Properties of unilevel block circulants

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Abstract

Let $\mathcal{A} = \{A_0, A_1, \ldots, A_{k-1}\} \subset \mathbb{C}^{d_1 \times d_2}$, $\zeta = e^{-2\pi i/k}$, $F_\ell = \sum_{m=0}^{k-1} \zeta^{\ell m} A_m$, $0 \leq \ell \leq k - 1$, and $\mathcal{F}_A = \bigoplus_{\ell=0}^{k-1} F_\ell$. All operations in indices are modulo $k$. It is well known that if $d_1 = d_2 = 1$ then $[A_s - r]_{r,s=0}^{k-1} = \Phi F_A \Phi^*$, where $\Phi = \frac{1}{\sqrt{k}} \sum_{\ell,m=0}^{k-1} [\zeta^{\ell m}]^{k-1}$. However, to our knowledge it has not been emphasized that $F_A$ plays a fundamental role in connection with all the matrices $[A_s - r]_{r,s=0}^{k-1}$, $0 \leq \alpha \leq k - 1$, with $d_1$, $d_2$ arbitrary. We begin by adapting a theorem of Ablow and Brenner with $d_1 = d_2 = 1$ to the case where $d_1$ and $d_2$ are arbitrary. We show that $A = [A_s - r]_{r,s=0}^{k-1}$ if and only if $A = U_{\alpha} F_A P^*$ where $U_{\alpha}$ and $P$ are related to $\Phi$, $P$ is unitary, and $U_{\alpha}$ is invertible (in fact, unitary) if and only if $\gcd(\alpha, k) = 1$, in which case we say that $A$ is a proper circulant. We prove the following for proper circulants $A = [A_s - r]_{r,s=0}^{k-1}$: (i) $A^\dagger = [B_{\ell - \alpha}]_{\ell,m=0}^{k-1}$ with $B_m = \frac{1}{k} \sum_{\ell=0}^{k-1} \zeta^{\ell m} F_\ell^\dagger$, $0 \leq m \leq k - 1$. (ii) Solving $A z = w$ reduces to solving $F_\ell u_\ell = v_{\alpha \ell}$, $0 \leq \ell \leq k - 1$, where $v_0, v_1, \ldots, v_{k-1}$ depend only on $w$. (iii) A singular value decomposition of $A$ can be obtained from singular value decompositions of $F_0, F_1, \ldots, F_{k-1}$. (iv) The least squares problem for $A$ reduces to independent least squares problems for $F_0, F_1, \ldots, F_{k-1}$. (v) If $d_1 = d_2 = d$, the eigenvalues of $[A_s - r]_{r,s=0}^{k-1}$ are the eigenvalues of $F_0, F_1, \ldots, F_{k-1}$, and the corresponding eigenvectors of $A$ are easily obtainable from those of $F_0, F_1, \ldots, F_{k-1}$. (vi) If $d_1 = d_2 = d$ and $\alpha > 1$ then the eigenvalue problem for $[A_s - r]_{r,s=0}^{k-1}$ reduces to eigenvalue problems for $d \times d$ matrices related to $F_0, F_1, \ldots, F_{k-1}$ in a manner depending upon $\alpha$.

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1 Introduction

Throughout this paper \( k \geq 2, d_1, d_2 \geq 1 \) are integers, \( \alpha \in \{0, 1, \ldots, k-1\} \), and

\[
C_{k:d_1 \times d_2} = \{ C = [C_{rs}]_{r,s=0}^{k-1} | C_{rs} \in \mathbb{C}^{d_1 \times d_2}, 0 \leq r, s \leq k-1 \}.
\]

All arithmetic operations in indices are modulo \( k \).

We call \( A = [A_{rs}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_1 \times d_2} \) an \( \alpha \)-circulant. We say that \( A \) is a proper \( \alpha \)-circulant, or simply a proper circulant, if \( \gcd(\alpha, k) = 1 \). We will say that \( A \) is a standard \( \alpha \)-circulant if \( d_1 = d_2 = 1 \) and denote it by \( A = [a_{rs}]_{r,s=0}^{k-1} \).

Of course, there is already a vast literature on standard \( \alpha \)-circulants. Matrices of the form

\[
A = [A_{rs}]_{r,s=0}^{k-1} \quad \text{where} \quad A_{rs} = \begin{cases} 
A_{s-r}, & 0 \leq r \leq s \leq k-1, \\
kA_{s-r}, & 0 \leq s < r \leq k-1,
\end{cases}
\]

are also called \( k \)-circulants; see e.g., [4]. We will not consider them.

We call \( [B_{rs}]_{r,s=0}^{k-1} \) an \( \alpha \)-cocirculant, again proper if \( \gcd(\alpha, k) = 1 \). This eliminates awkward terminology such as “the conjugate transpose of the Moore-Penrose inverse of an \( \alpha \)-circulant matrix is an \( \alpha \)-circulant.” The Moore-Penrose inverse of an \( \alpha \)-circulant is an \( \alpha \)-cocirculant (Theorem 4).

**Remark 1** Obviously, \( B \) is an \( \alpha \)-cocirculant if and only if \( B^* \) is an \( \alpha \)-circulant. Therefore any result concerning \( \alpha \)-circulants can be applied to \( B^* \) to obtain a result concerning \( B \).

**Remark 2** A proper \( \alpha \)-circulant \( A = [A_{s-\alpha r}]_{r,s=0}^{k-1} \) is also a \( \beta \)-cocirculant where \( \alpha \beta \equiv 1 \pmod{k} \), since

\[
A_{s-\alpha r} = A_{\alpha \beta s - \alpha r} = A_{-\alpha(r-\beta s)} = B_{r-\beta s}
\]

with \( B_m = A_{-\alpha m}, 0 \leq m \leq k-1 \). Similarly, a proper \( \beta \)-cocirculant \( B = [B_{r-\beta s}]_{r,s=0}^{k-1} \) is also an \( \alpha \)-circulant, since

\[
B_{r-\beta s} = B_{\alpha \beta r - \beta s} = B_{-\beta(s-\alpha r)} = C_{s-\alpha r}
\]

with \( C_m = B_{-\beta m}, 0 \leq m \leq k-1 \).

