Lin. Alg. Appl. 431 (2009), 1833-1847

Properties of multilevel block $\underline{\alpha}$-circulants

William F. Trench*

Trinity University, San Antonio, Texas 78212-7200, USA
Mailing address: 659 Hopkinton Road, Hopkinton, NH 03229 USA

Abstract

In a previous paper we characterized unilevel block α-circulants $A = [A_{s-\alpha r}]_{r,s=0}^{n-1}$, $A_m \in \mathbb{C}^{d_1 \times d_2}$, $0 \le m \le n-1$, in terms of the discrete Fourier transform $\mathcal{F}_A = \{F_0, F_1, \dots, F_{n-1}\}$ of $\mathcal{A} = \{A_0, A_1 \dots, A_{n-1}\}$, defined by $F_\ell = \frac{1}{n}\sum_{m=0}^{n-1} e^{-2\pi i \ell m/n} A_m$. We showed that most theoretical and computational problems concerning A can be conveniently studied in terms of corresponding problems concerning the Fourier coefficients $F_0, F_1, \dots, F_{n-1}$ individually. In this paper we show that analogous results hold for $(k+1)$-level matrices, where the first k levels have block circulant structure and the entries at the $(k+1)$-st level are unstructured rectangular matrices.

MSC: 15A09; 15A15; 15A18; 15A99

Keywords: Circulant; Block circulant; Multilevel; Discrete Fourier transform; Moore–Penrose Inverse; Eigenvalue problem;

1 Introduction

We consider $(k+1)$-level block matrices where the first k levels are circulant with orders $n_1, n_2, \dots, n_k \ge 2$ and the entries in the $(k+1)$-st level are arbitrary $d_1 \times d_2$ matrices with $d_1, d_2 \ge 1$. The systematic study of multilevel matrices was initiated by Voevodin and Tyrtyshnikov in the Russian publication [11], and in the English mathematical literature by Tyrtyshnikov [9, 10].

If $p \ge 2$ is an integer, let $\mathbb{Z}_p = \{0, 1, \dots, p-1\}$. Suppose $n_1, n_2, \dots, n_k$ are integers ≥ 2 and let

$$\mathcal{M}_{\underline{n}} = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}.$$

We denote members of $\mathcal{M}_{\underline{n}}$ by $\underline{r} = (r_1, r_2, \dots, r_k)$, $\underline{s} = (s_1, s_2, \dots, s_k)$, etc.; in particular, $\underline{0} = (0, 0, \dots, 0)$ and $\underline{1} = (1, 1, \dots, 1)$.

*e-mail:wtrench@trinity.edu

Let

$$c(\underline{r}) = \prod_{j=1}^{k} r_j, \quad \mu_j = \prod_{i=1}^{j-1} n_i, \text{ and } \nu_j = \prod_{i=j+1}^{k} n_i, \ 1 \le j \le k, \text{ with } \mu_1 = \nu_k = 1. \tag{1}$$

Following Tyrtyshnikov, we call members of $\mathcal{M}_{\underline{n}}$ multiindices. Henceforth it is be understood that multiindices are ordered lexicographically; i.e., $\underline{r} = \underline{s}$ if $r_j = s_j$, $1 \le j \le k$; $\underline{r} \prec \underline{s}$ (which we also write as $\underline{s} \succ \underline{r}$) if $r_1 < s_1$ or $r_j = s_j$, $1 \le j \le i < k$ and $r_{i+1} < s_{i+1}$; and $\underline{r} \preceq \underline{s}$ if $\underline{r} = \underline{s}$ or $\underline{r} \prec \underline{s}$. If the members of $\mathcal{M}_{\underline{n}}$ are listed in lexicographic order then the position of $\underline{r}$ in the list is

$$\gamma(\underline{r}) = \sum_{j=1}^{k} r_j \prod_{i=j+1}^{k} n_i, \quad \underline{0} \preceq \underline{r} \preceq \underline{n} - \underline{1}.$$

