = 0 \;; \quad \varepsilon > 0 .$$

Here

$$F(x) = \varepsilon \sin x \;, \quad G(x) = 1 - \cos x$$

and

$$V = \frac{1}{2}\left[\dot{x} + \varepsilon \sin x\right]^2 + (1 - \cos x) ,$$

$$\dot{V} = -\varepsilon \sin^2 x .$$

This is the same Liapunov function as obtained by Infante.

Example D. A globally stable oscillator.

$$\ddot{x} + \varepsilon(1 - x^2 + x^4)\,\dot{x} + x^3 = 0$$

Here

$$F(x) = \varepsilon\left(x - \frac{x^3}{3} + \frac{x^5}{5}\right) , \quad G(x) = \frac{x^4}{4}$$

and

$$V = \frac{1}{2}\left[\dot{x} + \varepsilon\left(x - \frac{x^3}{3} + \frac{x^5}{5}\right)\right]^2 + \frac{x^4}{4}$$

$$\dot{V} = -\varepsilon x^4\left(1 - \frac{x^2}{3} + \frac{x^5}{5}\right)$$

Again, our Liapunov function is the same as that given by Infante {2;p 82}. Szego {7;p 46} using a method not related to the energy concept obtains the function

$$V_s = \frac{1}{2} x^4 + \dot{x}^2 ,$$

$$\dot{V}_s = 2 \varepsilon \dot{x}^2 (1 - x^2 + x^4) .$$

Example E. Schultz and Gibson {5;p 28} consider the following example by means of the variable gradient technique.

$$\ddot{x} + (1 + f(x) + x f'(x)) \dot{x} + \beta x f(x) = 0$$

Here

$$F(x) = x + \int_0^x f(s) ds + \int_0^x s f'(s) ds = x (1 + f(x))$$

$$G(x) = \beta \int_0^x s f(s) ds .$$

In accordance with (3) we set

$$V = \left[\dot{x} + x(1 + f(x))\right]^2 + \beta \int_0^x s f(s) ds$$

$$\dot{V} = - \beta x^2 f(x) (1 + f(x)).$$

The conditions for global asymptotic stability are

$$x f(x) > 0 \quad ; \quad x \neq 0$$

$$\beta > 0$$

$$|f(x)| \geq 1$$

The Liapunov function obtained by Schultz and Gibson is

$$W = (x + \dot{x})^2 + 2 \int_0^x s\,[f(s) + s\,f'(s)]\,ds + 2\beta \int_0^x s\,f(s)\,ds$$

$$\dot{W} = -\,2\left[(f(x) + x\,f'(x))\,\dot{x}^2 + \beta\,x^2\,f(x)\right]$$

The Liapunov function W gives the following conditions for global asymptotic stability

$$\beta > 0$$

$$f(x) + xf'(x) \geq 0$$

$$f(x) > 0 \quad \text{for} \quad x \neq 0$$

3. A more general second order equation.

The Equation

$$(4) \qquad \ddot{x} + f(x)\,\dot{x}^2 + g(x)\,\dot{x} + h(x) = 0$$

is one step removed from the Liénard equation; we therefore aim to put it into a form similar to (2).

Let $y(x) = \exp\left[\int_0^x f(s)\,ds\right] = \exp\left[F(x)\right]$.

Multiplying (4) by y gives

$$(5) \qquad (\ddot{x} + f(x)\,\dot{x}^2)\,y(x) + g(x)\,y(x)\,\dot{x} + h(x)\,y(x) = 0$$

Define

$$\hat{G}(x) = \int_0^x g(s)\,y(s)\,ds\,, \quad \hat{H}(x) = \int_0^x h(s)\,y(s)\,ds\,.$$

Since $\dot{y} = \dot{x}\, f(x)\, y(x)$ we may write (5) in the form

$$\frac{d}{dt}\left[\dot{x}\, y + \hat{G}\right] + h(x)\, y(x) = 0$$

By analogy with (3) we take as a candidate for a Liapunov function

$$V = \frac{1}{2}\left[\dot{x}\, y(x) + \hat{G}(x)\right]^2 + \hat{H}(x)$$

Differentiating with respect to t along a trajectory of (4) yields

$$\dot{V} = \dot{x}\, h(x)\left(\exp\,[F(x)] - \exp\,[2F(x)]\right) - h(x)\, \hat{G}(x)\, \exp\,[F(x)]\ .$$

$\dot{V}$ would have the same form as in (3) if the first term above were zero. This can be accomplished by modifying V by use of the function

$$\tilde{H}(x) = \int_0^x h(s)\, \exp\,[2F(s)]\, ds$$

in place of $\hat{H}(x)$.

Our Liapunov function now becomes

$$(6) \qquad V = \frac{1}{2}\left[\dot{x}\, \exp\,[F(x)] + \hat{G}(x)\right] + \tilde{H}(x) \quad ,$$

$$\dot{V} = -\, h(x)\, \hat{G}(x)\, \exp\,[F(x)] \quad .$$

The function $\hat{G}(x)$ is not really needed in (6); if we delete it from the definition of V in (6) we get the alternate form

$$(7) \qquad V = \frac{1}{2}\dot{x}^2 \exp\,[2F(x)] + \tilde{H}(x) \quad ,$$

$$\dot{V} = -\dot{x}^2\, g(x) \exp\,[2F(x)] \quad .$$

Example. Infante {2;p 83} considered the problem of liquid motion in a surge tank, and derived the governing differential equation

$$\ddot{x} + \frac{\alpha^2}{\beta}\quad \dot{x}^2 + \frac{\beta}{\alpha(1+x^2)}\left[\frac{2\alpha^2}{\beta} - 1 + \frac{2\alpha^2}{\beta}\quad x\right]\dot{x}$$

$$+ \left[x - \beta \quad + \frac{\beta}{(1+x)^2}\right] = 0$$

Using the alternate form (7) we obtain the function

$$V = \frac{1}{2}\dot{x}^2 \exp\left[2\,\frac{\alpha^2}{\beta}\,x\right] + \int_0^x \left[s - \beta + \frac{\beta}{(1+s)^2}\right]\exp\left[2\quad\frac{\alpha^2}{\beta}\,s\right]\,ds,$$

$$\dot{V} = -\frac{\beta}{\alpha}\,\frac{\dot{x}^2}{(1+x)^2}\left[\frac{2\alpha^2}{\beta} - 1 + \frac{2\alpha^2}{\beta}\quad x\right] \quad \exp\left[2\;\frac{\alpha^2}{\beta}\quad x\right]$$

for the analysis of the stability of (8), which is the same as the Liapunov function given by Infante.