Henceforth \( \zeta = e^{-2\pi i/k} \),

\[
E = [\delta_{t,m-1}]_{t,m=0}^{k-1}, \quad \text{and} \quad \Phi = \frac{1}{\sqrt{k}} [\zeta^m]_{t,m=0}^{k-1} = \left[ \begin{array}{c} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_{k-1} \end{array} \right]
\]

(1)

(the Fourier matrix), with

\[
\phi_m = \frac{1}{\sqrt{k}} \begin{bmatrix} 1 \\ \zeta^m \\ \zeta^{2m} \\ \vdots \\ \zeta^{(k-1)m} \end{bmatrix}, \quad 0 \leq m \leq k-1.
\]

(2)
It is straightforward to verify that if indices are reduced modulo \( k \) then
\[
E^p \left( [g_{\ell m}]_{\ell,m=0}^{k-1} \right) E^{-q} = [g_{\ell+p,m+q}]_{\ell,m=0}^{k-1}.
\] (3)

Setting \( p = 1 \) and \( q = 0 \) and invoking (1) yields
\[
E \Phi = \frac{1}{\sqrt{k}} \sum_{l=0}^{k-1} [\zeta^{(l+1)m}]_{l,m=0}^{k-1} = \Phi D \text{ with } D = \text{diag}(1, \zeta, \zeta^2, \ldots, \zeta^{k-1}).
\] (4)

Therefore
\[
E \Phi = \Phi D \Phi^*.
\]

The discrete Fourier transform (DFT) of \( \{A_0, A_1, \ldots, A_{k-1}\} \subset \mathbb{C}^{d_1 \times d_2} \) is \( \{F_0, F_1, \ldots, F_{k-1}\} \) where
\[
F_\ell = \sum_{m=0}^{k-1} \zeta^{\ell m} A_m \in \mathbb{C}^{d_1 \times d_2}, \quad 0 \leq \ell \leq k - 1.
\] (5)

Since \( \Phi^{-1} = \Phi^* \),
\[
A_m = \frac{1}{k} \sum_{\ell=0}^{k-1} \zeta^{-\ell m} F_\ell, \quad 0 \leq m \leq k - 1.
\] (6)

We denote
\[
\mathcal{F}_A = \bigoplus_{\ell=0}^{k-1} F_\ell \in \mathbb{C}^{k \times d_1 \times d_2}.
\] (7)

For standard circulants (5)–(7) reduce to
\[
f_\ell = \sum_{m=0}^{k-1} a_m \zeta^{\ell m}, \quad a_m = \frac{1}{k} \sum_{\ell=0}^{k-1} f_\ell \zeta^{-\ell m}, \quad \text{and } \mathcal{F}_A = \text{diag}(f_0, f_1, \ldots, f_{k-1}).
\]

It is well known (see, e.g., [7]) that a standard 1-circulant \( A = [a_{s-r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k \times k} \) can be written as
\[
A = \Phi \mathcal{F}_A \Phi^* = \sum_{\ell=0}^{k-1} f_\ell \phi_\ell \phi_\ell^*.
\]

However, to our knowledge it has not been emphasized that \( \mathcal{F}_A \) plays a fundamental role in connection with all the standard circulants \( [a_{s-r}]_{r,s=0}^{k-1} \). (See Remark 3.)

In Section 2 we reformulate a result of Ablow and Brenner [1, Theorem 2.1] for standard \( \alpha \)-circulants to characterize \( \alpha \)-circulants in \( \mathbb{C}^{k \times d_1 \times d_2} \). We give a different characterization in Section 3: \( A = [A_{s-r}]_{r,s=0}^{k-1} \) if and only if \( A = U_\alpha \mathcal{F}_A P^* \), where \( U_\alpha \) and \( P \) are related to the Fourier matrix, \( P \) is unitary, and \( U_\alpha \) is invertible (in fact, unitary) if and only if \( \gcd(\alpha, k) = 1 \).
Since \( \mathcal{F}_A \) is independent of \( \alpha \), some computational results concerning \( \mathcal{F}_A \) apply simultaneously to all the proper \( \alpha \)-circulants \([A_{s}^{-\alpha}]_{r,s=0}^{k-1}\). For example, in Section 4 we show that

\[
A^{\dagger} = [B_{r^{-\alpha}s}]_{r,s=0}^{k-1} \quad \text{where} \quad B_{m} = \frac{1}{k} \sum_{\ell=0}^{k-1} \zeta^{\ell m} F_{\ell}^{\dagger}, \quad 0 \leq m \leq k - 1.
\]

We also prove the following for proper \( \alpha \)-circulants: (i) Solving \( Az = w \) reduces to solving \( F_\ell u_\ell = v_\alpha \ell \), \( 0 \leq \ell \leq k - 1 \), where \( v_0, v_1, \ldots, v_{k-1} \) depend only on \( w \). (ii) A singular value decomposition of \( A \) can be obtained from singular value decompositions of \( F_0, F_1, \ldots, F_{k-1} \). (iii) The least squares problem for \( A \) reduces to independent least squares problems for \( F_0, F_1, \ldots, F_{k-1} \). (iv) If \( d_1 = d_2 = d \), the eigenvalues of \([A_{s}^{-\alpha}]_{r,s=0}^{k-1}\) are the eigenvalues of \( F_0, F_1, \ldots, F_{k-1} \), and the corresponding eigenvectors of \( A \) are easily obtainable from those of \( F_0, F_1, \ldots, F_{k-1} \). (v) If \( d_1 = d_2 = d \) and \( \alpha > 1 \), the eigenvalue problem for \([A_{s}^{-\alpha}]_{r,s=0}^{k-1}\) reduces to eigenvalue problems for \( d \times d \) matrices related to \( F_0, F_1, \ldots, F_{k-1} \) in a manner depending upon \( \alpha \).

Block circulant 1-matrices \([A_{s}^{-\alpha}]_{r,s=0}^{k-1}\) have applications in preconditioning of block Toeplitz matrices; see, e.g. \([8, 9]\).

2 The Ablow–Brenner theorem revisited

Recall that \( E \) and \( \Phi \) are defined in (1) and (2). Let

\[
R = E \otimes I_{d_1}, \quad P_m = \phi_m \otimes I_{d_1}, \quad 0 \leq m \leq k - 1, \tag{8}
\]

\[
S = E \otimes I_{d_2}, \quad Q_m = \phi_m \otimes I_{d_2}, \quad 0 \leq m \leq k - 1, \tag{9}
\]

\[
P = \begin{bmatrix} P_0 & P_1 & \cdots & P_{k-1} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_0 & Q_1 & \cdots & Q_{k-1} \end{bmatrix}, \tag{10}
\]

and

\[
U_\alpha = \begin{bmatrix} P_0 & P_\alpha & P_{2\alpha} & \cdots & P_{(k-1)\alpha} \end{bmatrix}. \tag{11}
\]

Since

\[
P_{\ell}^{*} P_m = \delta_{\ell m} I_{d_1} \quad \text{and} \quad Q_{\ell}^{*} Q_m = \delta_{\ell m} I_{d_2}, \quad 0 \leq \ell, m \leq k - 1,
\]

\( P \) and \( Q \) are unitary, while \( U_\alpha \) is unitary if \( \gcd(\alpha, k) = 1 \); however, if \( \gcd(\alpha, k) = q > 1 \) and \( p = k/q \) then

\[
U_\alpha = \begin{bmatrix} P_0 P_\alpha \cdots P_{(p-1)\alpha} \cdots P_0 P_\alpha \cdots P_{(p-1)\alpha} \end{bmatrix}_{q}
\]

(i.e., the first \( p \) block columns are repeated \( q \) times) is not invertible. From (4) and (8)–(11),

\[
RP_{\ell} = \zeta^{\ell} P_{\ell} \quad \text{and} \quad SQ_{\ell} = \zeta^{\ell} Q_{\ell}, \quad 0 \leq \ell \leq k - 1. \tag{12}
\]

