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# Properties of unilevel block circulants

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**Abstract**

Let $\mathcal{A}=\{A_0,A_1,\dots,A_{k-1}\}\subset\mathbb{C}^{d_1\times d_2}$, $\zeta=e^{-2\pi i/k}$, $F_\ell=\sum_{m=0}^{k-1}\zeta^{\ell m}A_m$, $0\le\ell\le k-1$, and $\mathcal{F}_A=\bigoplus_{\ell=0}^{k-1}F_\ell$. All operations in indices are modulo $k$. It is well known that if $d_1=d_2=1$ then $[A_{s-r}]_{r,s=0}^{k-1}=\Phi\mathcal{F}_A\Phi^*$, where $\Phi=\frac{1}{\sqrt{k}}[\zeta^{\ell m}]_{\ell,m=0}^{k-1}$. However, to our knowledge it has not been emphasized that $\mathcal{F}_A$ plays a fundamental role in connection with all the matrices $[A_{s-\alpha r}]_{r,s=0}^{k-1}$, $0\le\alpha\le k-1$, with $d_1$, $d_2$ arbitrary. We begin by adapting a theorem of Ablow and Brenner with $d_1=d_2=1$ to the case where $d_1$ and $d_2$ are arbitrary. We show that $A=[A_{s-\alpha r}]_{r,s=0}^{k-1}$ if and only if $A=U_\alpha\mathcal{F}_AP^*$ where $U_\alpha$ and $P$ are related to $\Phi$, $P$ is unitary, and $U_\alpha$ is invertible (in fact, unitary) if and only if $\gcd(\alpha,k)=1$, in which case we say that $A$ is a proper circulant. We prove the following for proper circulants $A=[A_{s-\alpha r}]_{r,s=0}^{k-1}$: (i) $A^\dagger=[B_{r-\alpha s}]_{r,s=0}^{k-1}$ with $B_m=\frac{1}{k}\sum_{\ell=0}^{k-1}\zeta^{\ell m}F_\ell^\dagger$, $0\le m\le k-1$. (ii) Solving $Az=w$ reduces to solving $F_\ell u_\ell=v_{\alpha\ell}$, $0\le\ell\le k-1$, where $v_0$, $v_1$, …, $v_{k-1}$ depend only on $w$. (iii) A singular value decomposition of $A$ can be obtained from singular value decompositions of $F_0$, $F_1$, …, $F_{k-1}$. (iv) The least squares problem for $A$ reduces to independent least squares problems for $F_0$, $F_1$, …, $F_{k-1}$. (v) If $d_1=d_2=d$, the eigenvalues of $[A_{s-r}]_{r,s=0}^{k-1}$ are the eigenvalues of $F_0$, $F_1$, …, $F_{k-1}$, and the corresponding eigenvectors of $A$ are easily obtainable from those of $F_0$, $F_1$, …, $F_{k-1}$. (vi) If $d_1=d_2=d$ and $\alpha>1$ then the eigenvalue problem for $[A_{s-\alpha r}]_{r,s=0}^{k-1}$ reduces to eigenvalue problems for $d\times d$ matrices related to $F_0$, $F_1$, …, $F_{k-1}$ in a manner depending upon $\alpha$.

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## 1 Introduction

Throughout this paper $k \geq 2$, $d_1, d_2 \geq 1$ are integers, $\alpha \in \{0, 1, \ldots, k-1\}$, and

$$\mathbb{C}^{k:d_1 \times d_2} = \left\{ C = [C_{rs}]_{r,s=0}^{k-1} \,\middle|\, C_{rs} \in \mathbb{C}^{d_1 \times d_2}, 0 \leq r, s \leq k-1 \right\}.$$

All arithmetic operations in indices are modulo $k$.

We call $A = [A_{s-\alpha r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_1 \times d_2}$ an $\alpha$-circulant. We say that $A$ is a proper $\alpha$-circulant, or simply a proper circulant, if $\gcd(\alpha, k) = 1$. We will say that $A$ is a standard $\alpha$-circulant if $d_1 = d_2 = 1$ and denote it by $A = [a_{s-ar}]_{r,s=0}^{k-1}$. Of course, there is already a vast literature on standard $\alpha$-circulants. Matrices of the form

$$A = [A_{rs}]_{r,s=0}^{k-1} \quad \text{where} \quad A_{rs} = \begin{cases} A_{s-r}, & 0 \leq r \leq s \leq k-1, \\ kA_{s-r}, & 0 \leq s < r \leq k-1, \end{cases}$$

are also called $k$-circulants; see e.g., [4]. We will not consider them.

We call $[B_{r-\alpha s}]_{r,s=0}^{k-1}$ an $\alpha$-cocirculant, again proper if $\gcd(\alpha, k) = 1$. This eliminates awkward terminology such as " the conjugate transpose of the Moore-Penrose inverse of an $\alpha$-circulant matrix is an $\alpha$-circulant." The Moore-Penrose inverse of an $\alpha$-circulant is an $\alpha$-cocirculant (Theorem 4).

**Remark 1** Obviously, $B$ is an $\alpha$-cocirculant if and only if $B^*$ is an $\alpha$-circulant. Therefore any result concerning $\alpha$-circulants can be applied to $B^*$ to obtain a result concerning $B$.

**Remark 2** A proper $\alpha$-circulant $A = [A_{s-\alpha r}]_{r,s=0}^{k-1}$ is also a $\beta$-cocirculant where $\alpha\beta \equiv 1 \pmod{k}$, since

$$A_{s-\alpha r} = A_{\alpha\beta s - \alpha r} = A_{-\alpha(r-\beta s)} = B_{r-\beta s}$$

with $B_m = A_{-\alpha m}$, $0 \leq m \leq k-1$. Similarly, a proper $\beta$-cocirculant $B = [B_{r-\beta s}]_{r,s=0}^{k-1}$ is also an $\alpha$-circulant, since

$$B_{r-\beta s} = B_{\alpha\beta r - \beta s} = B_{-\beta(s-\alpha r)} = C_{s-\alpha r}$$

with $C_m = B_{-\beta m}$, $0 \leq m \leq k-1$.

Henceforth $\zeta = e^{-2\pi i/k}$,

$$E = [\delta_{\ell,m-1}]_{\ell,m=0}^{k-1}, \quad \text{and} \quad \Phi = \frac{1}{\sqrt{k}} [\zeta^{\ell m}]_{\ell,m=0}^{k-1} = \begin{bmatrix} \phi_0 & \phi_1 & \cdots & \phi_{k-1} \end{bmatrix} \tag{1}$$

(the Fourier matrix), with

$$\phi_m = \frac{1}{\sqrt{k}} \begin{bmatrix} 1 \\ \zeta^m \\ \zeta^{2m} \\ \vdots \\ \zeta^{(k-1)m} \end{bmatrix}, \quad 0 \leq m \leq k-1. \tag{2}$$

It is straightforward to verify that if indices are reduced modulo $k$ then

$$E^p \left( [g_{\ell m}]_{\ell,m=0}^{k-1} \right) E^{-q} = [g_{\ell+p,m+q}]_{\ell,m=0}^{k-1}. \tag{3}$$

Setting $p = 1$ and $q = 0$ and invoking (1) yields

$$E\Phi = \frac{1}{\sqrt{k}}[\zeta^{(\ell+1)m}]_{\ell,m=0}^{k-1} = \Phi D \quad \text{with} \quad D = \text{diag}(1, \zeta, \zeta^2, \dots, \zeta^{k-1}). \tag{4}$$

Therefore $E = \Phi D \Phi^*$.