4. The general second order equation.

We turn now to the most general second order equation,

$$(8) \qquad \ddot{x} + X(x,\dot{x}) = 0 \quad ; \quad X(0,0) = 0 \quad .$$

In order to get equation (9) into a form amenable to our previous analysis we assume that $\frac{\partial X}{\partial \dot{x}}$ and $\frac{\partial^2 X}{\partial \dot{x}^2}$ exist and are continuous. Write

$$X(x,\dot{x}) = h(x) + g(x)\,\dot{x} + f(x)\,\dot{x}^2 + \hat{X}(x,\dot{x}) \quad ,$$

where $\quad \hat{X}\,(x,\dot{x}) = o(\dot{x}^2)$ as $\dot{x} \to 0$. Equation (8) now takes the form

$$\ddot{x} + f(x)\,\dot{x}^2 + g(x)\,\dot{x} + h(x) + \hat{X}(x,\dot{x}) = 0 \; .$$

By analogy with the analysis of equation (4) we adopt the following form as a candidate for a Liapunov function,

$$\text{(9)} \qquad V = \frac{\dot{x}^2}{2} \exp\,[2F(x)] + \tilde{H}(x) \quad ,$$

where

$$F(x) = \int_0^x f(s)\,ds \quad ; \quad \tilde{H}(x) = \int_0^x h(s)\,\exp\,[2F(s)]\,ds.$$

$$\dot{V} = -\left[\dot{x}^2\,g(x) + \dot{x}\,\hat{X}(x,\dot{x})\right] \exp\,[2F(x)] \; .$$

The function (9) will be a Liapunov function for (8) if $h(x)$ and $g(x)$ are positive in some neighborhood of the origin; the extent of the region of asymptotic stability will depend on the nature of the function $\hat{X}(x,\dot{x})$.

Example. Rayleigh's equation.

$$\ddot{x} - \frac{\mu}{3} \dot{x}^3 + \mu\dot{x} + x = 0.$$

Here $h(x) = x$, $g(x) = \mu$ and $f(x) = 0$; thus

$$V = \frac{\dot{x}^2}{2} + \int_0^x sds = \frac{1}{2}\left(\dot{x}^2 + x^2\right)$$

$$\dot{V} = -\mu\dot{x}^2\left(1 - \frac{\dot{x}^2}{3}\right)$$

In this case V is the usual quadratic form.

5. Third order equations.

In this section we consider the third order equation

$$\dddot{x} + f(\dot{x})\,\ddot{x} + g(\dot{x}) + h(x) = 0. \tag{10}$$

Again, our aim is to put the new equation into a form resembling a previously treated equation. Define $F(\dot{x})$, $G(\dot{x})$, and $H(x)$ in the usual way, and write equation (10) as

$$\frac{d}{dt}\left[\ddot{x} + F(\dot{x})\right] + g(\dot{x}) + h(x) = 0 \tag{11}$$

The first two terms of (11) suggest the function

$$V_1 = \frac{1}{2}\left[\ddot{x} + F(\dot{x})\right]^2 + G(\dot{x}) .$$

This, however takes no account of the third term $h(x)$. We therefore, seek a Liapunov function in the form

$$V = V_1 + V_2 \quad ,$$

where V_1 is as above, and V_2 is to be determined so as to simplify the resulting expression for $\dot{V}$. Since $\dot{V} = \dot{V}_1 + \dot{V}_2$ it is simpler to choose $\dot{V}_2$ so as to simplify $\dot{V}$ and further require that $\dot{V}_2$ be given as a derivative.

Differentiating V with respect to time along a trajectory of (10) yields

$$\dot{V} = \left[\ddot{x} + F(\dot{x})\right]\left[-g(\dot{x}) - h(x)\right] + g(\dot{x})\ddot{x} + \dot{V}_2$$

$$= -g(\dot{x})F(\dot{x}) + \left[\dot{V}_2 - h(x)\ddot{x} - h(x)F(\dot{x})\right] \quad .$$

The first term above, $-g(\dot{x})F(\dot{x})$, corresponds to previous results and should be retained.

In order to choose $\dot{V}_2$ we assume that $F(\dot{x})$ has a continuous first derivative; write

$$F(\dot{x}) = a\dot{x} + \ldots ,$$

and

$$F(\dot{x})\,h(x) = a\dot{x}\,h(x) + [F(\dot{x}) - a\dot{x}]\;h(x)$$

$$= a\frac{d}{dt}[a\,H(x)] + \left(F(\dot{x}) - a\dot{x}\right)h(x) \quad .$$

This suggests that we take

$$\dot{V}_2 = \frac{d}{dt}\left[\dot{x}\,h(x) + a\,H(x)\right] \quad .$$

Then

$$\dot{V}_2 - h(x)\,\ddot{x} - h(x)\,F(\dot{x}) = \dot{x}^2\,h'(x) + \left[a\dot{x} - F(\dot{x})\right]h(x) \quad .$$

We thus, adopt as a candidate for a Liapunov function for equation (1),

$$V = \frac{1}{2}\left[\ddot{x} + F(\dot{x})\right]^2 + G(\dot{x}) + \dot{x}\,h(x) + a\,H(x) \tag{12}$$

$$\dot{V} = -\left[g(\dot{x}) + h(x)\right]F(\dot{x}) + \dot{x}^2\,h'(x) + a\,\dot{x}\,h(x) \quad .$$

Example. Barabashin's equation.

For the equation

$$\dddot{x} + a\,\ddot{x} + g(\dot{x}) + h(x) = 0 \quad , \tag{13}$$

The above analysis gives the Liapunov function

$$V = \frac{1}{2}\left[\ddot{x} + a\,\dot{x}\right]^2 + G(\dot{x}) + a\,H(x) + \dot{x}\,h(x)$$

$$\dot{V} = -\,a\,\dot{x}\,g(\dot{x}) + \dot{x}^2\,h'(x) \quad .$$

The V given here is the same as that obtained by Walker {2;p 87} using his modification of Infante's integral method; it yields the following set of sufficient conditions for global asymptotic stability

$$a\,\frac{g(\dot{x})}{\dot{x}} - h'(x) > 0 \quad ; \quad x \neq 0$$

$$a > 0 \;; \quad x\,h(x) > 0 \;; \quad x \neq 0$$

$$V \to \infty \quad \text{for} \quad |\underline{x}| \to \infty$$

$$\frac{1}{2}\left[\ddot{x} + a\dot{x}\right]^2 + \int_0^{\dot{x}} g(s)\,ds + a\int_0^{x} h(s)\,ds > \dot{x}\,h(x)$$

Schultz {5;p 39} and Puri {5;p 46} treat Barabashin's equation by the variable gradient method. Puri obtains the function

$$V_p = \frac{1}{2}(\ddot{x} + a\dot{x})^2 + a\int_0^{x} h(s)\,ds + \int_0^{\dot{x}} g(s)\,ds + h(x)\,\dot{x}$$

$$\dot{V}_p = -\ddot{x}^2\left[a\,\frac{g(\dot{x})}{\dot{x}} - h'(x)\right],$$

which yields almost the same conditions for global asymptotic stability as does the function V.

6. Some other forms.

In this section we present three examples which are not in any of the forms specified in the previous sections. These examples are intended to illustrate the gaps that develop as a method for generating Liapunov functions is extended.

Example A. Walker {2;p 89} studied the equation

$$\dddot{x} + b\ddot{x} + (x + c\dot{x})^m = 0. \tag{14}$$

This equation is not of the form of equation (10), but bears some similarity to it.