Ablow and Brenner [1, Theorem 2.1] showed that \( A \in \mathbb{C}^{k \times k} \) is a standard \( \alpha \)-circulant if and only if \( EAE^{-\alpha} = A \). We need the following adaptation of this result.
**Theorem 1** If \( A = [G_{rs}]_{r,s=0}^{k-1} \) with \( G_{rs} \in \mathbb{C}^{d_1 \times d_2} \), then \( RAS^{-\alpha} = A \) if and only if \( A \) is an \( \alpha \)-circulant; more precisely, if and only if
\[
G_{rs} = A_{s-\alpha r}, \quad 0 \leq r, s \leq k - 1, \tag{13}
\]
with
\[
A_s = G_{0s}, \quad 0 \leq s \leq k - 1. \tag{14}
\]

**Proof.** From (3), (8), and (9), \( RAS^{-\alpha} = [G_{r+1,s+\alpha}]_{r,s=0}^{k-1} \). Therefore we must show that (13) is equivalent to
\[
G_{r+1,s+\alpha} = G_{rs}, \quad 0 \leq r, s \leq k - 1. \tag{15}
\]
If (13) holds, then
\[
G_{r+1,s+\alpha} = A_{s+\alpha-(r+1)\alpha} = A_{s-\alpha r} = G_{rs}, \quad 0 \leq r, s \leq k - 1.
\]
For the converse we must show that (14) and (15) imply (13). We prove this by finite induction on \( r \). From (14),
\[
G_{rs} = A_{s-\alpha r}, \quad 0 \leq s \leq k - 1, \tag{16}
\]
if \( r = 0 \). Suppose (16) is true for some \( r \in \{0, \ldots, k - 2\} \). Replacing \( s \) by \( s - \alpha \) in (15) and (16) yields
\[
G_{r+1,s} = G_{r,s-\alpha}, \quad 0 \leq r, s \leq k - 1,
\]
and
\[
G_{r,s-\alpha} = A_{s-\alpha(r+1)}, \quad 0 \leq s \leq k - 1.
\]
Therefore
\[
G_{r+1,s} = A_{s-\alpha(r+1)}, \quad 0 \leq s \leq k - 1,
\]
which completes the induction. \( \Box \)

Theorem 1 with \( A = B^\ast \) yields the following corollary.

**Corollary 1** If \( B \in \mathbb{C}^{k \times d_2 \times d_1} \) then \( B \) is an \( \alpha \)-cocirculant if and only if \( S^\alpha BR^{-1} = B \).

The following corollary extends [10, Corollary 1].

**Corollary 2** (i) If \( A = [A_{r,s}]_{r,s=0}^{k-1} \in \mathbb{C}^{k \times d_1 \times d_2} \) and \( B = [B_{r,s}]_{r,s=0}^{k-1} \in \mathbb{C}^{k \times d_2 \times d_1} \), then \( AB = [C_{r,s}]_{r,s=0}^{k-1} \in \mathbb{C}^{k \times d_1 \times d_1} \) with \( C_m = \sum_{\ell=0}^{k-1} A_{r,s}B_{r,s} \), \( 0 \leq m \leq k - 1 \).
(ii) If \( \gcd(\alpha, k) = 1 \) and \( \alpha \beta \equiv 1 (\text{mod } k) \), then \( BA = [D_{r,s}]_{r,s=0}^{k-1} \in \mathbb{C}^{k \times d_2 \times d_2} \) with
\[
D_m = \sum_{\ell=0}^{k-1} B_{r,s}A_{r,s} \beta^\ell, \quad 0 \leq m \leq k - 1. \tag{17}
\]
Proof. (i) From Theorem 1 and Corollary 1, \( A = RAS^{-\alpha} \) and \( B = S^\alpha BR^{-1} \). Therefore \( AB = RABR^{-1} \), so Theorem 1 with \( R = S \) implies that \( AB \) is a 1-circulant. The stated formula for \( C_0, C_1, \ldots, C_{k-1} \) can be obtained by computing first block row entries of \( AB \).

(ii) Also, \( BA = S^\alpha BAS^{-\alpha} \). Applying this \( \beta \) times yields \( BA = SBAS^{-1} \), so Theorem 1 with \( R = S \) implies that \( BA \) is a 1-circulant. Computing the first block row entries of \( BA \) yields \( D_m = \sum_{\ell=0}^{k-1} B_{-\alpha\ell}A_{m-\alpha\ell} \) and replacing \( \ell \) by \( -\beta\ell \) yields (17).

Theorem 2

If

\[
A = [A_{s-\alpha r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_1 \times d_2}
\]

and

\[
B = [B_{s-\alpha_2 r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_2 \otimes d_3},
\]

then

\[
AB = [C_{s-\alpha_1 \alpha_2 r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_1 \otimes d_3},
\]

with

\[
C_m = \sum_{\ell=0}^{k-1} A_{\ell} B_{m-\alpha_2 \ell}, \quad 0 \leq m \leq k - 1.
\]

Proof. Let \( R = E \otimes I_{d_1} \), \( S = E \otimes I_{d_2} \), and \( T = E \otimes I_{d_3} \). From (18), (19), and Theorem 1,

(a) \( A = RAS^{-\alpha_1} \) and (b) \( B = SB^{-\alpha_2} \).

Applying (b) \( \alpha_1 \) times yields \( B = S^{\alpha_1} BT^{-\alpha_1 \alpha_2} \). From this and (a), \( RAB^{-\alpha_1 \alpha_2} = AB \). Now Theorem 1 implies (20), with (21) obtained by computing the entries in the first block row of \( AB \).

Theorem 2 generalizes [1, Theorem 3.1]; namely, the product of a standard \( \alpha \)-circulant and a standard \( \beta \)-circulant is an \( \alpha\beta \)-circulant. However, [1] does not include (21).

3 A DFT characterization of \( \alpha \)-circulants

Theorem 3

A matrix \( A \in \mathbb{C}^{k:d_1 \times d_2} \) is an \( \alpha \)-circulant \( A = [A_{s-\alpha r}]_{r,s=0}^{k-1} \) if and only if it can be written as

\[
A = \sum_{\ell=0}^{k-1} P_{\alpha\ell} F_{\ell} Q_{\ell}^*,
\]

where \( \{F_0, F_1, \ldots, F_{k-1}\} \) and \( \{A_0, A_1, \ldots, A_{k-1}\} \) are related as in (5) and (6) and \( P, Q, \) and \( U_{\alpha} \) are as in (8) – (11).
Proof. Eqns. (7)–(11) imply the second equality in (22). Therefore we need only justify the first. Suppose \( A = [a_{s-\alpha}]_{r,s=0}^{k-1} \) and define \( F_0, F_1, \ldots, F_{k-1} \) by (5). From (6),

\[
A_{s-\alpha r} = \frac{1}{k} \sum_{\ell=0}^{k-1} \zeta^{-\ell(s-\alpha)} F_\ell, \quad 0 \leq r, s \leq k - 1,
\]

so (8)–(11) imply that

\[
A = \frac{1}{k} \sum_{\ell=0}^{k-1} \begin{bmatrix} 1 \otimes I_{d_1} & \zeta^{\alpha \ell} \otimes I_{d_1} & \cdots & \zeta^{(k-1)\alpha \ell} \otimes I_{d_1} \\ \zeta^{(k-1)\alpha \ell} \otimes I_{d_2} & \zeta^{\alpha \ell} \otimes I_{d_2} & \cdots & \zeta^{(k-1)\alpha \ell} \otimes I_{d_2} \end{bmatrix} H F_\ell = \sum_{\ell=0}^{k-1} P_{\alpha \ell} F_\ell Q_\ell^*.
\]