The discrete Fourier transform (DFT) of $\{A_0, A_1, \dots, A_{k-1}\} \subset \mathbb{C}^{d_1 \times d_2}$ is $\{F_0, F_1, \dots, F_{k-1}\}$ where

$$F_\ell = \sum_{m=0}^{k-1} \zeta^{\ell m} A_m \in \mathbb{C}^{d_1 \times d_2}, \quad 0 \le \ell \le k-1. \tag{5}$$

Since $\Phi^{-1} = \Phi^*$,

$$A_m = \frac{1}{k} \sum_{\ell=0}^{k-1} \zeta^{-\ell m} F_\ell, \quad 0 \le m \le k-1. \tag{6}$$

We denote

$$\mathcal{F}_A = \bigoplus_{\ell=0}^{k-1} F_\ell \in \mathbb{C}^{k:d_1 \times d_2}. \tag{7}$$

For standard circulants (5)–(7) reduce to

$$f_\ell = \sum_{m=0}^{k-1} a_m \zeta^{\ell m}, \quad a_m = \frac{1}{k} \sum_{\ell=0}^{k-1} f_\ell \zeta^{-\ell m}, \quad \text{and} \quad \mathcal{F}_A = \text{diag}(f_0, f_1, \dots, f_{k-1}).$$

It is well known (see, e.g., [7]) that a standard 1-circulant $A = [a_{s-r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k \times k}$ can be written as

$$A = \Phi \mathcal{F}_A \Phi^* = \sum_{\ell=0}^{k-1} f_\ell \phi_\ell \phi_\ell^*.$$

However, to our knowledge it has not been emphasized that $\mathcal{F}_A$ plays a fundamental role in connection with all the standard circulants $[a_{s-\alpha r}]_{r,s=0}^{k-1}$. (See Remark 3.)

In Section 2 we reformulate a result of Ablow and Brenner [1, Theorem 2.1] for standard $\alpha$-circulants to characterize $\alpha$-circulants in $\mathbb{C}^{k:d_1 \times d_2}$. We give a different characterization in Section 3: $A = [A_{s-\alpha r}]_{r,s=0}^{k-1}$ if and only if $A = U_\alpha \mathcal{F}_A P^*$, where $U_\alpha$ and $P$ are related to the Fourier matrix, $P$ is unitary, and $U_\alpha$ is invertible (in fact, unitary) if and only if $\gcd(\alpha, k) = 1$.

Since $\mathcal{F}_A$ is independent of $\alpha$, some computational results concerning $\mathcal{F}_A$ apply simultaneously to all the proper $\alpha$-circulants $[A_{s-\alpha r}]_{r,s=0}^{k-1}$. For example, in Section 4 we show that

$$A^\dagger = [B_{r-\alpha s}]_{r,s=0}^{k-1} \quad \text{where} \quad B_m = \frac{1}{k}\sum_{\ell=0}^{k-1} \zeta^{\ell m} F_\ell^\dagger, \quad 0 \le m \le k-1.$$

We also prove the following for proper $\alpha$-circulants: (i) Solving $Az = w$ reduces to solving $F_\ell u_\ell = v_{\alpha\ell}$, $0 \le \ell \le k-1$, where $v_0$, $v_1$, ..., $v_{k-1}$ depend only on $w$. (ii) A singular value decomposition of $A$ can be obtained from singular value decompositions of $F_0$, $F_1$, ..., $F_{k-1}$. (iii) The least squares problem for $A$ reduces to independent least squares problems for $F_0$, $F_1$, ..., $F_{k-1}$. (iv) If $d_1 = d_2 = d$, the eigenvalues of $[A_{s-r}]_{r,s=0}^{k-1}$ are the eigenvalues of $F_0$, $F_1$, ..., $F_{k-1}$, and the corresponding eigenvectors of $A$ are easily obtainable from those of $F_0$, $F_1$, ..., $F_{k-1}$. (v) If $d_1 = d_2 = d$ and $\alpha > 1$, the eigenvalue problem for $[A_{s-\alpha r}]_{r,s=0}^{k-1}$ reduces to eigenvalue problems for $d \times d$ matrices related to $F_0$, $F_1$, ..., $F_{k-1}$ in a manner depending upon $\alpha$.

Block circulant 1-matrices $[A_{s-r}]_{r,s=0}^{k-1}$ have applications in preconditioning of block Toeplitz matrices; see, e.g. [8, 9].

## 2 The Ablow–Brenner theorem revisited

Recall that $E$ and $\Phi$ are defined in (1) and (2). Let

$$R = E \otimes I_{d_1}, \quad P_m = \phi_m \otimes I_{d_1}, \quad 0 \le m \le k-1, \tag{8}$$

$$S = E \otimes I_{d_2}, \quad Q_m = \phi_m \otimes I_{d_2}, \quad 0 \le m \le k-1, \tag{9}$$

$$P = \left[\begin{array}{ccc} P_0 & P_1 & \cdots P_{k-1} \end{array}\right], \quad Q = \left[\begin{array}{ccc} Q_0 & Q_1 & \cdots Q_{k-1} \end{array}\right], \tag{10}$$

and

$$U_\alpha = \left[\begin{array}{ccccc} P_0 & P_\alpha & P_{2\alpha} & \cdots & P_{(k-1)\alpha} \end{array}\right]. \tag{11}$$

Since

$$P_\ell^* P_m = \delta_{\ell m} I_{d_1} \quad \text{and} \quad Q_\ell^* Q_m = \delta_{\ell m} I_{d_2}, \quad 0 \le \ell, m \le k-1,$$

$P$ and $Q$ are unitary, while $U_\alpha$ is unitary if $\gcd(\alpha, k) = 1$; however, if $\gcd(\alpha, k) = q > 1$ and $p = k/q$ then

$$U_\alpha = [\underbrace{P_0\, P_\alpha\, \cdots\, P_{(p-1)\alpha} \cdots P_0\, P_\alpha\, \cdots\, P_{(p-1)\alpha}}_{q}]$$

(i.e., the first $p$ block columns are repeated $q$ times) is not invertible. From (4) and (8)–(11),

$$RP_\ell = \zeta^\ell P_\ell \quad \text{and} \quad SQ_\ell = \zeta^\ell Q_\ell, \quad 0 \le \ell \le k-1. \tag{12}$$

Ablow and Brenner [1, Theorem 2.1] showed that $A \in \mathbb{C}^{k\times k}$ is a standard $\alpha$-circulant if and only if $EAE^{-\alpha} = A$. We need the following adaptation of this result.