Write (14) as

$$\frac{d}{dt} \quad (\ddot{x} + b\dot{x}) + (x + c\dot{x})^m = 0 \quad ,$$

and let

$$V = \frac{1}{2} [\ddot{x} + b\dot{x}]^2 + \frac{(x + c\dot{x})^{m+1}}{C(m + 1)} + V_2 \quad ,$$

where

$$\frac{(x + c\dot{x})^{m+1}}{C(m + 1)} = G(\dot{x}) = \int_o^{\dot{x}} (x + c\dot{x}) \, d\dot{x}$$

Differentiating we get

$$\dot{V} = - (b - \frac{1}{c}) \quad \dot{x} \; (x + c\dot{x})^m + \dot{V}_2$$

This form of $\dot{V}$ does not suggest an appropriate form for $\dot{V}_2$. If, however, we do not replace $\dddot{x}$ by $- b\ddot{x} - (x+c\dot{x})^m$ in the evaluation of $\dot{V}$ we obtain the expression

$$\text{(15)} \qquad \dot{V} = (b - \frac{1}{c}) \; \dot{x} \; \dddot{x} + b \; (b - \frac{1}{c}) \; \dot{x} \; \ddot{x} + \dot{V}_2$$

The order of $\dddot{x}$ is now reduced by extracting an exact differential from the first two terms of (15). Thus,

$$\dot{V} = \frac{d}{dt}\left[\dot{x} \;\; \ddot{x} + b\dot{x}^2\right]\left(b - \frac{1}{c}\right) \; \dot{x}^2 + \dot{V}_2 \qquad .$$

The choice of $\dot{V}_2$ is now clear,

$$\dot{V}_2 = \frac{d}{dt}\left[\dot{x} \;\; \ddot{x} + b\dot{x}^2\right]\left(b - \frac{1}{c}\right) \quad .$$

With this choice of $\dot{V}_2$ we get

$$V = \frac{1}{2}\left[\ddot{x} + \frac{1}{c}\dot{x}\right]^2 + \frac{1}{2c^2}(bc - 1)\ \dot{x}^2 + \frac{1}{c(m+1)}\ (x + c\dot{x})^{m+1} ,$$

$$\dot{V} = -\left(b - \frac{1}{c}\right)\dot{x}^2 .$$

The Liapunov function obtained here is the same as that obtained by Walker.

Ingwerson {7;p 39} and Szego {8;p 52} consider equation (14) for the case $m = 3$, and obtain very similar results. Ingwerson uses a linear analogy in his work while Szego uses a modification of Zubov's technique.

Example B. Ingwerson {7;p 45} considers the third order system

$$\text{(16)} \qquad \dddot{x} + b_1\ddot{x} + (b_2 + c_2 b_3)\dot{x} + b_4(x + c_2\dot{x})^2 + b_3 x = 0 .$$

Let

$$y = x + c_2\dot{x}$$

With this notation equation (16) becomes

$$\frac{d}{dt}\left[\ddot{x} + b_1\dot{x}\right] + b_2\dot{x} + b_3 y + b_4 y^2 = 0 .$$

Since the coefficient of $\ddot{x}$ in the above equation is unity, it would be more convenient to make the coefficient of $\dot{x}$ in y equal to unity. Thus, let

$$z = \dot{x} + \frac{x}{c_2} \quad ; \quad \dot{z} = \ddot{x} + \frac{\dot{x}}{c_2} ,$$

and our equation now becomes

$$\frac{d}{dt}\left[\ddot{x} + b_1\dot{x}\right] + b_2\dot{x} + b_3\, c_2\, z \;+ b_4\, c_2^{\,2}\;\; z^2 = 0 .$$

We now take V of the form

$$V = \frac{1}{2}\left[\ddot{x} + b_1\,\dot{x}\right] + b_2\,\frac{\dot{x}^2}{2} + b_3\;c_2\,\frac{z^2}{2} + b_4\;c_2^{\,2}\;\frac{z^3}{3} + V_2 .$$

Differentiation yields

$$\text{(17)}\qquad \dot{V} = -\,b_1 b_2 \dot{x}^2 - \left[b_1 - \frac{1}{c_2}\right]\left[b_3 c_2\;\;\dot{x} + \frac{x}{c_2} + b_4 c_2^{\,2}\;\;\dot{x} + \frac{x}{c_2}\right]\dot{x} + \dot{V}_2$$

We now look for an appropriate derivative in the second term of (17) to choose as $\dot{V}_2$. After some rearrangement of terms we find that

$$\left[b_1 - \frac{1}{c_2}\right]\left[b_3 c_2\left(\dot{x} + \frac{x}{c_2}\right) + b_4 c_2^{\,2}\left(\dot{x} + \frac{x}{c_2}\right)^2\right]\dot{x} =$$

$$\left[b_1 - \frac{1}{c_2}\right]\left[b_3 c_2\;\dot{x}^2\;\left(1 + \frac{b_4 c_2}{b_3}\;\;\dot{x}\;\; + \frac{2b_4 c_2}{b_3 c_2}\;\;x\right)\right] +$$

$$\left[b_1 - \frac{1}{c_2}\right]\;\frac{d}{dt}\left[\frac{b_3}{2}\;\;x^2 + \frac{b_4}{3}\;\;x^3\right] .$$

Choose

$$\dot{V}_2 = \left[b_1 - \frac{1}{c_2}\right]\;\frac{d}{dt}\left[\frac{b_3}{2}\,x^2 + \frac{b_4}{3}\;\;x^3\right] ;$$

Then

$$\dot{V} = -\,b_1 b_2\,\dot{x}^2 - (b_1 c_2 - 1)\left[\,b_3\dot{x}^2\left(1 + \frac{b_4 c_2}{b_3}\,\dot{x} + \frac{2b_4}{b_3}\,x\right)\right]$$

$$V = \frac{1}{2}\,(\ddot{x} + b_1\dot{x})^2 + \frac{b_2}{2}\,\dot{x}^2 + \frac{b_3 c_2}{2}\,z^2\left[1 + \frac{2b_4 c_2}{b_3}\,z\right]$$

$$+\left(b_1 - \frac{1}{c_2}\right)\frac{b_3}{2}\,x^2\left(1 + \frac{2b_4}{3b_3}\,x\right).$$

The above function gives the following sufficient conditions for asymptotic stability:

$$b_i > 0 \quad ; \quad i = 1, 2, 3, 4$$

$$b_1\,c_2 > 1$$

$$1 + \frac{b_4 c_2}{b_3}\left(\dot{x} + \frac{2x}{c_2}\right) > 0$$

$$1 + \frac{2b_4 c_2}{b_3}\,z > 0$$

$$1 + \frac{2b_4}{3b_3}\,x > 0 \quad .$$

Ingwerson claims global stability, but his Liapunov function is not given in the report.

Example C. Our last example is a fourth order equation considered by Walker {2;p 96} :

(18) $$\ddddot{x} + 4\dddot{x} + 5\ddot{x} + 2\dot{x} + cx^3 = 0 .$$

Even though equation (18) is of a relatively simple form, containing only one nonlinearity, a good deal of ingenuity is required to produce a Liapunov function.