For the converse, if (22) holds then (12) implies that

\[
RAS^{-\alpha} = \sum_{\ell=0}^{k-1} (RP_{\alpha \ell}) F_\ell (S^\alpha Q_\ell)^* = \sum_{\ell=0}^{k-1} (\zeta^{\alpha \ell} P_{\alpha \ell}) F_\ell (\zeta^{-\alpha \ell} Q_\ell^*) = A.
\]

Therefore \( A \) is an \( \alpha \)-circulant, by Theorem 1; hence, \( A = [a_{s-\alpha}]_{r,s=0}^{k-1} \) with \( A_0, A_1, \ldots, A_{k-1} \) as in (6). \( \square \)

Remark 3 Theorem 3 implies that \( A \in \mathbb{C}^{k \times k} \) is a standard \( \alpha \)-circulant \([a_{s-\alpha}]_{r,s=0}^{k-1}\) if and only if

\[
A = \Phi_0 \mathcal{F} A \Phi^* = \sum_{\ell=0}^{k-1} f_\ell \phi_{\alpha \ell} \phi_{\alpha \ell}^*,
\]

where \( \Phi \) is as in (1), \( \Phi_\alpha = \begin{bmatrix} \phi_0 & \phi_\alpha & \cdots & \phi_{(k-1)\alpha} \end{bmatrix} \), and

\[
f_\ell = \sum_{m=0}^{k-1} a_m \zeta^{\ell m}, \quad 0 \leq \ell \leq k - 1.
\]

Corollary 3 A matrix \( B \in \mathbb{C}^{d_1 \times d_1} \) is an \( \alpha \)-cocirculant if and only if it can be written as \( B = \sum_{\ell=0}^{k-1} Q_\ell G_\ell P_{\alpha \ell}^* \), where

\[
G_\ell = \sum_{m=0}^{k-1} \zeta^{-\ell m} B_m, \quad 0 \leq \ell \leq k - 1, \quad \text{and} \quad B_m = \frac{1}{k} \sum_{\ell=0}^{k-1} \zeta^{\ell m} G_\ell, \quad 0 \leq m \leq k - 1.
\]

Proof. Apply Theorem 3 to \( B^* \). \( \square \)

It is well known that standard 1-circulants commute. The following corollary extends this.

Corollary 4 Suppose \( d_1 = d_2 \), \( \gcd(\alpha, k) = 1 \), and \( \alpha \beta \equiv 1 \pmod{k} \). Let \( A = [a_{s-\alpha}]_{r,s=0}^{k-1}, B = [b_{s-\beta}]_{r,s=0}^{k-1}, \)

\[
F_\ell = \sum_{m=0}^{k-1} \zeta^{\ell m} A_m \quad \text{and} \quad G_\ell = \sum_{m=0}^{k-1} \zeta^{\ell m} B_m.
\]

Then \( AB = BA \) if and only if \( F_{\beta \ell} G_\ell = G_\alpha \ell F_\ell, 0 \leq \ell \leq k - 1. \)
Proof. Since gcd(α, k) = gcd(β, k) = 1, we may change summation indices ℓ → αℓ and ℓ → βℓ. Therefore, from Theorem 3 with Q = P,

\[ A = \sum_{\ell=0}^{k-1} P_{\alpha \ell} F_{\ell} P_{\ell}^*, \quad B = \sum_{\ell=0}^{k-1} P_{\beta \ell} G_{\ell} P_{\ell}^*, \quad AB = \sum_{\ell=0}^{k-1} P_{\ell} F_{\ell} G_{\ell} P_{\ell}^*, \quad \text{and} \quad BA = \sum_{\ell=0}^{k-1} P_{\ell} G_{\ell} F_{\ell} P_{\ell}^*, \]

which implies the conclusion.

4 Moore-Penrose inversion and singular value decomposition

Recall that the Moore-Penrose inverse \( X^\dagger \) of a matrix \( X \) is the unique matrix \( Y \) that satisfies the Penrose conditions

\[(XY)^* = XY, \quad (YX)^* = YX, \quad XYX = X, \quad \text{and} \quad YXY = Y.\]

**Theorem 4** The Moore–Penrose inverse of an \( \alpha \)-circulant is an \( \alpha \)-cocirculant.

Proof. From Theorem 1, if \( A \) is an \( \alpha \)-circulant then \( A = RAS^{-\alpha} \). Let \( B = S^\alpha A^\dagger R^{-1} \). We will show that \( A \) and \( B \) satisfy the Penrose conditions:

\[ AB = (RAS^{-\alpha})(S^\alpha A^\dagger R^{-1}) = RAA^\dagger R^* = R(AA^\dagger)^* R^* = (AB)^*, \]

\[ BA = (S^\alpha A^\dagger R^{-1})(RAS^{-\alpha}) = S^\alpha A^\dagger A(S^\alpha)^* = S^\alpha (A^\dagger A)^* (S^\alpha)^* = (BA)^*, \]

\[ ABA = (RAA^\dagger R^{-1})(RAS^{-\alpha}) = R(AA^\dagger A)S^{-\alpha} = RAS^{-\alpha} = A, \]

and

\[ BAB = (S^\alpha A^\dagger AS^{-\alpha})(S^\alpha A^\dagger R^{-1}) = S^\alpha (A^\dagger AA^\dagger) R^{-1} = S^\alpha A^\dagger R^{-1} = B. \]

Therefore \( B = A^\dagger \) or, equivalently, \( S^\alpha A^\dagger R^{-1} = A^\dagger \). Now Corollary 1 implies that \( A^\dagger \) is an \( \alpha \)-cocirculant.

We can be more explicit if gcd(\( \alpha, k \)) = 1.