**Theorem 1** *If* $A = [G_{rs}]_{r,s=0}^{k-1}$ *with* $G_{rs} \in \mathbb{C}^{d_1 \times d_2}$, *then* $RAS^{-\alpha} = A$ *if and only if* $A$ *is an* $\alpha$*-circulant; more precisely, if and only if*

$$G_{rs} = A_{s-\alpha r}, \quad 0 \le r, s \le k-1, \tag{13}$$

*with*

$$A_s = G_{0s}, \quad 0 \le s \le k-1. \tag{14}$$

Proof. From (3), (8), and (9), $RAS^{-\alpha} = [G_{r+1,s+\alpha}]_{r,s=0}^{k-1}$. Therefore we must show that (13) is equivalent to

$$G_{r+1,s+\alpha} = G_{rs}, \quad 0 \le r, s \le k-1. \tag{15}$$

If (13) holds, then

$$G_{r+1,s+\alpha} = A_{(s+\alpha)-(r+1)\alpha} = A_{s-\alpha r} = G_{rs}, \quad 0 \le r, s \le k-1.$$

For the converse we must show that (14) and (15) imply (13). We prove this by finite induction on $r$. From (14),

$$G_{rs} = A_{s-\alpha r}, \quad 0 \le s \le k-1, \tag{16}$$

if $r = 0$. Suppose (16) is true for some $r \in \{0, \dots, k-2\}$. Replacing $s$ by $s - \alpha$ in (15) and (16) yields

$$G_{r+1,s} = G_{r,s-\alpha}, \quad 0 \le r, s \le k-1,$$

and

$$G_{r,s-\alpha} = A_{s-\alpha(r+1)}, \quad 0 \le s \le k-1.$$

Therefore

$$G_{r+1,s} = A_{s-\alpha(r+1)}, \quad 0 \le s \le k-1,$$

which completes the induction. $\square$

Theorem 1 with $A = B^*$ yields the following corollary.

**Corollary 1** *If* $B \in \mathbb{C}^{k:d_2 \times d_1}$ *then* $B$ *is an* $\alpha$*-cocirculant if and only if* $S^\alpha B R^{-1} = B$.

The following corollary extends [10, Corollary 1].

**Corollary 2** (i) *If* $A = [A_{s-\alpha r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_1 \times d_2}$ *and* $B = [B_{r-\alpha s}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_2 \times d_1}$, *then* $AB = [C_{s-r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_1 \times d_1}$ *with* $C_m = \sum_{\ell=0}^{k-1} A_\ell B_{\ell-\alpha m}$, $0 \le m \le k-1$.

(ii) *If* $\gcd(\alpha, k) = 1$ *and* $\alpha\beta \equiv 1 \pmod{k}$, *then* $BA = [D_{s-r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_2 \times d_2}$ *with*

$$D_m = \sum_{\ell=0}^{k-1} B_\ell A_{m+\ell}, \quad 0 \le m \le k-1. \tag{17}$$

Proof. (i) From Theorem 1 and Corollary 1, $A = RAS^{-\alpha}$ and $B = S^{\alpha}BR^{-1}$. Therefore $AB = RABR^{-1}$, so Theorem 1 with $R = S$ implies that $AB$ is a 1-circulant. The stated formula for $C_0$, $C_1$, …, $C_{k-1}$ can be obtained by computing first block row entries of $AB$.

(ii) Also, $BA = S^{\alpha}BAS^{-\alpha}$. Applying this $\beta$ times yields $BA = SBAS^{-1}$, so Theorem 1 with $R = S$ implies that $BA$ is a 1-circulant. Computing the first block row entries of $BA$ yields $D_m = \sum_{\ell=0}^{k-1} B_{-\alpha\ell}A_{m-\alpha\ell}$ and replacing $\ell$ by $-\beta\ell$ yields (17). □

**Theorem 2** *If*

$$A = [A_{s-\alpha_1 r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_1\times d_2} \tag{18}$$

*and*

$$B = [B_{s-\alpha_2 r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_2\otimes d_3}, \tag{19}$$

*then*

$$AB = [C_{s-\alpha_1\alpha_2 r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_1\otimes d_3}, \tag{20}$$

*with*

$$C_m = \sum_{\ell=0}^{k-1} A_\ell B_{m-\alpha_2\ell}, \quad 0 \le m \le k-1. \tag{21}$$

Proof. Let $R = E \otimes I_{d_1}$, $S = E \otimes I_{d_2}$, and $T = E \otimes I_{d_3}$. From (18), (19), and Theorem 1,

$$\text{(a)} \quad A = RAS^{-\alpha_1} \quad \text{and} \quad \text{(b)} \quad B = SBT^{-\alpha_2}.$$

Applying (b) $\alpha_1$ times yields $B = S^{\alpha_1}BT^{-\alpha_1\alpha_2}$. From this and (a), $RABT^{-\alpha_1\alpha_2} = AB$. Now Theorem 1 implies (20), with (21) obtained by computing the entries in the first block row of $AB$. □

Theorem 2 generalizes [1, Theorem 3.1]; namely, the product of a standard $\alpha$-circulant and a standard $\beta$-circulant is an $\alpha\beta$-circulant. However, [1] does not include (21).

## 3 A DFT characterization of $\alpha$-circulants

**Theorem 3** *A matrix $A \in \mathbb{C}^{k:d_1\times d_2}$ is an $\alpha$-circulant $A = [A_{s-\alpha r}]_{r,s=0}^{k-1}$ if and only if it can be written as*

$$A = \sum_{\ell=0}^{k-1} P_{\alpha\ell}F_\ell Q_\ell^* = U_\alpha \mathcal{F}_A Q^*, \tag{22}$$

*where $\{F_0, F_1, \ldots, F_{k-1}\}$ and $\{A_0, A_1, \ldots, A_{k-1}\}$ are related as in (5) and (6) and $P$, $Q$, and $U_\alpha$ are as in (8)–(11).*

Proof. Eqns. (7)–(11) imply the second equality in (22). Therefore we need only justify the first. Suppose $A = [A_{s-\alpha r}]_{r,s=0}^{k-1}$ and define $F_0, F_1, \dots, F_{k-1}$ by (5). From (6),

$$A_{s-\alpha r} = \frac{1}{k}\sum_{\ell=0}^{k-1} \zeta^{-\ell(s-\alpha r)} F_\ell, \quad 0 \le r, s \le k-1,$$

so (8)–(11) imply that

$$A = \frac{1}{k}\sum_{\ell=0}^{k-1} \begin{bmatrix} 1 \otimes I_{d_1} \\ \zeta^{\alpha\ell} \otimes I_{d_1} \\ \vdots \\ \zeta^{(k-1)\alpha\ell} \otimes I_{d_1} \end{bmatrix} F_\ell \begin{bmatrix} 1 \otimes I_{d_2} \\ \zeta^{\ell} \otimes I_{d_2} \\ \vdots \\ \zeta^{(k-1)\ell} \otimes I_{d_2} \end{bmatrix}^H = \sum_{\ell=0}^{k-1} P_{\alpha\ell} F_\ell Q_\ell^*.$$

For the converse, if (22) holds then (12) implies that

$$RAS^{-\alpha} = \sum_{\ell=0}^{k-1} (RP_{\alpha\ell}) F_\ell (S^\alpha Q_\ell)^* = \sum_{\ell=0}^{k-1} (\zeta^{\alpha\ell} P_{\alpha\ell}) F_\ell (\zeta^{-\alpha\ell} Q_\ell^*) = A.$$