The terms in (18) may be grouped, and the equation rewritten as

$$(\ddddot{x} + \dddot{x}) + 3(\dddot{x} + \ddot{x}) + 2(\ddot{x} + \dot{x}) + cx^3 = 0 .$$

This suggests that we let

$$y = \dot{x} + x ,$$

and write the equation in the form

(19) $$\frac{d}{dt}(\ddot{y} + 3\dot{y}) + 2\dot{y} + cx^3 = 0$$

Equation (19) suggests that we try to construct a Liapunov function in the form

$$V = \frac{1}{2}(\ddot{y} + 3\dot{y})^2 + \dot{y}^2 + V_2 .$$

Differentiating,

$$\dot{V} = -\,6\dot{y}^2 - cx^3\,(\dddot{x} + \ddot{x} + 3\ddot{x} + 3\dot{x}) + \dot{V}_2$$

$$= -\,6\dot{y}^2 + 3cx^2\,(\ddot{x}\,\dot{x} + 4\dot{x}^2) - c\,\frac{d}{dt}\left[x^3\,(\ddot{x} + 4\dot{x}) + \frac{3}{4}\,x^4\right] + \dot{V}_2 \quad .$$

We thus take

$$\dot{V}_2 = c\,\frac{d}{dt}\left[x^3\,\ddot{x} + 4\dot{x} + \frac{3}{4}\,x^4\right] \quad .$$

which yields

$$\dot{V} = -\,6(\ddot{x} + \dot{x})^2 + 3cx^2\,(\ddot{x}\,\dot{x} + 4\dot{x}^2) \tag{20}$$

The above form of $\dot{V}$ does not lead to satisfactory stability criteria; therefore we shall redefine $\dot{V}_2$. Equation (20) may be written as

$$\dot{V} = -\,6(1-2\,x^2)\left(\dot{x} + \frac{\ddot{x}}{8}\right)^2 - \frac{12}{64}\,cx^2\,\ddot{x}^2 - \frac{21}{2}\,\dot{x}\,\ddot{x} - \frac{189}{32}\,\ddot{x}^2$$

Redefine $\dot{V}_2$ as

$$\dot{V}_2 = c\,\frac{d}{dt}\left[x^3\,(\ddot{x} + 4\dot{x}) + \frac{3}{4}\,x^4\right] + \frac{21}{4}\,\frac{d}{dt}\,(\dot{x}^2) \quad .$$

The last term of the new $\dot{V}_2$ is designed to simplify the last three terms in the expression for $\dot{V}$. With this definition of $\dot{V}_2$ we get the final result,

$$V = \frac{1}{2}(\ddot{y} + 3\dot{y})^2 + (x, \dot{x}, \ddot{x}) \begin{pmatrix} \frac{3cx^2}{4} & 2cx^2 & \frac{cx^2}{2} \\ 2cx^2 & \frac{25}{4} & 1 \\ \frac{cx^2}{2} & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \\ \ddot{x} \end{pmatrix} \tag{21}$$

$$\dot{V} = -6\,(1-2cx^2)\left(\dot{x} + \frac{\ddot{x}}{8}\right)^2 - \frac{189}{32}\,\ddot{x}^2 - \frac{13}{6}\,x^2\,\ddot{x}^2 .$$

The Liapunov function defined in (21) yields the following stability criteria:

V is positive definite if

$c > 0$ and $cx^2 < 1$

$\dot{V}$ is negative definite if

$c > 0$ and $2cx^2 < 1.$

II. An attempt was made by Dr. J. P. Clay to find out if any theory could be developed regarding the underlying structure of the class of all Liapunov functions (yielding asymptotic stability) admitted by a given system of differential equations. Coincident with this, the class of all systems of differential equations (of a given type) which admitted a given Liapunov function was studied.

This attempt yielded only a few results of an elementary nature, and was abandoned as being premature. The generality of the approach was far too great to yield concrete results until more is known about specific classes of equations and their associated Liapunov functions.

The following is a brief description of Dr. Clay's ideas.

Let W be a closed neighborhood of the origin in n-dimensional Euclidean space, and let $V(\underline{x})$ be a positive definite mapping of W into the reals having continuous first partial derivatives.

We define

$$L_V(W) = \{\underline{f}/\underline{f}: W \rightarrow E^n;\ \underline{f}\ (0) = 0;\ \nabla V \cdot f \text{ is negative definite}\} .$$

Basically $L_V(W)$ represents the set of all autonomous systems of differential equations for which the origin is an asymptotically stable isolated singular point, and for which V is a Liapunov function.

The following results hold:

(1) $f, g \in L_V(w) \Longrightarrow f + g \in L_V(W)$

(2) $f \in L_V(w)$ and $\alpha > 0 \Longrightarrow \alpha f \in L_V(W)$

(3) $L_V(w) \subset C(W; E^n)$ = the set of continuous functions on W into E^n.

(4) $L_V(w)$ is a convex topological semi-group of $C(W; E^n)$.

Let $\underline{f}:W \to E^n$ with $\underline{f}(0) = 0$ and f continuous.
We define

$$L_{\underline{f}}(W) = \{V/V:W \to R;\ V \text{ has continuous first partial derivatives; } V \text{ is positive definite; } \nabla V\cdot\underline{f} \text{ is negative definite}\}.$$

Assuming that the system $\dot{\underline{x}} = \underline{f}(\underline{x})$ has an asymptotically stable singular point at the origin, then $L_{\underline{f}}(W)$ is the set of all Liapunov functions defined on W for the system $\dot{\underline{x}} = f(\underline{x})$.

The following results hold:

(1) For any V_1, $V_2 \in L_{\underline{f}}(W)$, and $\alpha > 0$

(i) $V_1 + V_2 \in L_{\underline{f}}(W)$

(ii) $\alpha V_1 \in L_{\underline{f}}(w)$

(iii) $V_1 V_2 \in L_{\underline{f}}(W)$.

(2) If $V \in L_f(W)$, then $S_\varepsilon(V) \cap L_{\underline{f}}(W) \neq \phi$ for any $\varepsilon > 0$. Here

$$S_\varepsilon(V) = \{V^* / \sup_{\underline{x}} |V(\underline{x}) - V^*(\underline{x})| < \varepsilon\}$$

(3) $L_f(W)$ is a convex topological semi-ring of the ring of continuous functions on W into R.

An attempt was made to investigate the structure of $L_{\underline{f}}(W)$ and $L_V(W)$ using topology, algebra, and functional analysis but no substantive results were obtained.

The main body of this part of the report consists of our efforts to uncover some structure of a more specific class of differential equations. Specifically, we wished to classify in a meaningful way those systems of autonomous differential equations which admit of a quadratic form as a Liapunov function for the determination of the stability or instability of the origin.

We considered the system

(a) $\dot{\underline{x}} = \underline{f}(\underline{x})$; $f(0) = 0$,

where $\underline{f}(\underline{x})$ is an analytic function of $\underline{x}$.

It is well known that if (a) has the form

(b) $\dot{x} = A\underline{x} + \underline{F}(\underline{x})$,

where A is a constant nxn matrix and $\underline{F}$ is analytic with no terms of degree less than two, then (a) admits of a quadratic form as a Liapunov function if the eigenvalues of A either all have negative real parts or at least one eigenvalue has a positive real part. We conjectured that if the matrix A has no eigenvalue with a positive real part, and also no eigenvalue of multiplicity greater than one with zero real part, then (b) admits of a quadratic Liapunov function. This conjecture turned out to be false as the following example shows.

$$\dot{x} = y$$

(c)
$$\dot{y} = -y + x^4 .$$

For the system (c)

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

with eigenvalues $\lambda_1 = 0$, $\lambda_2 = -1$. However, for any quadratic form

$$V = \frac{1}{2} \, (ax^2 + 2bxy + cy^2)$$

we get

$$\dot{V} = (b-c)\, y^2 + (a-b)\, xy + bx^5 + cyx^4 .$$

Along the line $y = 0$, $\dot{V} = bx^5$ which may take on both positive and negative values in any neighborhood of the origin.