**Theorem 5** The Moore–Penrose inverse of a proper \( \alpha \)-circulant \( A = [A_{s-\alpha}]_{r,s=0}^{k-1} \) is the \( \alpha \)-cocirculant \( B = [B_{r-\alpha}]_{r,s=0}^{k-1} \), where

\[ B_m = \frac{1}{k} \sum_{\ell=0}^{k-1} \zeta^{\ell m} F_{\ell}^\dagger, \quad 0 \leq m \leq k - 1, \quad (23) \]

with

\[ F_{\ell} = \sum_{m=0}^{k-1} \zeta^{\ell m} A_m, \quad 0 \leq \ell \leq k - 1. \]
let define

\[ A^\dagger = B = \sum_{\ell=0}^{k-1} Q_{\ell} F_{\ell} P_{\alpha_{\ell}} = \sum_{\ell=0}^{k-1} (\phi_{\ell} \otimes I_{d_2}) F_{\ell} (\phi_{\alpha_{\ell}} \otimes I_{d_1})^* \]

\[ = \frac{1}{k} \left[ \sum_{\ell=0}^{k-1} \zeta^{\ell(r-s)} F_{\ell} \right]_{r,s=0}^{k-1} = [B_{r,\alpha_{\ell}}]_{r,s=0}^{k-1} \]

from (8)–(11) and (23). 

\[ \text{Remark 4} \quad \text{Theorem 5 can also be proved by using (6) and (23) to express the entries of } AB, BA, ABA, \text{ and } BAB \text{ explicitly in terms of } F_0, F_1, \ldots, F_{k-1} \text{ and } F_0^\dagger, F_1^\dagger, \ldots, F_{k-1}^\dagger, \text{ noting that} \]

\[ \sum_{\ell=0}^{k-1} \zeta^{\ell(r-s)} = \sum_{\ell=0}^{k-1} \zeta^{\ell(r-s)} = \delta_{rs}, \quad 0 \leq r, s \leq k-1, \]

the latter because gcd(\(\alpha, k\)) = 1. However, this is tedious.

\[ \text{Remark 5} \quad \text{Theorem 5 extends a result of Davis [6]: If } A = [a_{s-\alpha}]_{r,s=0}^{k-1} \in \mathbb{C}^{k \times k} \text{ then } A^\dagger = \Phi \text{ diag}(a_0^\dagger, a_1^\dagger, \ldots, a_{k-1}^\dagger) \Phi^*, \text{ where } \Phi \text{ is the Fourier matrix (1), } 0^\dagger = 0, \text{ and } a^\dagger = 1/a \text{ if } a \neq 0. \]

\[ \text{Theorem 6} \quad \text{Suppose gcd}(\alpha, k) = 1 \text{ and} \]

\[ A = [a_{s-\alpha}]_{r,s=0}^{k-1} = \sum_{\ell=0}^{k-1} P_{\alpha_{\ell}} F_{\ell} Q_{\ell}^* = U_{\alpha} F_{A} Q^*. \]

Let \( F_{\ell} = \Omega_{\ell} \Sigma_{\ell} \Psi_{\ell}^* \) be a singular value decomposition of \( F_{\ell}, 0 \leq \ell \leq k-1, \) and define

\[ M_{\alpha} = \left[ \begin{array}{cccc} P_0 \Omega_0 & P_1 \Omega_1 & \cdots & P_{(k-1)\alpha} \Omega_{k-1} \end{array} \right] \]

and

\[ N = \left[ \begin{array}{cccc} Q_0 \Psi_0 & Q_1 \Psi_1 & \cdots & Q_{k-1} \Psi_{k-1} \end{array} \right]. \]
Then

$$A = M_{\alpha} \left( \bigoplus_{\ell=0}^{k-1} \Sigma_{\ell} \right) N^*$$

is a singular value decomposition of $A$, except that the singular values are not necessarily ordered.

5 The least squares problem

Suppose $G \in \mathbb{C}^{d_1 \times d_2}$ and consider the least squares problem for $G$: If $v \in \mathbb{C}^{d_1}$, find $u \in \mathbb{C}^{d_2}$ such that

$$\|Gu - v\| = \min_{\xi \in \mathbb{C}^{d_2}} \|G\xi - v\|, \quad (24)$$

where $\| \cdot \|$ is the 2-norm. This problem has a unique solution if and only if $\text{rank}(G) = d_2$, in which case $u = (G^*G)^{-1}G^*v$. In any case, the optimal solution of (24) is the unique $u_0 \in \mathbb{C}^{d_2}$ of minimum norm that satisfies (24); thus, $u_0 = G^1v$. The general solution of (24) is $u = u_0 + q$ where $Gq = 0$, and

$$\|Gu - v\| = \|(G G^1 - I)v\|$$

for all such $u$.

Now consider the following least squares problem: if $A = [A_{s-r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_1 \times d_2}$ with $\gcd(\alpha, k) = 1$ and $w \in \mathbb{C}^{kd_1}$, find $z \in \mathbb{C}^{kd_2}$ such that

$$\|Az - w\| = \min_{\xi \in \mathbb{C}^{kd_2}} \|A\xi - w\|.$$

We write

$$z = \sum_{\ell=0}^{k-1} Q_{\ell} u_{\ell} \quad \text{and} \quad w = \sum_{\ell=0}^{k-1} P_{\ell} v_{\ell} = \sum_{\ell=0}^{k-1} P_{\alpha \ell} v_{\alpha \ell}, \quad (25)$$

since substituting $\alpha \ell$ for $\ell$ is legitimate because $\gcd(\alpha, k) = 1$. Since $A = \sum_{\ell=0}^{k-1} P_{\alpha \ell} F_\ell Q_\ell^*$ and $Q_\ell^* Q_m = \delta_{\ell m} I_{kd_2}$,

$$Az - w = \sum_{\ell=0}^{k-1} P_{\alpha \ell} (F_\ell u_{\ell} - v_{\alpha \ell}).$$

Since $P_{\alpha \ell} P_{\alpha m} = \delta_{\ell m} I_{d_1}$ (because $\gcd(\alpha, k) = 1$), it follows that

$$\|Az - w\|^2 = \sum_{\ell=0}^{k-1} \|F_\ell u_{\ell} - v_{\alpha \ell}\|^2. \quad (26)$$

This implies the following theorem.
Theorem 7 Suppose $A$ is a proper $\alpha$-circulant and let $z$ and $w$ be as in (25). Then
\[ \|Az - w\| = \min_{\xi \in \mathbb{C}^{kd}} \|A\xi - w\| \] (27)
if and only if
\[ \|F_\ell u_\ell - v_{\alpha\ell}\| = \min_{\psi_\ell \in \mathbb{C}^{d_2}} \|F_\ell \psi_\ell - v_{\alpha\ell}\|, \quad 0 \leq \ell \leq k - 1. \]

Therefore (27) has a unique solution, given by
\[ z = \sum_{\ell=0}^{k-1} Q_\ell (F_\ell^* F_\ell)^{-1} F_\ell^* v_{\alpha\ell}, \]
if and only rank$(F_\ell) = d_2$, $0 \leq \ell \leq k - 1$. In any case, the optimal solution of (27) is
\[ z_0 = \sum_{\ell=0}^{k-1} Q_\ell F_\ell^* v_{\alpha\ell}. \]
The general solution of (27) is $z = z_0 + \sum_{\ell=0}^{k-1} Q_\ell u_\ell$, where $F_\ell u_\ell = 0$, $0 \leq \ell \leq k - 1$, and
\[ \|Az - w\|^2 = \sum_{\ell=0}^{k-1} \|(F_\ell F_\ell^*)^{-1} - I_{d_2}) v_{\alpha\ell}\|^2 \]
for all such $z$.