Therefore $A$ is an $\alpha$-circulant, by Theorem 1; hence, $A = [A_{s-\alpha r}]_{r,s=0}^{k-1}$ with $A_0$, $A_1, \dots, A_{k-1}$ as in (6). □

**Remark 3** Theorem 3 implies that $A \in \mathbb{C}^{k\times k}$ is a standard $\alpha$-circulant $[a_{s-\alpha r}]_{r,s=0}^{k-1}$ if and only if

$$A = \Phi_\alpha \mathcal{F}_A \Phi^* = \sum_{\ell=0}^{k-1} f_\ell \phi_{\alpha\ell} \phi_\ell^*,$$

where $\Phi$ is as in (1), $\Phi_\alpha = \left[\begin{array}{cccc} \phi_0 & \phi_\alpha & \cdots & \phi_{(k-1)\alpha} \end{array}\right]$, and

$$f_\ell = \sum_{m=0}^{k-1} a_m \zeta^{\ell m}, \quad 0 \le \ell \le k-1.$$

**Corollary 3** *A matrix $B \in \mathbb{C}^{k:d_2\times d_1}$ is an $\alpha$-cocirculant if and only if it can be written as $B = \sum_{\ell=0}^{k-1} Q_\ell G_\ell P_{\alpha\ell}^*$, where*

$$G_\ell = \sum_{m=0}^{k-1} \zeta^{-\ell m} B_m, \quad 0 \le \ell \le k-1, \quad \textit{and} \quad B_m = \frac{1}{k}\sum_{\ell=0}^{k-1} \zeta^{\ell m} G_\ell, \quad 0 \le m \le k-1.$$

Proof. Apply Theorem 3 to $B^*$. □

It is well known that standard 1-circulants commute. The following corollary extends this.

**Corollary 4** *Suppose $d_1 = d_2$, $\gcd(\alpha, k) = 1$, and $\alpha\beta \equiv 1 \pmod{k}$. Let $A = [A_{s-\alpha r}]_{r,s=0}^{k-1}$, $B = [B_{s-\beta r}]_{r,s=0}^{k-1}$,*

$$F_\ell = \sum_{m=0}^{k-1} \zeta^{\ell m} A_m \quad \textit{and} \quad G_\ell = \sum_{m=0}^{k-1} \zeta^{\ell m} B_m.$$

*Then $AB = BA$ if and only if $F_{\beta\ell} G_\ell = G_{\alpha\ell} F_\ell$, $0 \le \ell \le k-1$.*

Proof. Since $\gcd(\alpha,k)=\gcd(\beta,k)=1$, we may change summation indices $\ell\to\alpha\ell$ and $\ell\to\beta\ell$. Therefore, from Theorem 3 with $Q=P$,

$$A=\sum_{\ell=0}^{k-1}P_{\alpha\ell}F_\ell P_\ell^*=\sum_{\ell=0}^{k-1}P_\ell F_{\beta\ell}P_{\beta\ell}^*,\quad B=\sum_{\ell=0}^{k-1}P_{\beta\ell}G_\ell P_\ell^*=\sum_{\ell=0}^{k-1}P_\ell G_{\alpha\ell}P_{\alpha\ell}^*,$$

$$AB=\sum_{\ell=0}^{k-1}P_\ell F_{\beta\ell}G_\ell P_\ell^*,\quad\text{and}\quad BA=\sum_{\ell=0}^{k-1}P_\ell G_{\alpha\ell}F_\ell P_\ell^*,$$

which implies the conclusion. □

## 4 Moore-Penrose inversion and singular value decomposition

Recall that the Moore-Penrose inverse $X^\dagger$ of a matrix $X$ is the unique matrix $Y$ that satisfies the Penrose conditions

$$(XY)^*=XY,\quad (YX)^*=YX,\quad XYX=X,\ \text{and}\ YXY=Y.$$

**Theorem 4** *The Moore–Penrose inverse of an $\alpha$-circulant is an $\alpha$-cocirculant.*

Proof. From Theorem 1, if $A$ is an $\alpha$-circulant then $A=RAS^{-\alpha}$. Let $B=S^\alpha A^\dagger R^{-1}$. We will show that $A$ and $B$ satisfy the Penrose conditions:

$$AB=(RAS^{-\alpha})(S^\alpha A^\dagger R^{-1})=RAA^\dagger R^*=R(AA^\dagger)^*R^*=(AB)^*,$$

$$BA=(S^\alpha A^\dagger R^{-1})(RAS^{-\alpha})=S^\alpha A^\dagger A(S^\alpha)^*=S^\alpha(A^\dagger A)^*(S^\alpha)^*=(BA)^*,$$

$$ABA=(RAA^\dagger R^{-1})(RAS^{-\alpha})=R(AA^\dagger A)S^{-\alpha}=RAS^{-\alpha}=A,$$

and

$$BAB=(S^\alpha A^\dagger AS^{-\alpha})(S^\alpha A^\dagger R^{-1})=S^\alpha(A^\dagger AA^\dagger)R^{-1}=S^\alpha A^\dagger R^{-1}=B.$$

Therefore $B=A^\dagger$ or, equivalently, $S^\alpha A^\dagger R^{-1}=A^\dagger$. Now Corollary 1 implies that $A^\dagger$ is an $\alpha$-cocirculant. □

We can be more explicit if $\gcd(\alpha,k)=1$.

**Theorem 5** *The Moore–Penrose inverse of a proper $\alpha$-circulant $A=[A_{s-\alpha r}]_{r,s=0}^{k-1}$ is the $\alpha$-cocirculant $B=[B_{r-\alpha s}]_{r,s=0}^{k-1}$, where*

$$B_m=\frac{1}{k}\sum_{\ell=0}^{k-1}\zeta^{\ell m}F_\ell^\dagger,\quad 0\le m\le k-1,\tag{23}$$

*with*

$$F_\ell=\sum_{m=0}^{k-1}\zeta^{\ell m}A_m,\quad 0\le\ell\le k-1.$$

PROOF. From Theorem 3, $A = U_\alpha \mathcal{F}_A Q^*$ where $Q$ and $U_\alpha$ are unitary, the latter since $\gcd(\alpha, k) = 1$. We will first show that $A$ and $B = Q\mathcal{F}_A^\dagger U_\alpha^*$ satisfy the Penrose conditions:

$$AB = (U_\alpha \mathcal{F}_A Q^*)(Q\mathcal{F}_A^\dagger U_\alpha^*) = U_\alpha \mathcal{F}_A \mathcal{F}_A^\dagger U_\alpha^* = U_\alpha (\mathcal{F}_A \mathcal{F}_A^\dagger)^* U_\alpha^* = (AB)^*,$$

$$BA = (Q\mathcal{F}_A^\dagger U_\alpha^*)(U_\alpha \mathcal{F}_A Q^*) = Q\mathcal{F}_A^\dagger \mathcal{F}_A Q^* = Q(\mathcal{F}_A^\dagger \mathcal{F}_A)^* Q^* = (BA)^*,$$

$$ABA = (U_\alpha \mathcal{F}_A \mathcal{F}_A^\dagger U_\alpha^*)(U_\alpha \mathcal{F}_A Q^*) = U_\alpha (\mathcal{F}_A \mathcal{F}_A^\dagger \mathcal{F}_A) Q^* = U_\alpha \mathcal{F}_A Q^* = A,$$

and

$$BAB = (Q\mathcal{F}_A^\dagger \mathcal{F}_A Q^*)(Q\mathcal{F}_A^\dagger U_\alpha^*) = Q(\mathcal{F}_A^\dagger \mathcal{F}_A \mathcal{F}_A^\dagger) U_\alpha^* = Q\mathcal{F}_A^\dagger U_\alpha^* = B.$$