On the other hand, one can produce any number of examples of equations which admit of a quadratic Liapunov function (or, in fact, of any given Liapunov function) by the following device.

Let $V(\underline{x})$ be a given positive definite function, and let $P(\underline{x})$ be any sign constant scalar function. Then if $\underline{G}(\underline{x})$ is such that $\underline{G}(\underline{x}) \cdot \nabla V(\underline{x}) = 0$ the equation

(d) $\dot{x} = P(\underline{x}) \, \nabla V + \underline{G}(\underline{x})$

has $V(\underline{x})$ as a Liapunov function. The characterization given in (d) is, of course, not in a useful form except to produce examples.

The next section of this report describes our efforts to broaden the scope of the problem in order to achieve meaningful results.

Equations with homogeneous right hand sides

1. Introduction.

We devoted considerable effort to systems of the form

(1) $$\dot{x}_i = b_i(\underline{x});\ i = 1,\ldots,n,$$

where $b_1, b_2, \ldots, b_n$ are homogeneous polynomials as a first step toward finding conditions on the more general system

(2) $$\dot{x}_i = f_i(\underline{x});\ i = 1,\ldots,n$$

which imply the existence of a homogeneous polynomial Liapunov function.

Suppose the right side of (2) is analytic in $(x_1,\ldots,x_n)$. Then

$$\dot{x}_i = \sum_{j=1}^{\infty} f_{ij}(\underline{x});\ i = 1,\ldots,n,$$

where $f_{ij}(\underline{x})$ is a homogeneous polynomial of degree j.

If $f_{i1} \not\equiv 0$ for some i, then (2) has a linear part; this is the classical situation studied by Liapunov. Suppose $k > 1$ is the smallest integer for which $f_{ik} \not\equiv 0$ for some i. If k is even, (2) is unstable. [1;p96]. Hence we assume that $k = 2m + 1$, in which case (2) can be written

(3) $$\dot{x}_i = b_i(\underline{x}) + g_i(\underline{x});\ i = 1,\ldots,n,$$

where b_i is a homogeneous polynomial of degree $2m + 1$, and g_i contains only higher degree terms. If $V(\underline{x})$ is a Liapunov function for (1) (V positive definite, and $\dot{V} < 0$) it is also a Liapunov function for (3); hence our interest in (1).

It follows immediately from homogeneity [1;p97] that if (1) is asymptotically stable, it is asymptotically stable in the large; thus we shall always refer merely to asymptotic stability.

As a starting point for the investigation of the existence of homogeneous Liapunov functions for (1) we considered generalizing the following theorem of Liapunov [2]:

Theorem 1. If the eigenvalues of the matrix $A = (a_{ij})_{i,j=1}^{n}$ have negative real parts and if U is a negative definite homogeneous form of degree 2k, then there exists a positive definite homogeneous form Q of degree 2k such that

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right) \frac{\partial Q}{\partial x_i} = U. \tag{4}$$

This theorem guarantees the existence of Liapunov functions (and asymptotic stability) for the linear system

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j \; ; \; i=1,\ldots,n. \tag{5}$$

An analogue for the nonlinear system (1) would give conditions on the functions $b_1,\ldots,b_n$ such that, for a given negative definite form U of degree 2k + 2m, there is a positive definite homogeneous form Q of degree 2k that satisfies

$$\sum_{i=1}^{n} b_i(\underline{x}) \frac{\partial Q}{\partial x_i} = U(\underline{x}) \tag{6}$$

This question, however, is not well posed unless we restrict U; if V is of degree 2m + 2k it has

$$\binom{2m + 2k + n - 1}{n - 1}$$

coefficients, while, if Q is of degree 2k it has only

$$\binom{2k+n-1}{n-1}$$

coefficients. Hence for $m > 0$, the dimension of the space of homogeneous polynomials which can be written in the form of the left side of (6) is lower than the dimension of the space of homogeneous U's of degree $2m + 2k$. Thus it is necessary to add the explicit assumption that U lies in the lower dimensional subspace of homogeneous polynomials which can be written in the form (6) for any homogeneous Q of degree 2k.

2. The eigen-polynomial problem.

The hypothesis of theorem 1 on the eigenvalues of A is necessary and sufficient for the origin to be asymptotically stable for (5). The classical notion of eigenvalue can be generalized for (1) by calling a homogeneous polynomial $\lambda(\underline{x})$ of degree 2m an eigen-polynomial of the set $\{b_1,\ldots,b_n\}$ of index 2k if there exists a non-trivial homogeneous polynomial $P(\underline{x})$ of degree $2k + 1$ such that

$$(7) \qquad \sum_{i=1}^{n} b_i(\underline{x}) \frac{\partial P}{\partial x_i} \equiv \lambda(\underline{x})P(\underline{x}).$$

We call P an eigen-form of $\{b_1,\ldots,b_n\}$. (It is not necessary for $\{b_1,\ldots,b_n\}$ and P to be odd and λ even to make good sense algebraically, but we restrict our attention to this case because it is the only one of interest in stability theory.)

This is a generalization of the classical eigenvalue problem: if $k = m = 0$, λ reduces to a constant, and

$$b_i(\underline{x}) = \sum_{j=1}^{n} b_{ij}\, x_j$$

$$P(\underline{x}) = \sum_{j=1}^{n} \alpha_j\, x_j,$$

where b_{ij} and α_j are constants. Now (7) reduces to

$$\sum_{i=1}^{n} \alpha_i \Big(\sum_{j=1}^{n} b_{ij}\, x_j\Big) \equiv \lambda \sum_{j=1}^{n} \alpha_j\, x_j.$$

Thus λ is an eigenvalue of $B = (b_{ij})_{i,j=1}^{n}$ and $A = (\alpha_1, \ldots \alpha_n)^T$ an eigenvector of B^T.

The relevance of the above discussion to systems of the form (1) is as follows:

If $\underline{x}$ (t) is a trajectory of (1), relation (7) implies that

$$\frac{d}{dt} P(\underline{x}(t)) = \lambda(\underline{x}(t))\, P(\underline{x}(t));$$

hence

$$P(\underline{x}(t)) = P(\underline{x}(0)) \exp\Big[\int_0^t \lambda(x(t)\, ds\Big].$$

So that if λ is negative definite, $P(\underline{x})$ approaches zero along any trajectory of (1).

As far as we know there is no well developed theory of eigen-polynomials and eigen-forms as defined here. We have confined our attention to the case $k = 0$, for which (7) takes the form

(8) $$\sum_{i=1}^{n} \alpha_i b_i(\underline{x}) \equiv \lambda(\underline{x})\Big(\sum_{i=1}^{n} \alpha_i\, x_i\Big),$$

with constants $\alpha_1,\ldots,\alpha_n$. The eigen-form

$$y = \sum_{i=1}^{n} \alpha_i \, x_i$$

satisfies the equation

$$\dot{y} = \lambda(\underline{x}(t)) \, y$$

along a trajectory of (1).