6 The case where $d_1 = d_2$

Throughout this section $d_1 = d_2 = d$ and $A = [A_{s-r}]_{r,s=0}^{k-1}$ is a proper circulant. Then (26) implies the following theorem, which reduces the problem of solving the $kd \times kd$ system $Az = w$ to solving $k$ independent $d \times d$ systems.

Theorem 8 If $A$ is a proper $\alpha$-circulant, $z = \sum_{\ell=0}^{k-1} P_\ell u_\ell$, and $w = \sum_{\ell=0}^{k-1} P_\ell v_\ell$, then $Az = w$ if and only if $F_\ell u_\ell = v_{\alpha\ell}$, $0 \leq \ell \leq k - 1$.

This and Theorem 5 imply the following theorem.

Theorem 9 A proper $\alpha$-circulant
\[ A = [A_{s-r}]_{r,s=0}^{k-1} = \sum_{\ell=0}^{k-1} P_{\alpha\ell} F_\ell^* \] (28)
is invertible if and only $F_0, F_1, \ldots, F_{k-1}$ are all invertible. In this case
\[ A^{-1} = [B_{r-\alpha}]_{r,s=0}^{k-1} \quad \text{with} \quad B_m = \frac{1}{k} \sum_{\ell=0}^{k-1} \zeta_{\ell m} F_\ell^{-1}, \quad 0 \leq m \leq k - 1, \]
and the solution of $Az = w$ is $z = \sum_{\ell=0}^{k-1} P_\ell F_\ell^{-1} v_{\alpha\ell}$.
Remark 6 Theorem 9 and Remark 2 extend [5, Theorem 1]: the inverse of a standard nonsingular \( \alpha \)-circulant is a \( \beta \)-circulant, where \( \alpha \beta \equiv 1 \) (mod \( k \)).

**Theorem 10** Suppose \( A \) is a proper \( \alpha \)-circulant as in (28) and \( \alpha \beta \equiv 1 \) (mod \( k \)).

(i) \( A \) is Hermitian if and only if \( P_{\beta \ell} F_{\beta \ell}^{*} P_{\beta \ell} = P_{\alpha \ell} F_{\ell} \), \( 0 \leq \ell \leq k - 1 \).

(ii) \( A \) is normal if and only if \( F_{\beta \ell} F_{\beta \ell}^{*} = F_{\ell}^{*} F_{\ell} \), \( 0 \leq \ell \leq k - 1 \).

(iii) \( A \) is EP (i.e., \( A^{\dagger} A = A A^{\dagger} \)) if and only if \( F_{\beta \ell} F_{\beta \ell}^{*} = F_{\ell}^{*} F_{\ell} \), \( 0 \leq \ell \leq k - 1 \).

**Proof.**  From (28) and Theorem 5,

\[
A = \sum_{\ell=0}^{k-1} P_{\alpha \ell} F_{\ell} P_{\alpha \ell}^{*}, \quad A^{*} = \sum_{\ell=0}^{k-1} P_{\beta \ell} F_{\beta \ell}^{*} P_{\beta \ell}^{*}, \quad \text{and} \quad A^{\dagger} = \sum_{\ell=0}^{k-1} P_{\ell} F_{\ell}^{*} P_{\alpha \ell}, \quad (29)
\]

(i) Since \( \alpha \beta \equiv 1 \) (mod \( k \)), replacing \( \ell \) by \( \beta \ell \) in the second sum in (29) yields \( A^{*} = \sum_{\ell=0}^{k-1} P_{\beta \ell} F_{\beta \ell}^{*} P_{\beta \ell}^{*} \), and comparing this with the first sum in (29) yields (i).

(ii) From (29),

\[
A A^{*} = \sum_{\ell=0}^{k-1} P_{\alpha \ell} F_{\ell} F_{\ell}^{*} P_{\alpha \ell}^{*} = \sum_{\ell=0}^{k-1} P_{\beta \ell} F_{\beta \ell} F_{\beta \ell}^{*} P_{\beta \ell}^{*} \quad \text{and} \quad A^{*} A = \sum_{\ell=0}^{k-1} P_{\ell} F_{\ell}^{*} F_{\ell} P_{\ell}^{*},
\]

which implies (ii).

(iii) From (29),

\[
A A^{\dagger} = \sum_{\ell=0}^{k-1} P_{\alpha \ell} F_{\ell} F_{\ell}^{\dagger} P_{\alpha \ell}^{*} = \sum_{\ell=0}^{k-1} P_{\beta \ell} F_{\beta \ell} F_{\beta \ell}^{\dagger} P_{\beta \ell}^{*} \quad \text{and} \quad A^{\dagger} A = \sum_{\ell=0}^{k-1} P_{\ell} F_{\ell}^{\dagger} F_{\ell} P_{\ell}^{*},
\]

which implies (iii). \( \square \)

**Remark 7** If \( A \) is a square matrix and there is a matrix \( B \) such that \( A B A = A \), \( B A B = B \), and \( A B = B A \), then \( B \) is unique and is called the group inverse of \( A \), which is usually denoted by \( A^{\#} \). Davis [6] noted that if \( A \in \mathbb{C}^{k \times k} \) is a standard 1-circulant then \( A^{\dagger} = A^{\#} \). Theorem 10(iii) extends this: If \( A \in \mathbb{C}^{k \times d} \) is a proper \( \alpha \)-circulant and \( \alpha \beta \equiv 1 \) (mod \( k \)), then \( A^{\dagger} = A^{\#} \) if and only if \( F_{\ell}^{\dagger} F_{\ell} = F_{\beta \ell} F_{\beta \ell}^{\dagger} \), \( 0 \leq \ell \leq k - 1 \).

7 The eigenvalue problem with \( \alpha = 1 \)

In this section we assume that \( \alpha = 1 \) and \( d_{1} = d_{2} = d \). The following theorem and its proof are motivated by [2, Theorem 2].

**Theorem 11** Let

\[
S \mathcal{R} = \bigcup_{\ell=0}^{k-1} \{ z \mid R z = \zeta^{\ell} z \}.
\]
If $\lambda$ is an eigenvalue of $A$, let $E_A(\lambda)$ be the $\lambda$-eigenspace of $A$; i.e.,

$$E_A(\lambda) = \{ z \mid Az = \lambda z \}.$$ 

(i) If $\lambda$ is an eigenvalue of $A = [A_{s-r}]_{r,s=0}^{k-1}$ then $E_A(\lambda)$ has a basis in $S_R$.

(ii) If $A \in \mathbb{C}^{k:d \times d}$ and has $kd$ linearly independent eigenvectors in $S_R$, then $A$ is a 1-circulant.

**Proof.** (i) From Theorem 8, $z = \sum_{\ell=0}^{k-1} P_{\ell} u_{\ell} \in E_A(\lambda)$ if and only if $F_{\ell} u_{\ell} = \lambda u_{\ell}, \quad 0 \leq \ell \leq k-1$. Therefore $\lambda$ is an eigenvalue of $A$ if and only if it is an eigenvalue of $F_\ell$ for some $\ell \in \{0,1,\ldots,k-1\}$. Let $T_\lambda$ be the subset of $\{0,1,\ldots,k-1\}$ for which this is true. Then $E_A(\lambda)$ consists of linear combinations of the vectors of the form $P_{\ell} u_{\ell}$ with $\ell \in T_\lambda$ and $(\lambda, u_{\ell})$ an eigenpair of $F_\ell$. Since $RP_\ell = \zeta^\ell P_\ell$ (recall (12)), this completes the proof of (i).