Therefore

$$\begin{aligned} A^\dagger &= B = \sum_{\ell=0}^{k-1} Q_\ell F_\ell^\dagger P_{\alpha\ell}^* = \sum_{\ell=0}^{k-1} (\phi_\ell \otimes I_{d_2}) F_\ell^\dagger (\phi_{\alpha\ell} \otimes I_{d_1})^* \\ &= \frac{1}{k} \left[ \sum_{\ell=0}^{k-1} \zeta^{\ell(r-\alpha s)} F_\ell^\dagger \right]_{r,s=0}^{k-1} = [B_{r-\alpha s}]_{r,s=0}^{k-1}, \end{aligned}$$

from (8)–(11) and (23). □

**Remark 4** Theorem 5 can also be proved by using (6) and (23) to express the entries of $AB$, $BA$, $ABA$, and $BAB$ explicitly in terms of $F_0$, $F_1$, ..., $F_{k-1}$ and $F_0^\dagger$, $F_1^\dagger$, ..., $F_{k-1}^\dagger$, noting that

$$\sum_{\ell=0}^{k-1} \zeta^{\ell(r-s)} = \sum_{\ell=0}^{k-1} \zeta^{\alpha\ell(r-s)} = \delta_{rs}, \quad 0 \le r, s \le k-1,$$

the latter because $\gcd(\alpha, k) = 1$. However, this is tedious.

**Remark 5** Theorem 5 extends a result of Davis [6]: If $A = [a_{s-r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k\times k}$ then $A^\dagger = \Phi \operatorname{diag}(a_0^\dagger, a_1^\dagger, \ldots, a_{k-1}^\dagger)\Phi^*$, where $\Phi$ is the Fourier matrix (1), $0^\dagger = 0$, and $a^\dagger = 1/a$ if $a \ne 0$.

**Theorem 6** *Suppose* $\gcd(\alpha, k) = 1$ *and*

$$A = [A_{s-\alpha r}]_{r,s=0}^{k-1} = \sum_{\ell=0}^{k-1} P_{\alpha\ell} F_\ell Q_\ell^* = U_\alpha \mathcal{F}_A Q^*.$$

*Let* $F_\ell = \Omega_\ell \Sigma_\ell \Psi_\ell^*$ *be a singular value decomposition of* $F_\ell$, $0 \le \ell \le k-1$, *and define*

$$M_\alpha = \left[ \begin{array}{cccc} P_0\Omega_0 & P_\alpha\Omega_1 & \cdots & P_{(k-1)\alpha}\Omega_{k-1} \end{array} \right]$$

*and*

$$N = \left[ \begin{array}{cccc} Q_0\Psi_0 & Q_1\Psi_1 & \cdots & Q_{k-1}\Psi_{k-1} \end{array} \right].$$

*Then*

$$A = M_\alpha \left( \bigoplus_{\ell=0}^{k-1} \Sigma_\ell \right) N^*$$

*is a singular value decomposition of* $A$*, except that the singular values are not necessarily ordered.*

# 5 The least squares problem

Suppose $G \in \mathbb{C}^{d_1 \times d_2}$ and consider the least squares problem for $G$: If $v \in \mathbb{C}^{d_1}$, find $u \in \mathbb{C}^{d_2}$ such that

$$\|Gu - v\| = \min_{\xi \in \mathbb{C}^{d_2}} \|G\xi - v\|, \tag{24}$$

where $\|\cdot\|$ is the 2-norm. This problem has a unique solution if and only if $\operatorname{rank}(G) = d_2$, in which case $u = (G^*G)^{-1}G^*v$. In any case, the optimal solution of (24) is the unique $u_0 \in \mathbb{C}^{d_2}$ of minimum norm that satisfies (24); thus, $u_0 = G^\dagger v$. The general solution of (24) is $u = u_0 + q$ where $Gq = 0$, and

$$\|Gu - v\| = \|(GG^\dagger - I)v\|$$

for all such $u$.

Now consider the following least squares problem: if $A = [A_{s-\alpha r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_1 \times d_2}$ with $\gcd(\alpha, k) = 1$ and $w \in \mathbb{C}^{kd_1}$, find $z \in \mathbb{C}^{kd_2}$ such that

$$\|Az - w\| = \min_{\xi \in \mathbb{C}^{kd_2}} \|A\xi - w\|.$$

We write

$$z = \sum_{\ell=0}^{k-1} Q_\ell u_\ell \quad \text{and} \quad w = \sum_{\ell=0}^{k-1} P_\ell v_\ell = \sum_{\ell=0}^{k-1} P_{\alpha\ell} v_{\alpha\ell}, \tag{25}$$

since substituting $\alpha\ell$ for $\ell$ is legitimate because $\gcd(\alpha, k) = 1$. Since $A = \sum_{\ell=0}^{k-1} P_{\alpha\ell} F_\ell Q_\ell^*$ and $Q_\ell^* Q_m = \delta_{\ell m} I_{kd_2}$,

$$Az - w = \sum_{\ell=0}^{k-1} P_{\alpha\ell}(F_\ell u_\ell - v_{\alpha\ell}).$$

Since $P_{\alpha\ell}^* P_{\alpha m} = \delta_{\ell m} I_{d_1}$ (because $\gcd(\alpha, k) = 1$), it follows that

$$\|Az - w\|^2 = \sum_{\ell=0}^{k-1} \|F_\ell u_\ell - v_{\alpha\ell}\|^2. \tag{26}$$

This implies the following theorem.

**Theorem 7** *Suppose $A$ is a proper $\alpha$-circulant and let $z$ and $w$ be as in* (25)*. Then*

$$\|Az - w\| = \min_{\xi \in \mathbb{C}^{kd_2}} \|A\xi - w\| \tag{27}$$

*if and only if*

$$\|F_\ell u_\ell - v_{\alpha\ell}\| = \min_{\psi_\ell \in C^{d_2}} \|F_\ell \psi_\ell - v_{\alpha\ell}\|, \quad 0 \le \ell \le k-1.$$

*Therefore* (27) *has a unique solution, given by*

$$z = \sum_{\ell=0}^{k-1} Q_\ell (F_\ell^* F_\ell)^{-1} F_\ell^* v_{\alpha\ell},$$

*if and only* $\operatorname{rank}(F_\ell) = d_2$, $0 \le \ell \le k-1$. *In any case, the optimal solution of* (27) *is*

$$z_0 = \sum_{\ell=0}^{k-1} Q_\ell F_\ell^\dagger v_{\alpha\ell}.$$

*The general solution of* (27) *is $z = z_0 + \sum_{\ell=0}^{k-1} Q_\ell u_\ell$, where $F_\ell u_\ell = 0$, $0 \le \ell \le k-1$, and*

$$\|Az - w\|^2 = \sum_{\ell=0}^{k-1} \|(F_\ell F_\ell^\dagger - I_{d_1}) v_{\alpha\ell}\|^2$$

*for all such $z$.*

# 6 The case where $d_1 = d_2$

Throughout this section $d_1 = d_2 = d$ and $A = [A_{s-\alpha r}]_{r,s=0}^{k-1}$ is a proper circulant. Then (26) implies the following theorem, which reduces the problem of solving the $kd \times kd$ system $Az = w$ to solving $k$ independent $d \times d$ systems.