Suppose now that there are n linearly independent eigen-forms

$$y_i = \sum_{j=1}^{n} \alpha_{ij} \, x_j \; ; \; i = 1,\ldots,n$$

(which implies that the matrix $A = (\alpha_{ij})$ is non-singular) with associated eigen-polynomials $\lambda_1, \lambda_2,\ldots,\lambda_n$, then (1) is equivalent to

$$\dot{y}_i = \lambda_i \, [A^{-1}\underline{y}(t)] \, y_i \; ; \; i = 1,\ldots,n,$$

where $\underline{y}(t) = (y_1(t),\ldots y_n(t))$.

This is the analogue of the diagonal form for (5). The question of the existence of n linearly independent eigen-forms, or more appropriately, the determination of all canonical forms for (1), requires an analogue of the Jordan canonical form for matrices. We have not attempted to develop such a theory because we thought it wise to see first if we could settle the question of stability for the diagonal system. Our efforts in this direction are reported in sections 4 through 8. However, we did obtain an algebraic formulation of the eigen-polynomial problem for k=0, which we report here for completeness.

a) $\lambda_1\ (x_1,0) < 0;\ \lambda_2\ (0,x_2) < 0$

b) at every $(x_1,x_2) \neq (0,0)$ at least one of $\lambda_1\ (x_1,x_2)$ and $\lambda_2\ (x_1,x_2)$ is negative.

<u>Proof:</u> The proof makes use of the basic relation (15) which holds along any trajectory of (14).

$$\text{(15)} \qquad \frac{d}{dt}\,(\lambda_1 - \lambda_2) = (\lambda_1 - \lambda_2)\left(\frac{\partial\lambda_1}{\partial x_1}\ x_1 + \frac{\partial\lambda_2}{\partial x_2}\ x_2\right)$$

This relation is obtained by applying Euler's identity for homogeneous functions:

$$\frac{\partial\lambda}{\partial x_1}\,x_1 + \frac{\partial\lambda}{\partial x_2}\,x_2 = 2m\lambda$$

to the derivative of $\lambda_1 - \lambda_2$ along a trajectory. It is clear from (15) that a line along which $\lambda_1 = \lambda_2$ is a trajectory of (14). Moreover (15) shows that $\lambda_1 - \lambda_2$ is equal to its initial value times an exponential, and hence $\lambda_1 - \lambda_2$ does not change sign along any trajectory.

We prove first that every trajectory of (14) approaches the origin (quasi-asymptotic stability). It follows immediately from the representation (13) and hypothesis (a) that any trajectory which starts on one of the coordinate axes must approach the origin, and in fact, monotonically. Now assume that $x_1(0) \neq 0$ and $x_2(0) \neq 0$. Since $\lambda_2 - \lambda_1$ does not change sign along a trajectory we may assume without loss of generality that

$\lambda_2 - \lambda_1\ \geq 0.$

From (13)

$$\frac{x_2(t)}{x_1(t)} = \frac{x_2(0)}{x_1(0)}\ \exp\left[\int_0^t \{\lambda_2(\underline{x}\ (s)) - \lambda_1\ (\underline{x}\ (s))\}\ ds\right],$$

so that $\frac{x_2}{x_1}$ approaches a limit ℓ (possibly infinite, but not zero). Define

$$\xi_1 = \frac{x_1}{r}\ ; \quad \xi_2 = \frac{x_2}{r}\ ; \ r = \sqrt{x_1^{\,2} + x_2^{\,2}}\ .$$

Then ξ_1 and ξ_2 approach limits:

(16)
$$\lim_{t \to \infty} \xi_1 = \bar{\xi}_1 = \frac{1}{\sqrt{1+\ell^2}}$$

$$\lim_{t \to \infty} \xi_2 = \bar{\xi}_2 = \frac{\ell}{\sqrt{1+\ell^2}}\ .$$

Now, $\lambda_2\ (\bar{\xi}_1,\bar{\xi}_2) - \lambda_1\ (\xi_1,\xi_2) \geq 0$ and it follows from hypothesis (b) that $\lambda_1\ (\bar{\xi}_1,\bar{\xi}_2) = -\sigma$ where $\sigma > 0$. Thus there exists a T such that $t \geq T$ implies.

(17)
$$\lambda_1(x_1,x_2) < -\frac{\sigma}{2}\ r^{2m}$$

From (13)

$$|x_1(t)| = |x_1(T)| \exp\left[\int_T^t \lambda,\ (\underline{x}(s))\ ds\right]$$

$$\leq |x_1(T)| \exp\left[-\frac{\sigma}{2}\int_T^t (r(s))^{2m}\ ds\right]$$

$$\leq |x_1(T)| \exp\left[-\frac{\sigma}{2}\int_T^t (x_1(s))^{2m}\ ds\right]$$

From (17), $|x_1|$ is decreasing for $t \geq T$; therefore it has a limit $|\bar{x}_1|$. The assumption that $|\bar{x}_1| \neq 0$, along with the last inequality of (18), leads to a contradiction. Hence $\lim_{t \to \infty} x_1(t) = 0$. Since $\xi_1 \neq 0$ (because $\ell \neq 0$ in (16) and $r = \frac{x_1}{\xi_1}$, $\lim_{t \to 0} r\ (t) = 0$, which completes the first part of the proof.

The above proof shows that once a trajectory of (14) enters a portion of the plane in which both λ_1 and λ_2 are negative it remains there and $r(t) \to 0$ monotonically. Thus, starting at a point where $r(x_1(0), x_2(0)) = r_o$ and say $\lambda_1(x_1(0),x_2(0)) < 0$ the maximum value r can assume is less than $x_1^2(0) + \eta^2$ where η is the number of smallest absolute value such that $\lambda_2(x_1(0),\eta) = 0$. Hypothesis (a) now assures us that $x_1^2(0) + \eta^2$ can be made as small as desired by choosing r_o sufficiently small. This completes the proof of the theorem.

5. A conjecture on the diagonal case for general n.

We sought to generalize theorem 2 with the following conjecture:

Conjecture 1. The system

$$\dot{x}_i = \lambda_i(\underline{x})\, x_i \; ; \; i = 1, 2, \ldots, n, \tag{19}$$

where $\lambda_1, \ldots, \lambda_n$ are real homogeneous polynomials of degree 2m, is asymptotically stable if and only if for any partition $[i_1, \ldots, i_k]$, $[i_{k+1}, \ldots, i_n]$ of $N = [1, \ldots, n]$ into disjoint subsets, (either of which may be empty) at least one of the polynomials $\lambda_{i_1}, \ldots, \lambda_{i_k}$ is negative at any non-zero point on the hyperplane $x_{i_{k+1}} = x_{i_{k+2}} = \ldots = x_{i_n} = 0$.

We expended considerable effort to prove this conjecture, but without success. At this point we can only surmise that either the hypothesis is too weak, or the theorem is very deep. The crucial defect in our attempts to prove it seems to be the lack of a generalization of the relation (15) which implied that $\lambda_1 - \lambda_2$ is sign constant along a trajectory.

We have obtained some partial results which are presented here.

In the following discussion we may assume that no trajectory intersects a coordinate hyper-plane, for if $x_n(t_o) = 0$ for some $t_o \geq 0$, then $x_n(t) \equiv 0$ for $t \geq t_o$, and (19) reduces to a system of n - 1 equations which satisfy the hyperthesis of the conjecture with n replaced by n - 1.