If $(e_{0m}, e_{1m}, \dots, e_{m-1,m})$ is the natural basis for $\mathbb{C}^m$ and

$$e_{\underline{r}} = e_{r_1 n_1} \otimes e_{r_2 n_2} \otimes \cdots \otimes e_{r_k n_k}, \quad \underline{0} \preceq \underline{r} \preceq \underline{n} - \underline{1},$$

then $\mathcal{B} = (e_{\underline{0}}, \dots, e_{\underline{r}}, \dots, e_{\underline{n}-\underline{1}})$ is a multilevel basis for $\mathbb{C}^{c(\underline{n})}$. For later reference we note that

$$\text{(a)} \quad (e_{\underline{r}} \otimes e_{\underline{s}}^T)\, e_{\underline{\ell}} = \delta_{\underline{\ell}\,\underline{s}}\, e_{\underline{r}} \quad \text{and (b)} \quad (e_{\underline{r}} \otimes e_{\underline{\ell}}^T)(e_{\underline{m}} \otimes e_{\underline{s}}^T) = \delta_{\underline{\ell}\,\underline{m}}\, e_{\underline{r}} \otimes e_{\underline{s}}^T. \tag{2}$$

If d_1 and d_2 are positive integers then arbitrary vectors $x \in \mathbb{C}^{d_2 c(\underline{n})}$ and $y \in \mathbb{C}^{d_1 c(\underline{n})}$ can be written uniquely as

$$x = \sum_{\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} (e_{\underline{s}} \otimes x_{\underline{s}}) = \begin{bmatrix} x_{\underline{0}} \\ \vdots \\ x_{\underline{r}} \\ \vdots \\ x_{\underline{n}-\underline{1}} \end{bmatrix} \quad \text{with} \quad x_{\underline{s}} \in \mathbb{C}^{d_2}, \quad \underline{0} \preceq \underline{s} \preceq \underline{n} - \underline{1},$$

and

$$y = \sum_{\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} (e_{\underline{s}} \otimes y_{\underline{s}}) = \begin{bmatrix} y_{\underline{0}} \\ \vdots \\ y_{\underline{r}} \\ \vdots \\ y_{\underline{n}-\underline{1}} \end{bmatrix} \quad \text{with} \quad y_{\underline{s}} \in \mathbb{C}^{d_1}, \quad \underline{0} \preceq \underline{s} \preceq \underline{n} - \underline{1}.$$

Henceforth we denote the sets of vectors in $\mathbb{C}^{c(\underline{n})d_2}$ and $\mathbb{C}^{c(\underline{n})d_1}$ written in these forms as $\mathbb{C}^{\underline{n}:d_2}$ and $\mathbb{C}^{\underline{n}:d_1}$, respectively. A linear transformation $L : \mathbb{C}^{\underline{n}:d_2} \to \mathbb{C}^{\underline{n}:d_1}$ can be written uniquely as $y = Hx$, where

$$H = \sum_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} (e_{\underline{r}} \otimes e_{\underline{s}}^T) \otimes H_{\underline{r}\,\underline{s}} = [H_{\underline{r}\,\underline{s}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \text{ with } H_{\underline{r}\,\underline{s}} \in \mathbb{C}^{d_1 \times d_2}, \quad \underline{0} \preceq \underline{r}, \underline{s} \preceq \underline{n} - \underline{1}; \tag{3}$$

thus,

$$\begin{aligned} y &= Hx = \left(\sum_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} (e_{\underline{r}} \otimes e_{\underline{s}}^T) \otimes H_{\underline{r}\,\underline{s}}\right)\left(\sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} e_{\underline{\ell}} \otimes x_{\underline{\ell}}\right) \\ &= \sum_{\underline{r},\underline{s},\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} (e_{\underline{r}} \otimes e_{\underline{s}}^T) e_{\underline{\ell}} \otimes H_{\underline{r}\,\underline{s}} x_{\underline{\ell}} = \sum_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} e_{\underline{r}} \otimes H_{\underline{r}\,\underline{s}} x_{\underline{s}}, \end{aligned}$$

$\underline{0} \preceq \underline{r} \preceq \underline{n}-\underline{1}$, from (2)(a). We will denote the set of matrices in $\mathbb{C}^{c(\underline{n})d_1 \times c(\underline{n})d_2}$ written in the form (3) by $\mathbb{C}^{\underline{n}:d_1 \times d_2}$.