**Theorem 8** *If $A$ is a proper $\alpha$-circulant, $z = \sum_{\ell=0}^{k-1} P_\ell u_\ell$, and $w = \sum_{\ell=0}^{k-1} P_\ell v_\ell$, then $Az = w$ if and only if*

$$F_\ell u_\ell = v_{\alpha\ell}, \quad 0 \le \ell \le k-1.$$

This and Theorem 5 imply the following theorem.

**Theorem 9** *A proper $\alpha$-circulant*

$$A = [A_{s-\alpha r}]_{r,s=0}^{k-1} = \sum_{\ell=0}^{k-1} P_{\alpha\ell} F_\ell P_\ell^* \tag{28}$$

*is invertible if and only $F_0, F_1, \dots, F_{k-1}$ are all invertible. In this case*

$$A^{-1} = [B_{r-\alpha s}]_{r,s=0}^{k-1} \quad \text{with} \quad B_m = \frac{1}{k} \sum_{\ell=0}^{k-1} \zeta^{\ell m} F_\ell^{-1}, \quad 0 \le m \le k-1,$$

*and the solution of $Az = w$ is $z = \sum_{\ell=0}^{k-1} P_\ell F_\ell^{-1} v_{\alpha\ell}$.*

**Remark 6** Theorem 9 and Remark 2 extend [5, Theorem 1]: the inverse of a standard nonsingular $\alpha$-circulant is a $\beta$-circulant, where $\alpha\beta \equiv 1 \pmod{k}$.

**Theorem 10** *Suppose $A$ is a proper $\alpha$-circulant as in* (28) *and* $\alpha\beta \equiv 1 \pmod{k}$.

(i) *$A$ is Hermitian if and only if* $P_{\beta\ell}F^*_{\beta\ell} = P_{\alpha\ell}F_\ell$, $0 \le \ell \le k-1$.

(ii) *$A$ is normal if and only if* $F_{\beta\ell}F^*_{\beta\ell} = F^*_\ell F_\ell$, $0 \le \ell \le k-1$.

(iii) *$A$ is EP* (*i.e.*, $A^\dagger A = AA^\dagger$) *if and only if* $F_{\beta\ell}F^\dagger_{\beta\ell} = F^\dagger_\ell F_\ell$, $0 \le \ell \le k-1$.

Proof.

From (28) and Theorem 5,

$$A = \sum_{\ell=0}^{k-1} P_{\alpha\ell}F_\ell P^*_\ell, \quad A^* = \sum_{\ell=0}^{k-1} P_\ell F^*_\ell P^*_{\alpha\ell}, \quad \text{and} \quad A^\dagger = \sum_{\ell=0}^{k-1} P_\ell F^\dagger_\ell P^*_{\alpha\ell}. \tag{29}$$

(i) Since $\alpha\beta \equiv 1 \pmod{k}$, replacing $\ell$ by $\beta\ell$ in the second sum in (29) yields $A^* = \sum_{\ell=0}^{k-1} P_{\beta\ell}F^*_{\beta\ell}P^*_\ell$, and comparing this with the first sum in (29) yields (i).

(ii) From (29),

$$AA^* = \sum_{\ell=0}^{k-1} P_{\alpha\ell}F_\ell F^*_\ell P^*_{\alpha\ell} = \sum_{\ell=0}^{k-1} P_\ell F_{\beta\ell}F^*_{\beta\ell}P^*_\ell \quad \text{and} \quad A^*A = \sum_{\ell=0}^{k-1} P_\ell F^*_\ell F_\ell P^*_\ell,$$

which implies (ii).

(iii) From (29),

$$AA^\dagger = \sum_{\ell=0}^{k-1} P_{\alpha\ell}F_\ell F^\dagger_\ell P^*_{\alpha\ell} = \sum_{\ell=0}^{k-1} P_\ell F_{\beta\ell}F^\dagger_{\beta\ell}P^*_\ell \quad \text{and} \quad A^\dagger A = \sum_{\ell=0}^{k-1} P_\ell F^\dagger_\ell F_\ell P^*_\ell.$$

which implies (iii). □

**Remark 7** If $A$ is a square matrix and there is a matrix $B$ such that $ABA = A$, $BAB = B$, and $AB = BA$, then $B$ is unique and is called the group inverse of $A$, which is usually denoted by $A^{\#}$. Davis [6] noted that if $A \in \mathbb{C}^{k\times k}$ is a standard 1-circulant then $A^\dagger = A^{\#}$. Theorem 10(iii) extends this: If $A \in \mathbb{C}^{k:d\times d}$ is a proper $\alpha$-circulant and $\alpha\beta \equiv 1 \pmod{k}$, then $A^\dagger = A^{\#}$ if and only if $F^\dagger_\ell F_\ell = F_{\beta\ell}F^\dagger_{\beta\ell}$, $0 \le \ell \le k-1$.

# 7 The eigenvalue problem with $\alpha = 1$

In this section we assume that $\alpha = 1$ and $d_1 = d_2 = d$. The following theorem and its proof are motivated by [2, Theorem 2].

**Theorem 11** *Let*

$$\mathcal{S}_R = \bigcup_{\ell=0}^{k-1} \left\{ z \,\middle|\, Rz = \zeta^\ell z \right\}.$$

*If $\lambda$ is an eigenvalue of $A$, let $\mathcal{E}_A(\lambda)$ be the $\lambda$-eigenspace of $A$; i.e,*

$$\mathcal{E}_A(\lambda) = \left\{ z \,\middle|\, Az = \lambda z \right\}.$$

(i) *If $\lambda$ is an eigenvalue of $A = [A_{s-r}]_{r,s=0}^{k-1}$ then $\mathcal{E}_A(\lambda)$ has a basis in $\mathcal{S}_R$.*

(ii) *If $A \in \mathbb{C}^{k:d\times d}$ and has kd linearly independent eigenvectors in $\mathcal{S}_R$, then $A$ is a 1-circulant.*

Proof. (i) From Theorem 8, $z = \sum_{\ell=0}^{k-1} P_\ell u_\ell \in \mathcal{E}_A(\lambda)$ if and only if $F_\ell u_\ell = \lambda u_\ell, \quad 0 \le \ell \le k-1$. Therefore $\lambda$ is an eigenvalue of $A$ if and only if it is an eigenvalue of $F_\ell$ for some $\ell \in \{0, 1, \dots, k-1\}$. Let $\mathcal{T}_\lambda$ be the subset of $\{0, 1, \dots, k-1\}$ for which this is true. Then $\mathcal{E}_A(\lambda)$ consists of linear combinations of the vectors of the form $P_\ell u_\ell$ with $\ell \in \mathcal{T}_\lambda$ and $(\lambda, u_\ell)$ an eigenpair of $F_\ell$. Since $RP_\ell = \zeta^\ell P_\ell$ (recall (12)), this completes the proof of (i).