Lemma 1. Let the system (19) satisfy the hypothesis of conjecture 1, and let $\underline{x}(t)$ be a trajectory along which one of the λ_i dominates the others for sufficiently large t. Then $\lim_{t\to\infty} \underline{x}(t) = 0$.

Proof: Without loss of generality, assume that

$$\lambda_n(\underline{x}(t)) - \lambda_i(\underline{x}(t)) \geq 0 \; ; \; t \geq T, \; i = 1,\ldots,n-1.$$

From (13), if $t \geq T$

$$\frac{x_n(t)}{x_i(t)} = \frac{x_n(T)}{x_i(T)} \exp\left[\int_T^t \{\lambda_n(\underline{x}(s)) - \lambda_i(\underline{x}(s))\} ds\right] ;$$

hence $\frac{x_n}{x_i}$, being non-decreasing for $t \geq T$, approaches a limit ℓ_i (possibly infinite, but not zero). Let

$$r = (x_1^2 + \ldots + x_n^2)^{1/2}$$

and

$$\xi_i = \frac{x_i}{r} \; ; \; i = 1,\ldots,n.$$

Then

$$\lim_{t\to\infty} \xi_i(t) = \overline{\xi}_i = \frac{1}{i}\left(\frac{1}{\ell_1^2} + \frac{1}{\ell_2^2} + \ldots + \frac{1}{\ell_{n-1}^2} + 1\right)^{-1/2},$$

where $\ell_n = 1$. Suppose $\xi_i \neq 0$ for i in the set $[i_1,\ldots,i_k]$ and $\xi_i = 0$ for i in $[i_{k+1},\ldots,i_n]$. The first set is not empty (since $\xi_1^2 + \ldots + \xi_n^2 = 1$), and by hypothesis contains an integer j such that $\lambda_j(\overline{\xi}_1,\ldots,\overline{\xi}_n) = -\sigma$ ($\sigma > 0$). As in (18), there is a $T_1 \geq T$ such that for $t \geq T_1$,

$$(20) \qquad |x_j(t)| \leq |x_j(T_1)| \exp\left[-\frac{\sigma}{2}\int_{T_1}^{t} x_j(s))^{2m} ds\right] .$$

$\lim\limits_{t\to\infty} |x_j(t)|$ exists because $|x_j(t)|$ is decreasing for $t \geq T_1$, and the assumption that this limit is not zero leads to a contradiction via (20). Thus $\lim\limits_{t\to\infty} x_j(t) = 0$, and since $r = \dfrac{x_j}{\xi_j}$ and $\overline{\xi_j} \neq 0$, it follows that $\lim\limits_{t\to\infty} r(t) = 0$. This completes the proof of the lemma.

Corollary 1. If $\lambda_1,\ldots,\lambda_n$ satisfy the hypothesis of conjecture 1 and $\lambda_n - \lambda_i$ is positive semi-definite for $1 \leq i \leq n-1$, then the system (19) is asymptotically stable.

Corollary 2. If $\lambda_1,\ldots, \lambda_n$ satisfy the hypothesis of conjecture 1 and if $\underline{x}(t)$ is a trajectory for which $\xi_1,\ldots,\xi_n$ approach limits, then $\lim\limits_{t\to\infty} \underline{x}(t) = 0$.

If $\lim\limits_{t\to\infty} \xi = \overline{\xi}$ exists for $i = 1,\ldots,n$, Corollary 2 implies that

$$\lambda_i(\overline{\xi}_1,\ldots,\overline{\xi}_n) \leq 0 \; ; \; i = 1,\ldots,n.$$

Thus the hypothesis of conjecture 1 and the assumption that $\xi_1,\ldots,\xi_n$ have limits are incompatable, unless the former implies the existence of at least one point where $\lambda_1,\ldots,\lambda_n$ are all non-positive. This is true for $n = 2$, but we have not been able to show it for $n > 2$. In conjecture 2 (Section 8) we strengthen our hypothesis so as to remove this difficulty. However, the stronger hypothesis brought us no closer to a solution.

6. An estimate.

In trying to translate the hypothesis of conjecture 1 into usable form, we discovered the following lemma.

Lemma 2. Let $\lambda_1,\ldots,\lambda_n$ satisfy the hypothesis of conjecture 1; let $S = [i_1, i_2, \ldots, i_k]$ and $N-S = [i_{k+1}, \ldots, i_n]$; let

$$r_S^2(\underline{x}) = \sum_{i \varepsilon S} x_i^2$$

and

$$r_{N-S}^2(\underline{x}) = \sum_{i \varepsilon N-S} x_i^2 .$$

For each $S \neq N$, there exists a constant $A(S)$ such that if

(21) $$\lambda_i(x) \geq 0 \quad ; i \varepsilon S$$

then

(22) $$r_S^2(\underline{x}) \leq A(S)\, r_{N-S}^2(\underline{x}) .$$

Proof. If (22) is false for some set S, then for each integer j there exists a point $\underline{x}^{(j)}$ which satisfies (21) and for which

$$r_S^2(\underline{x}^{(j)}) > j\, r_{N-S}^2(\underline{x}^{(j)}) .$$

From the homogeneity of λ_i, $\underline{x}^{(j)}$ can be normalized so that

(23) $$r_S^2(\underline{x}^{(j)}) = 1 .$$

Then

(24) $$r_{N-S}^2(\underline{x}^{(j)}) < \frac{1}{j}$$

and

$$r^2(\underline{x}^{(j)}) = r_S^2(\underline{x}^{(j)}) + r_{N-S}^2(\underline{x}^{(j)}) < 1 + \frac{1}{j} ;$$

hence $\{\underline{x}^{(j)}\}$ is a bounded sequence and has a limit point $\hat{\underline{x}}$. From (23) $r_S^2(\hat{\underline{x}}) = 1$, and from (24), $r_{N-S}^2(\hat{\underline{x}}) = 0$. Therefore $\hat{\underline{x}}$ is a non-zero point on the hyperplane defined by $x_i = 0$ for $i \varepsilon N-S$. From the hypothesis of conjecture 1, $\lambda_i(\hat{\underline{x}}) < 0$ for at least one $i \varepsilon S$. This is a contradiction; the definition of $\hat{\underline{x}}$ and the continuity of λ_i imply that $\lambda_i(\hat{\underline{x}}) \geq 0$. This proves lemma 2.

Lemma 2 at first seemed to be quite promising. Taking S to be the set of all i for which $\lambda_i \geq 0$ at any given point, and N-S the non-empty set of all i for which $\lambda_i < 0$, it can be seen that (22) gives an upper bound for the sum of the squares of those coordinates whose absolute values are increasing, in terms of a similar sum of terms whose absolute values are decreasing. In fact (22) can be replaced by

$$r^2(\underline{x}) \leq (1 + A(S))\, r^2_{N-S}(\underline{x}) .$$

Over any period of time during which no λ_i changes sign, this gives a decreasing upper bound for the distance from the trajectory to the origin.

We have not been able to use this seemingly favorable state of affairs to prove the conjecture.

7. Two theorems.

In attempting an induction proof of conjecture 1 we were led to write (19) in the form

$$\text{(25)} \quad \begin{aligned} \dot{x}_i &= (\lambda_i(x_1,\ldots,x_{n-1},0) + x_n\, \phi_i(\underline{x}))\, x_i \ ; \ i = 1,\ldots,n-1, \\ \dot{x}_n &= \lambda_n(\underline{x})\, x_i \ , \end{aligned}$$

where ϕ_i is homogeneous of degree 2m-1.