The usual rule for matrix multiplication applies; i.e., if H is as in (3) and

$$G = \sum_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} (e_{\underline{r}} \otimes e_{\underline{s}}^T) \otimes G_{\underline{r}\,\underline{s}} = [G_{\underline{r}\,\underline{s}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \text{ with } G_{\underline{r}\,\underline{s}} \in \mathbb{C}^{d_2 \times d_3}, \quad \underline{0} \preceq \underline{r}, \underline{s} \preceq \underline{n} - \underline{1},$$

then

$$\begin{aligned} HG &= \left(\sum_{\underline{r},\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} (e_{\underline{r}} \otimes e_{\underline{\ell}}^T) \otimes H_{\underline{r}\,\underline{\ell}}\right)\left(\sum_{\underline{m},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} (e_{\underline{m}} \otimes e_{\underline{s}}^T) \otimes G_{\underline{m}\,\underline{s}}\right) \\ &= \sum_{\underline{r},\underline{\ell},\underline{m},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} [(e_{\underline{r}} \otimes e_{\underline{\ell}}^T)(e_{\underline{m}} \otimes e_{\underline{s}}^T)] \otimes H_{\underline{r}\,\underline{\ell}} G_{\underline{m}\,\underline{s}} \\ &= \sum_{\underline{r},\underline{\ell},\underline{m},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \delta_{\underline{\ell}\,\underline{m}}\, (e_{\underline{r}} \otimes e_{\underline{s}}^T) \otimes H_{\underline{r}\,\underline{\ell}} G_{\underline{m}\,\underline{s}} \quad \text{by (2)(b)} \\ &= \sum_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} (e_{\underline{r}} \otimes e_{\underline{s}}^T) \otimes \left(\sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} H_{\underline{r}\,\underline{\ell}} G_{\underline{\ell}\,\underline{s}}\right) = \sum_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} (e_{\underline{r}} \otimes e_{\underline{s}}^T) \otimes K_{\underline{r}\,\underline{s}} = [K_{\underline{r}\,\underline{s}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}}, \end{aligned}$$

where

$$K_{\underline{r}\,\underline{s}} = \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} H_{\underline{r}\,\underline{\ell}} G_{\underline{\ell}\,\underline{s}}, \quad \underline{0} \preceq \underline{r}, \underline{s} \preceq \underline{n} - \underline{1}.$$

In this paper we consider multilevel block $\underline{\alpha}$-circulants

$$A = [A_{\underline{s}-\underline{\alpha}\,\underline{r}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \text{ where } \underline{\alpha} \in \mathcal{M}_{\underline{n}} \text{ and } A_{\underline{m}} \in \mathbb{C}^{d_1 \times d_2}, \quad \underline{0} \preceq \underline{m} \preceq \underline{n} - \underline{1}.$$

Multilevel $\underline{1}$-circulants $[A_{\underline{s}-\underline{r}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}}$ have important applications in preconditioning of multilevel and multilevel block Toeplitz matrices $T = [T_{\underline{s}-\underline{r}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}}$; see, e.g., [3]–[7], a very incomplete list. We are not aware of any published results on multilevel $\underline{\alpha}$-circulants with $\underline{\alpha} \succ \underline{1}$.

The proofs of some of our results are similar to results obtained in [8] for unilevel block circulants. Nevertheless, we include complete proofs here since we believe that simply referring to [8] would impede the presentation here and would not be convincing in the multilevel setting.

2 Preliminaries

Throughout the rest of this paper all arithmetic operations and relations involving multi-indices are entrywise and modulo $\underline{n}$, i.e., $\underline{r} \equiv \underline{s} \pmod{\underline{n}}$, $\gcd(\underline{\alpha}, \underline{n}) = \underline{q}$ and $\underline{p} = \underline{\alpha}/\underline{q}$ mean that

$$r_j \equiv s_j \pmod{n_j}, \quad \gcd(\alpha_j, n_j) = q_j, \quad \text{and} \quad p_j = \alpha_j/q_j, \quad 1 \le j \le k,$$

respectively. Also,

$$\underline{r} + \underline{s} = (r_1 + s_1 \,(\mathrm{mod}\, n_1), r_2 + s_2 \,(\mathrm{mod}\, n_2), \dots, r_k + s_k \,(\mathrm{mod}\, n_k))$$

and

$$\underline{r}\,\underline{s} = (r_1 s_1 \,(\mathrm{mod}\, n_1), r_2 s_2 \,(\mathrm{mod}\, n_2), \dots, r_k s_k \,(\mathrm{mod}\, n_k)).$$