(ii) From Theorem 1, we must show that $RA = AR$. If $Az = \lambda z$ and $Rz = \zeta^s z$ then $RAz = \lambda Rz = \lambda \zeta^s z$ and $ARz = \zeta^s Az = \zeta^s \lambda z$. Hence $ARz = RAz$ for all $z$ in a basis for $\mathbb{C}^{k:d\times d}$, so $AR = RA$. □

**Theorem 12** *Let $R$ and $P$ be as in (8) and (10). Then the 1-circulant $A = [A_{s-r}]_{r,s=0}^{k-1}$ is diagonalizable if and only if $F_0, F_1, \dots, F_{k-1}$ are all diagonalizable. In this case, if*

$$F_\ell = T_\ell D_\ell T_\ell^*, \quad 0 \le \ell \le k-1,$$

*are spectral decompositions of $F_0, F_1, \dots, F_{k-1}$ and*

$$\Psi = \left[ \begin{array}{cccc} P_0 T_0 & P_1 T_1 & \cdots & P_{k-1} T_{k-1} \end{array} \right],$$

*then*

$$A = \Psi \left( \bigoplus_{\ell=0}^{k-1} D_\ell \right) \Psi^*$$

*is a spectral decomposition of $A$.*

# 8 The eigenvalue problem with $\alpha > 1$

In this section we assume that $d_1 = d_2 = d$, $\alpha \in \{2, 3, \dots k-1\}$ and, $\gcd(\alpha, k) = 1$. From Theorem 8, $Az = \lambda z$ if and only if $z = \sum_{s=0}^{k-1} P_s u_s$, where

$$F_s u_s = \lambda u_{\alpha s}, \quad 0 \le s \le k-1. \tag{30}$$

Therefore $Az = 0$ if and only if $z = \sum_{s=0}^{k-1} P_s u_s$ where $F_s u_s = 0$, $0 \le s \le k-1$, so the makeup of the null space of $A$ is transparent. Hence, we assume that $\lambda \neq 0$. Then we must consider the orbits of the permutation on $\{0, \dots, k-1\}$ defined by $s \to \alpha s \pmod{k}$. We consider an example before presenting the general discussion.

Let $k = 10$ and $\alpha = 3$. The permutation of $\{0, 1, \ldots, 9\}$ defined by $s \to 3s$ (mod 10) is given by

$$\left( \begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 0 & 3 & 6 & 9 & 2 & 5 & 8 & 1 & 4 & 7 \end{array} \right).$$

The orbits of this permutation are

$$\mathcal{O}_0 = \{0\}, \quad \mathcal{O}_1 = \{1, 3, 9, 7\}, \quad \mathcal{O}_2 = \{2, 6, 8, 4\}, \quad \text{and} \quad \mathcal{O}_3 = \{5\}.$$

Therefore (30) divides into four independent systems:

$$\text{(i)} \quad F_0 u_0 = \lambda u_0; \quad \text{(ii)} \quad F_1 u_1 = \lambda u_3, \quad F_3 u_3 = \lambda u_9, \quad F_9 u_9 = \lambda u_7, \quad F_7 u_7 = \lambda u_1,$$

$$\text{(iii)} \quad F_5 u_5 = \lambda u_5; \quad \text{(iv)} \quad F_2 u_2 = \lambda u_6, \quad F_6 u_6 = \lambda u_8, \quad F_8 u_8 = \lambda u_4, \quad F_4 u_4 = \lambda u_2.$$

From (i), if $(\lambda, u_0)$ is an eigenpair of $F_0$ then $(\lambda, P_0 u_0)$ is an eigenpair of $A$. Similarly, from(iii), if $(\lambda, u_5)$ is an eigenpair of $F_5$ then $(\lambda, P_5 u_5)$ is an eigenpair of $A$. The analysis of (ii) and (iv) is more complicated, but identical. We will consider (ii), which is equivalent to

$$u_3 = \frac{1}{\lambda} F_1 u_1, \quad u_9 = \frac{1}{\lambda} F_3 u_3, \quad u_7 = \frac{1}{\lambda} F_9 u_9, \quad u_1 = \frac{1}{\lambda} F_7 u_7, \tag{31}$$

since $\lambda \neq 0$. Hence,

$$u_3 = \frac{1}{\lambda} G_3 u_1 \quad u_9 = \frac{1}{\lambda^2} G_9 u_1, \quad u_7 = \frac{1}{\lambda^3} G_7 u_1, \quad \text{and} \quad u_1 = \frac{1}{\lambda^4} G_1 u_1, \tag{32}$$

where

$$G_3 = F_1, \quad G_9 = F_3 F_1, \quad G_7 = F_9 F_3 F_1, \quad \text{and} \quad G_1 = F_7 F_9 F_3 F_1. \tag{33}$$

In particular, the last equalities in (32) and (33) are equivalent to $G_1 u_1 = \lambda^4 u_1$. Therefore, if $(\gamma, u_1)$ is an eigenpair of $G_1$ and $\gamma \neq 0$, then $\lambda = \gamma^{1/4}$ is an eigenvalue of $A$ with the associated eigenvector

$$\begin{aligned} z &= (P_1 + \gamma^{-1/4} P_3 G_3 + \gamma^{-2/4} P_9 G_9 + \gamma^{-3/4} P_7 G_7) u_1 \\ &= \left( P_1 + \sum_{m=1}^{3} \gamma^{-m/4} P_{3^m} G_{3^m} \right) u_1. \end{aligned} \tag{34}$$

(Recall that subscripts are taken modulo 10.) However, $\gamma^{1/4} e^{2\pi i r/4}$, $0 \le r \le 3$, are all fourth roots of $\gamma$ and therefore eigenvalues of $A$. Replacing $\gamma^{1/4}$ with $\gamma^{1/4} e^{2\pi i r/4}$ in (34) shows that

$$z_r = \left( P_1 + \sum_{m=1}^{3} \gamma^{-m/4} e^{-2\pi i r m/4} P_{3^m} G_{3^m} \right) u_1, \quad 0 \le r \le 3, \tag{35}$$

are the respective associated eigenvectors of $A$.