(ii) From Theorem 1, we must show that $RA = AR$. If $Az = \lambda z$ and $Rz = \zeta^s z$ then $RAz = \lambda Rz = \lambda \zeta^s z$ and $ARz = \zeta^s Az = \zeta^s \lambda z$. Hence $ARz = RAz$ for all $z$ in a basis for $\mathbb{C}^{k:d \times d}$, so $AR = RA$.

**Theorem 12** Let $R$ and $P$ be as in (8) and (10). Then the 1-circulant $A = [A_{s-r}]_{r,s=0}^{k-1}$ is diagonalizable if and only if $F_0, F_1, \ldots, F_{k-1}$ are all diagonalizable. In this case, if

$$F_\ell = T_\ell D_\ell T_\ell^*, \quad 0 \leq \ell \leq k-1,$$

are spectral decompositions of $F_0, F_1, \ldots, F_{k-1}$ and

$$\Psi = \begin{bmatrix} P_0 T_0 & P_1 T_1 & \cdots & P_{k-1} T_{k-1} \end{bmatrix},$$

then

$$A = \Psi \begin{bmatrix} D_0 \oplus D_1 \oplus \cdots \oplus D_{k-1} \end{bmatrix} \Psi^*$$

is a spectral decomposition of $A$.

8 The eigenvalue problem with $\alpha > 1$

In this section we assume that $d_1 = d_2 = d, \alpha \in \{2,3,\ldots,k-1\}$ and, gcd($\alpha,k) = 1$. From Theorem 8, $Az = \lambda z$ if and only if $z = \sum_{s=0}^{k-1} P_s u_s$, where

$$F_s u_s = \lambda u_{\alpha s}, \quad 0 \leq s \leq k-1.$$ (30)

Therefore $Az = 0$ if and only if $z = \sum_{s=0}^{k-1} P_s u_s$ where $F_s u_s = 0, 0 \leq s \leq k-1$, so the makeup of the null space of $A$ is transparent. Hence, we assume that $\lambda \neq 0$. Then we must consider the orbits of the permutation on $\{0,\ldots,k-1\}$ defined by $s \rightarrow \alpha s \pmod{k}$. We consider an example before presenting the general discussion.
Let $k = 10$ and $\alpha = 3$. The permutation of $\{0, 1, \ldots, 9\}$ defined by $s \to 3s \pmod{10}$ is given by
\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
0 & 3 & 6 & 9 & 2 & 5 & 8 & 1 & 4 & 7
\end{pmatrix}.
\]
The orbits of this permutation are
\[
O_0 = \{0\}, \quad O_1 = \{1, 3, 9, 7\}, \quad O_2 = \{2, 6, 8, 4\}, \quad \text{and} \quad O_3 = \{5\}.
\]
Therefore (30) divides into four independent systems:

(i) \(F_0u_0 = \lambda u_0\); (ii) \(F_1u_1 = \lambda u_3\), \(F_3u_3 = \lambda u_9\), \(F_9u_9 = \lambda u_7\), \(F_7u_7 = \lambda u_1\),

(iii) \(F_5u_5 = \lambda u_5\); (iv) \(F_2u_2 = \lambda u_6\), \(F_6u_6 = \lambda u_8\), \(F_8u_8 = \lambda u_4\), \(F_4u_4 = \lambda u_2\).

From (i), if \((\lambda, u_0)\) is an eigenpair of \(F_0\) then \((\lambda, P_0u_0)\) is an eigenpair of \(A\).

Similarly, from (iii), if \((\lambda, u_5)\) is an eigenpair of \(F_5\) then \((\lambda, P_5u_5)\) is an eigenpair of \(A\).

The analysis of (ii) and (iv) is more complicated, but identical. We will consider (ii), which is equivalent to

\[
\begin{align*}
(31) & \quad u_3 = \frac{1}{\lambda}F_1u_1, \quad u_9 = \frac{1}{\lambda}F_3u_3, \quad u_7 = \frac{1}{\lambda}F_9u_9, \quad u_1 = \frac{1}{\lambda}F_7u_7, \\
(32) & \quad \text{where} \quad G_3 = F_1, \quad G_9 = F_3F_1, \quad G_7 = F_9F_3F_1, \quad \text{and} \quad G_1 = F_7F_9F_3F_1.
\end{align*}
\]

In particular, the last equalities in (32) and (33) are equivalent to \(G_1u_1 = \lambda^4u_1\).

Therefore, if \((\gamma, u_1)\) is an eigenpair of \(G_1\) and \(\gamma \neq 0\), then \(\lambda = \gamma^{1/4}\) is an eigenvalue of \(A\) with the associated eigenvector

\[
\begin{align*}
z & = \left( P_1 + \gamma^{-1/4}P_3G_3 + \gamma^{-2/4}P_9G_9 + \gamma^{-3/4}P_7G_7 \right)u_1 \\
& = \left( P_1 + \sum_{m=1}^{3} \gamma^{-m/4} P_{3^m} G_{3^m} \right)u_1. \quad (34)
\end{align*}
\]

(Recall that subscripts are taken modulo 10.) However, \(\gamma^{1/4}e^{2\pi ir/4}, 0 \leq r \leq 3\), are all fourth roots of \(\gamma\) and therefore eigenvalues of \(A\). Replacing \(\gamma^{1/4}\) with \(\gamma^{1/4}e^{2\pi ir/4}\) in (34) shows that

\[
\begin{align*}
z_r & = \left( P_1 + \sum_{m=1}^{3} \gamma^{-m/4} e^{-2\pi irm/4} P_{3^m} G_{3^m} \right)u_1, \quad 0 \leq r \leq 3, \quad (35)
\end{align*}
\]

are the respective associated eigenvectors of \(A\).
Now suppose the permutation \( s \mapsto \alpha s \pmod{k} \) of \( \{0, 1, \ldots, k-1\} \) has \( p \) orbits \( \mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_{p-1} \), and let

\[
0 = s_0 < s_1 < s_2 < \cdots < s_{p-1} \quad \text{with} \quad s_\ell \in \mathcal{O}_\ell, \quad 0 \leq \ell \leq p - 1.
\]