We denote

$$\zeta_j = e^{-2\pi i/n_j},\ 1 \le j \le k, \quad \zeta^{\underline{s}} = \zeta_1^{s_1} \zeta_2^{s_2} \cdots \zeta_k^{s_k}, \quad \underline{0} \preceq \underline{s} \preceq \underline{n} - \underline{1},$$

and

$$\Phi = \frac{1}{\sqrt{c(\underline{n})}} [\zeta^{\underline{r}\,\underline{s}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} = \begin{bmatrix} \phi_{\underline{0}} & \cdots & \phi_{\underline{s}} & \cdots & \phi_{\underline{n}} \end{bmatrix},$$

with

$$\phi_{\underline{s}} = \frac{1}{\sqrt{c(\underline{n})}} \begin{bmatrix} 1 \\ \vdots \\ \zeta^{\underline{r}\,\underline{s}} \\ \vdots \\ \zeta^{(\underline{n}-\underline{1})(\underline{s})} \end{bmatrix}, \quad \underline{0} \preceq \underline{s} \preceq \underline{n} - \underline{1}. \tag{4}$$

Note that

$$\phi_{\underline{s}} = \psi_{s_1,1} \otimes \psi_{s_2,2} \otimes \cdots \otimes \psi_{s_k,k},$$

where

$$\psi_{s_j,j} = \frac{1}{\sqrt{n_j}} \begin{bmatrix} 1 \\ \zeta_j^{s_j} \\ \vdots \\ \zeta_j^{(n_j-1)s_j} \end{bmatrix}, \quad 0 \le s_j \le n_{j-1}, \quad 1 \le j \le k;$$

hence,

$$\phi_{\underline{s}}^* \phi_{\underline{r}} = \delta_{\underline{r}\,\underline{s}} =_{\mathrm{Def}} \begin{cases} 1 & \text{if} \quad \underline{r} = \underline{s}, \\ 0 & \text{if} \quad \underline{r} \ne \underline{s}, \end{cases} \quad \underline{0} \preceq \underline{r}, \underline{s} \preceq \underline{n} - \underline{1}. \tag{5}$$

Now let $E_j = [\delta_{r_j, s_j - 1}]_{r_j, s_j = 0}^{n_j - 1}$, $1 \le j \le k$,

$$E = E_1 \otimes E_2 \otimes \cdots \otimes E_k = [\delta_{\underline{r}, \underline{s} - \underline{1}}]_{\underline{r}, \underline{s} = \underline{0}}^{\underline{n} - \underline{1}}, \tag{6}$$

and

$$E^{\underline{u}} = E_1^{u_1} \otimes E_2^{u_2} \otimes \cdots \otimes E_k^{u_k} = [\delta_{\underline{r},\underline{s}-\underline{u}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}}. \tag{7}$$

It is straightforward to verify that

$$(E^{\underline{u}} \otimes I_{d_2}) \begin{bmatrix} x_{\underline{0}} \\ \vdots \\ x_{\underline{r}} \\ \vdots \\ x_{\underline{n}-\underline{1}} \end{bmatrix} = \begin{bmatrix} x_{\underline{u}} \\ \vdots \\ x_{\underline{r}+\underline{u}} \\ \vdots \\ x_{\underline{n}-\underline{1}+\underline{u}} \end{bmatrix} \text{ if } x \in \mathbb{C}^{\underline{n}:d_2} \tag{8}$$

and

$$(E^{\underline{u}} \otimes I_{d_1}) \left([B_{\underline{r}\,\underline{s}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}}\right) (E^{-\nu} \otimes I_{d_2}) = [B_{\underline{r}+\underline{u},\underline{s}+\underline{v}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \quad \text{if} \quad B \in \mathbb{C}^{\underline{n}:d_1 \times d_2}. \tag{9}$$

From (4), (6), and (8) with $\underline{u} = \underline{1}$,

$$E\phi_{\underline{s}} = \frac{1}{\sqrt{c(\underline{n})}} \begin{bmatrix} \zeta^{s} \\ \vdots \\ \zeta^{(\underline{r}+\underline{1})\underline{s}} \\ \vdots \\ \zeta^{(\underline{n}-\underline{1})\underline{s}} \end{bmatrix} = \zeta^{\underline{s}} \phi_{\underline{s}}, \quad \underline{0} \preceq \underline{s} \preceq \underline{n} - \underline{1}. \tag{10}$$