<u>Theorem 3.</u> Let the system

$$\text{(26)} \quad \dot{y}_i = \lambda_i(y_1,\ldots,y_{n-1},0)\, y_i \ ; \ i = 1,\ldots,n-1,$$

be asymptotically stable. Then there is a constant $0 < \delta < 1$ such that if

$|\xi_n(t)| \leq \delta$ along a trajectory of (25), then $\lim_{t\to\infty} \underline{x}(t) = 0$.

Proof. We use a device due to Zubov [1]. The asymptotic stability of (26) implies that any trajectory $\underline{y}(t)$ of (26) satisfies

(27) $$|\underline{y}(t)| = 0(t^{-1/2m}) \quad \text{as } t\to\infty \quad ,$$

(Krasonskii [3]).

Zubov showed that this guarantees the existence of a Liapunov function $V(y_1,\ldots,y_{n-1})$, continuously differentiable with respect to $(y_1,\ldots,y_{n-1})$, with derivative along a trajectory of (26) given by

(28) $$\frac{dV}{dt} = \sum_{i=1}^{n-1} \frac{\partial V}{\partial y_i} \lambda_i \,(y_1,\ldots,y_{n-1},0) = -\,(y_1^{\,2} + \ldots + y_{n-1}^{\,2})^{m+k}$$

where k is a sufficiently large positive integer.

Explicitly,

$$V\,(u) = \int_0^\infty |\,\underline{y}\,(t,\underline{u})\,|^{2m+2k}\, dt \;,$$

where $\underline{y}$ (t, u) is the solution of (26) for which $\underline{y}\,(o,\,\underline{u}) = \underline{u} = (u_1,\ldots,u_{n-1})$, and $|\quad|$ is the Eucledian norm. The existence of the integral follows from (27). V can be shown to be positive homogeneous of order 2k.

Suppose $x_1(t),\ldots,x_{n-1}(t)$ are taken to be the first n-1 components of a trajectory $\underline{x}(t)$ of (25); let

$$\psi(t) = V(x_1(t),\ldots,x_{n-1}(t)),$$

$$\rho^2 = x_1^{\,2} + \ldots + x_{n-1}^{\,2}$$

and

$$r^2 = x_1^{\,2} + \ldots + x_n^{\,2} = \rho^2 + x_n^{\,2}$$

From (25) and (28),

$$\dot{\psi} = - \rho^{2m+2k} + x_n \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} x_i \phi_i .$$

There is a constant $M > 0$ such that the second term on the right is dominated by

$$M \mid \xi_n \mid (1 - \xi_n{}^2)^{-(m+k)} \rho^{2m+2k} \quad ;$$

hence

$$\dot{\psi} < - \rho^{2m+2k} (1 - M \mid \xi_n \mid (1 - \xi_n{}^2)^{m+k}) \quad .$$

choose δ so that $\mid \xi_n \mid \leq \delta$ implies that

$$M \mid \xi_n \mid (1 - \xi_n{}^2) \leq 1 - \alpha \qquad (0 < \alpha < 1, \text{ arbitrary}) ;$$

then if $\mid \xi_n \mid \leq \delta$ for all t,

$$\psi(t) < \phi(0) - \alpha \int_0^t \rho(s)^{2m+2k} \, ds \tag{29}$$

and ψ is strictly decreasing. Let $\lim_{t\to\infty} \psi(t) = \bar{\psi}$; we want to show that $\bar{\psi} = 0$. If $\psi \neq 0$, then ρ is bounded away from zero, because for all t,

$$0 < \bar{\psi} \leq \psi(t) = V(x_1(t),\ldots,x_{n-1}(t)) \leq \mu \rho^{2k} (t),$$

where $\mu = \min V(x_1,\ldots,x_{n-1})$ on $\rho = 1$. Substituting in (29) yields

$$\psi(t) < \psi(0) - \alpha\left(\frac{\bar{\psi}}{\mu}\right)^{1+\frac{m}{k}} \tag{30}$$

a contradiction, since it implies that $\psi(t) < 0$ eventually. Since $\bar{\psi} = 0$, $\rho = 0$ because V is positive definite; therefore $\overline{\lim_{t\to\infty}} |x_n| = 0$ also (if not, then $\overline{\lim_{t\to\infty}} \mid \xi_n \mid = 1$, violating the assumption that $\mid \xi_n \mid \leq \delta < 1$ for all t). This completes the proof of theorem 3.

The hypotheses of theorem 3 can be weakened to require that

$|\xi_n| \leq \delta$ for all sufficiently large t.

Theorem 4. Let the system (26) be asymptotically stable and $\underline{x}(t)$ be a trajectory of (25) for which $\lim_{t\to\infty} x_n(t) = 0$. Then $\lim_{t\to\infty} x(t) = 0$.

Proof. Using the notation of theorem 3, there exists a δ in (0,1) such that either $|\xi_n| > \delta$ or

$$(31) \qquad \mathring{\psi} < -\alpha\,\rho^{2m+2k}$$

For any t such that (31) does not hold, $|\xi_n| > \delta$ and

$$(32) \qquad \psi(t) \leq \eta\,\rho^{2k} \leq \eta r^{2k} \leq \eta\left(\frac{|x_n|}{\delta}\right)^{2k} ,$$

where $\eta = \max V(x_1,\ldots,x_{n-1})$ on the sphere $\rho = 1$. If either (31) or (32) holds for all sufficiently large t, we are finished; if this is not the case, there exists a sequence $\tau_o < \tau_1 < \tau_2 < \ldots$ approaching infinity such that (32) holds for $\tau_{2j} \leq t \leq \tau_{2j+1}$, and (31) holds for

$\tau_{2j+1} \leq t \leq \tau_{2j+2}$. Then

$$\psi(t) \leq \eta\left(\frac{|x_n(t)|}{\delta}\right)^{2k} \quad ; \quad \tau_{2j} \leq t \leq \tau_{2j+1}$$

and

$$\psi(t) \leq \psi(\tau_{2j+1}) \leq \eta\left(\frac{|x_n(\tau_{2j+1})|}{\delta}\right)^{2k} \quad ; \quad \tau_{2j+1} \leq t \leq \tau_{2j+2} .$$

Since x_n approaches zero, so does ψ, and therefore ρ approaches zero.

This completes the proof of theorem 4.

Corollary. If the system (26) is asymptotically stable and λ_n is negative definite, then the origin is asymptotically stable for the system (25).

8. A stronger hypothesis.

The hypothesis of the following conjecture is stronger than that of conjecture 1; it has the advantage of guaranteeing the existence of a region where $\lambda_1,\ldots,\lambda_n$ are all negative.

Conjecture 2. Let $A_1, A_2,\ldots,A_n$ be a set of constants such that $A_i \geq 0$ and $A_1^2 +\ldots+A_n^2 < 1$, and let $\lambda_i < 0$ whenever $|\xi_i| > A_i$. Then the system

$$\dot{x}_i = \lambda_i(\underline{x})\, x_i \quad ; \quad i = 1,\ldots,n$$

is asymptotically stable.

We have not been able to obtain sharper results with this stronger hypothesis.

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