Suppose \( \mathcal{O}_\ell \) has \( r_\ell \) distinct members; thus,

\[
\mathcal{O}_\ell = \{ s_\ell, \alpha s_\ell, \ldots, \alpha^{r_\ell - 1} s_\ell \} \quad \text{where} \quad \alpha^{r_\ell} \equiv 1 \pmod{k}, \quad (36)
\]

and \( \bigcup_{\ell=0}^{p-1} \mathcal{O}_\ell = \{0, 1, \ldots, k-1\} \). If \( r_\ell = 1 \) and \((\lambda, u_{s_\ell})\) is an eigenpair of \( F_{s_\ell} \), then \((\lambda, P_{s_\ell} u_{s_\ell})\) is an eigenpair of \( A \). Now consider an orbit \( \mathcal{O}_\ell \) with \( r_\ell > 1 \), such as \( \mathcal{O}_2 \) and \( \mathcal{O}_4 \) in the example. The system associated with \( \mathcal{O}_\ell \) is

\[
F_{\alpha^r s_\ell} u_{\alpha^r s_\ell} = \lambda u_{\alpha^r s_\ell}, \quad 0 \leq r \leq r_\ell - 1, \quad \text{where} \quad \alpha^{r_\ell} = 1,
\]

which is analogous to (ii), where \( s_\ell = 1, \alpha = 3 \) and \( k = 10 \). Since \( \lambda \neq 0 \), this is equivalent to

\[
u_{\alpha^r s_\ell} = \frac{1}{\lambda} F_{\alpha^r s_\ell} u_{\alpha^r s_\ell}, \quad 0 \leq r \leq r_\ell - 1,
\]

which is analogous to (31). Therefore

\[
u_{\alpha^r s_\ell} = \frac{1}{\lambda} F_{\alpha^r s_\ell} G_{\alpha^r s_\ell} u_{s_\ell}, \quad 0 \leq r \leq r_\ell - 1,
\]

where

\[
G_{\alpha^r s_\ell} = F_{\alpha^r s_\ell} \cdots F_{s_\ell}, \quad 0 \leq r \leq r_\ell - 1,
\]

which is analogous to (32) and (33). In particular, setting \( r = r_\ell - 1 \) and noting that \( \alpha^{r_\ell} s_\ell = s_\ell \) yields

\[
u_{s_\ell} = \frac{1}{\lambda^{r_\ell}} G_{s_\ell} u_{s_\ell} \quad \text{where} \quad G_{s_\ell} = F_{\alpha^{r_\ell - 1} s_\ell} \cdots F_{s_\ell}.
\]

Therefore, if \((\gamma_{s_\ell}, u_{s_\ell})\) is an eigenvalue of \( G_{s_\ell} \), then \( \gamma_\ell^{1/r_\ell} \) is an eigenvalue of \( A \) with associated eigenvector

\[
z_\ell = \left( P_{s_\ell} + \sum_{m=1}^{r_\ell - 1} \gamma_\ell^{-m/r_\ell} P_{\alpha^m s_\ell} G_{\alpha^m s_\ell} \right) u_{s_\ell},
\]

which is analogous to (34). However, since \( \gamma_\ell^{1/r_\ell} e^{2\pi i m/r_\ell} \) are all \( r_\ell \)-th roots of \( \gamma \), they are all eigenvalues of \( A \). Replacing \( \gamma_\ell^{1/r_\ell} \) with \( \gamma_\ell^{1/r_\ell} e^{2\pi i m/4} \) in (38) yields associated eigenvectors

\[
z_{r_\ell} = \left( P_{s_\ell} + \sum_{m=1}^{r_\ell - 1} \gamma_\ell^{-m/r_\ell} e^{-2\pi i m/r_\ell} P_{\alpha^m s_\ell} G_{\alpha^m s_\ell} \right) u_{s_\ell}, \quad 0 \leq r \leq r_\ell - 1,
\]

which is analogous to (35).
Remark 8 Now we apply the preceding argument to a standard $\alpha$-circulant $A = [a_{\alpha r}]_{r,s=0}^{k-1}$ with $\text{gcd}(\alpha, k) = 1$. From Remark 3,

$$A = \sum_{s=0}^{k-1} f_s \phi_{\alpha s} \phi_s^* \quad \text{with} \quad f_s = \sum_{r=0}^{k-1} a_r \zeta_r^s, \quad 0 \leq r \leq k - 1,$$

and $\phi_0, \phi_1, \ldots, \phi_{k-1}$ as in (1). Then $z = \sum_{s=0}^{k-1} u_s \phi_s$ is $\lambda$-eigenvector of $A$ if and only if $f_s u_s = \lambda u_s$, $0 \leq s \leq k - 1$. Let $\mathcal{O}_\ell$ be as in (36) and assume that $f_{\alpha r s} \neq 0$, $0 \leq r \leq r_\ell - 1$. Let

$$g_{\alpha r^{+1}s} = \prod_{q=0}^{r} f_{\alpha q s}, \quad 0 \leq r \leq r_\ell - 1,$$

and

$$\gamma_\ell = g_{\alpha s^r} = f_{\alpha r^{+1}s} \cdots f_{s}.$$

From (37),

$$u_{\alpha r^{+1}s} = \frac{1}{\lambda^{r+1}} g_{\alpha r^{+1}s} u_s, \quad 0 \leq r \leq r_\ell - 2, \quad \text{and} \quad u_{\alpha s^r} = u_s = \lambda^{-r} \gamma_\ell u_s.$$

Therefore $\gamma_\ell e^{2\pi ir/r_\ell}$, $0 \leq r \leq r_\ell - 1$, are eigenvalues of $A$. From (39),

$$z_r = \left( \phi_{s} + \sum_{m=1}^{r-1} \gamma_\ell^{-m/r_\ell} e^{-2\pi irm/r_\ell} g_{\alpha m} \phi_{\alpha m} s \right), \quad 0 \leq r \leq r_\ell - 1,$$

are associated eigenvectors.

For example, let $\alpha = k - 1$, so $A = [a_{\alpha r}]_{r,s=0}^{k-1}$. If $k = 2p$ then $v_0 = f_0$, $v_\ell = \sqrt{f_\ell f_{k-\ell}}$, $1 \leq \ell \leq p - 1$, and $v_p = f_p$. Hence, $(f_0, \phi_0), (f_p, \phi_p)$,

$$\left( \sqrt{f_\ell f_{k-\ell}}, \phi_\ell + \frac{1}{\sqrt{f_\ell f_{k-\ell}}} \phi_{(k-1)\ell} \right), \quad \text{and} \quad \left( -\sqrt{f_\ell f_{k-\ell}}, \phi_\ell - \frac{1}{\sqrt{f_\ell f_{k-\ell}}} \phi_{(k-1)\ell} \right)$$

$1 \leq \ell \leq p - 1$, are eigenpairs of $A$. If $k = 2p + 1$ then $v_0 = f_0$ and $v_\ell = f_\ell f_{k-\ell}$, $1 \leq \ell \leq q$. Hence $(f_0, \phi_0), (f_p, \phi_p)$,

$$\left( \sqrt{f_\ell f_{k-\ell}}, \phi_\ell + \frac{1}{\sqrt{f_\ell f_{k-\ell}}} \phi_{(k-1)\ell} \right), \quad \text{and} \quad \left( -\sqrt{f_\ell f_{k-\ell}}, \phi_\ell - \frac{1}{\sqrt{f_\ell f_{k-\ell}}} \phi_{(k-1)\ell} \right)$$

$1 \leq \ell \leq p$, are eigenpairs of $A$.

The eigenvalues of $A$ were given in [5] without the associated eigenvectors.

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References