Now suppose the permutation $s \to \alpha s \pmod{k}$ of $\{0, 1, \dots, k-1\}$ has $p$ orbits $\mathcal{O}_0$, $\mathcal{O}_1$, …, $\mathcal{O}_{p-1}$, and let

$$0 = s_0 < s_1 < s_2 < \cdots < s_{p-1} \quad \text{with} \quad s_\ell \in \mathcal{O}_\ell, \quad 0 \le \ell \le p-1.$$

Suppose $\mathcal{O}_\ell$ has $r_\ell$ distinct members; thus,

$$\mathcal{O}_\ell = \{s_\ell, \alpha s_\ell, \dots, \alpha^{r_\ell - 1} s_\ell\} \quad \text{where} \quad \alpha^{r_\ell} \equiv 1 \pmod{k}, \tag{36}$$

and $\bigcup_{\ell=0}^{p-1} \mathcal{O}_\ell = \{0, 1, \dots, k-1\}$. If $r_\ell = 1$ and $(\lambda, u_{s_\ell})$ is an eigenpair of $F_{s_\ell}$, then $(\lambda, P_{s_\ell} u_{s_\ell})$ is an eigenpair of $A$. Now consider an orbit $\mathcal{O}_\ell$ with $r_\ell > 1$, such as $\mathcal{O}_2$ and $\mathcal{O}_4$ in the example. The system associated with $\mathcal{O}_\ell$ is

$$F_{\alpha^r s_\ell} u_{\alpha^r s_\ell} = \lambda u_{\alpha^{r+1} s_\ell}, \quad 0 \le r \le r_\ell - 1, \quad \text{where} \quad \alpha^{r_\ell} = 1,$$

which is analogous to (ii), where $s_\ell = 1$, $\alpha = 3$ and $k = 10$. Since $\lambda \ne 0$, this is equivalent to

$$u_{\alpha^{r+1} s_\ell} = \frac{1}{\lambda} F_{\alpha^r s_\ell} u_{\alpha^r s_\ell}, \quad 0 \le r_\ell - 1,$$

which is analogous to (31). Therefore

$$u_{\alpha^{r+1} s_\ell} = \frac{1}{\lambda^{r+1}} G_{\alpha^{r+1} s_\ell} u_{s_\ell}, \quad 0 \le r \le r_\ell - 1, \tag{37}$$

where

$$G_{\alpha^{r+1} s_\ell} = F_{\alpha^r s_\ell} \cdots F_{s_\ell}, \quad 0 \le r \le r_\ell - 1,$$

which is analogous to (32) and (33). In particular, setting $r = r_\ell - 1$ and noting that $\alpha^{r_\ell} s_\ell = s_\ell$ yields

$$u_{s_\ell} = \frac{1}{\lambda^{r_\ell}} G_{s_\ell} u_{s_\ell} \quad \text{where} \quad G_{s_\ell} = F_{\alpha^{r_\ell - 1} s_\ell} \cdots F_{s_\ell}.$$

Therefore, if $(\gamma_\ell, u_{s_\ell})$ is an eigenvalue of $G_{s_\ell}$, then $\gamma_\ell^{1/r_\ell}$ is an eigenvalue of $A$ with associated eigenvector

$$z_\ell = \left( P_{s_\ell} + \sum_{m=1}^{r_\ell - 1} \gamma_\ell^{-m/r_\ell} P_{\alpha^m s_\ell} G_{\alpha^m s_\ell} \right) u_{s_\ell}, \tag{38}$$

which is analogous to (34). However, since $\gamma^{1/r_\ell} e^{2\pi i r/r_\ell}$ are all $r_\ell$-th roots of $\gamma$, they are all eigenvalues of $A$. Replacing $\gamma^{1/r_\ell}$ with $\gamma^{1/r_\ell} e^{2\pi i r/4}$ in (38) yields associated eigenvectors

$$z_{r\ell} = \left( P_{s_\ell} + \sum_{m=1}^{r_\ell - 1} \gamma_\ell^{-m/r_\ell} e^{-2\pi i r m/r_\ell} P_{\alpha^m s_\ell} G_{\alpha^m s_\ell} \right) u_{s_\ell}, \quad 0 \le r \le r_\ell - 1, \tag{39}$$

which is analogous to (35).

**Remark 8** Now we apply the preceding argument to a standard $\alpha$-circulant $A = [a_{s-\alpha r}]_{r,s=0}^{k-1}$ with $\gcd(\alpha, k) = 1$. From Remark 3,

$$A = \sum_{s=0}^{k-1} f_s \phi_{\alpha s} \phi_s^* \quad \text{with} \quad f_s = \sum_{r=0}^{k-1} a_r \zeta^{rs}, \quad 0 \le r \le k-1,$$

and $\phi_0, \phi_1, \ldots, \phi_{k-1}$ as in (1). Then $z = \sum_{s=0}^{k-1} u_s \phi_s$ is $\lambda$-eigenvector of $A$ if and only if $f_s u_s = \lambda u_{\alpha s}$, $0 \le s \le k-1$. Let $\mathcal{O}_\ell$ be as in (36) and assume that $f_{\alpha^r s_\ell} \ne 0$, $0 \le r \le r_\ell - 1$. Let

$$g_{\alpha^{r+1} s_\ell} = \prod_{q=0}^{r} f_{\alpha^q s_\ell}, \quad 0 \le r \le r_\ell - 1,$$

and

$$\gamma_\ell = g_{\alpha^{r_\ell} s_\ell} = f_{\alpha^{r_\ell - 1} s_\ell} \cdots f_{s_\ell}.$$

From (37),

$$u_{\alpha^{r+1} s_\ell} = \frac{1}{\lambda^{r+1}} g_{\alpha^{r+1} s_\ell} u_{s_\ell}, \quad 0 \le r \le r_\ell - 2, \quad \text{and} \quad u_{\alpha^{r_\ell} s_\ell} = u_{s_\ell} = \lambda^{-r_\ell} \gamma_\ell u_{s_\ell}.$$

Therefore $\gamma_\ell^{r_\ell} e^{2\pi i r / r_\ell}$, $0 \le r \le r_\ell - 1$, are eigenvalues of $A$. From (39),

$$z_{r\ell} = \left( \phi_{s_\ell} + \sum_{m=1}^{r_\ell - 1} \gamma_\ell^{-m/r_\ell} e^{-2\pi i r m / r_\ell} g_{\alpha^m s_\ell} \phi_{\alpha^m s_\ell} \right), \quad 0 \le r \le r_\ell - 1,$$

are associated eigenvectors.

For example, let $\alpha = k-1$, so $A = [a_{s+r}]_{r,s=0}^{k-1}$. If $k = 2p$ then $v_0 = f_0$, $v_\ell = \sqrt{f_\ell f_{k-\ell}}$, $1 \le \ell \le p-1$, and $v_p = f_p$. Hence, $(f_0, \phi_0)$, $(f_p, \phi_p)$,

$$\left( \sqrt{f_\ell f_{k-\ell}}, \, \phi_\ell + \frac{1}{\sqrt{f_\ell f_{k-\ell}}} \phi_{(k-1)\ell} \right), \quad \text{and} \quad \left( -\sqrt{f_\ell f_{k-\ell}}, \, \phi_\ell - \frac{1}{\sqrt{f_\ell f_{k-\ell}}} \phi_{(k-1)\ell} \right)$$

$1 \le \ell \le p-1$, are eigenpairs of $A$. If $k = 2p+1$ then $v_0 = f_0$ and $v_\ell = f_\ell f_{k-\ell}$, $1 \le \ell \le q$. Hence $(f_0, \phi_0)$,

$$\left( \sqrt{f_\ell f_{k-\ell}}, \, \phi_\ell + \frac{1}{\sqrt{f_\ell f_{k-\ell}}} \phi_{(k-1)\ell} \right), \quad \text{and} \quad \left( -\sqrt{f_\ell f_{k-\ell}}, \, \phi_\ell - \frac{1}{\sqrt{f_\ell f_{k-\ell}}} \phi_{(k-1)\ell} \right)$$

$1 \le \ell \le p$, are eigenpairs of $A$.

The eigenvalues of $A$ were given in [5] without the associated eigenvectors.

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