Hence

$$E\Phi = \Phi D \quad \text{with} \quad D = \operatorname{diag}\left(1, \dots, \zeta^{\underline{s}}, \dots, \zeta^{(\underline{n}-\underline{1})\underline{s}}\right), \quad \text{so} \quad E = \Phi D \Phi^*.$$

Now let

$$R = E \otimes I_{d_1}, \quad S = E \otimes I_{d_2}, \tag{11}$$

$$P_{\underline{s}} = \phi_{\underline{s}} \otimes I_{d_1}, \quad Q_{\underline{s}} = \phi_{\underline{s}} \otimes I_{d_2}, \quad \underline{0} \preceq \underline{s} \preceq \underline{n} - \underline{1}. \tag{12}$$

From (5),

$$P_{\underline{r}}^* P_{\underline{s}} = \delta_{\underline{r}\,\underline{s}} I_{\underline{n}:d_1} \quad \text{and} \quad Q_{\underline{r}}^* Q_{\underline{s}} = \delta_{\underline{r}\,\underline{s}} I_{\underline{n}:d_2}, \quad \underline{0} \preceq \underline{r}, \underline{s} \preceq \underline{n} - \underline{1}. \tag{13}$$

From (10) and (11),

$$RP_{\underline{s}} = \zeta^{\underline{s}} P_{\underline{s}} \quad \text{and} \quad SQ_{\underline{s}} = \zeta^{\underline{s}} Q_{\underline{s}}, \quad \underline{0} \preceq \underline{s} \preceq \underline{n} - \underline{1}. \tag{14}$$

Also, let

$$P = \begin{bmatrix} P_{\underline{0}} & \cdots & P_{\underline{s}} & \cdots & P_{\underline{n}-\underline{1}} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{\underline{0}} & \cdots & Q_{\underline{s}} & \cdots & Q_{\underline{n}-\underline{1}} \end{bmatrix}, \tag{15}$$

and

$$U_{\underline{\alpha}} = \begin{bmatrix} P_{\underline{0}} & \cdots & P_{\underline{\alpha}\,\underline{s}} & \cdots & P_{\underline{\alpha}(\underline{n}-\underline{1})} \end{bmatrix}. \tag{16}$$

From (13), P and Q are unitary. If $\gcd(\underline{\alpha}, \underline{n}) = \underline{1}$ the mapping $\underline{s} \to \underline{\alpha}\,\underline{s}$ is a permutation of $\mathcal{M}_{\underline{n}}$, so $U_{\underline{\alpha}}$ is unitary. However, if $\gcd(\underline{\alpha}, \underline{n}) = \underline{q} \succ \underline{1}$ then the first $c(\underline{p})$

block columns $P_{\underline{0}}, \ldots, P_{\underline{\alpha}\,\underline{s}}, \ldots, P_{\underline{\alpha}(\underline{p}-\underline{1})}$ of $U_{\underline{\alpha}}$ are repeated $c(\underline{q})$ times, so $U_{\underline{\alpha}}$ is not invertible.

From (14) and (15),

$$R = PD_R P^* \quad \text{and} \quad S = QD_S Q^*$$

where

$$D_R = \bigoplus_{\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \zeta^{\underline{r}} I_{d_1} \quad \text{and} \quad D_S = \bigoplus_{\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \zeta^{\underline{s}} I_{d_2}.$$

3 The Ablow-Brenner theorem for multilevel block circulants

Ablow and Brenner [1] showed that $A \in \mathbb{C}^{n\times n}$ is a standard α-circulant $A = [a_{s-\alpha r}] \in \mathbb{C}^{n\times n}$ if and only if

$$\left([\delta_{r,s-1}]_{r,s=0}^{n-1}\right) A \left([\delta_{r,s-1}]_{r,s=0}^{n-1}\right)^{-\alpha} = A.$$

This was generalized to characterize unilevel block circulants in [8, Theorem 1]. Here we generalize it to multilevel block circulants.

Theorem 1 *If $A = [G_{\underline{r}\,\underline{s}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}}$ with $G_{\underline{r}\,\underline{s}} \in \mathbb{C}^{d_1\times d_2}$ then $RAS^{-\underline{\alpha}} = A$ (see (11)) if and only if A is an $\underline{\alpha}$-circulant; more precisely, if and only if*

$$G_{\underline{r}\,\underline{s}} = A_{\underline{s}-\underline{\alpha}\,\underline{r}}, \quad \underline{0} \preceq \underline{r}\,, \underline{s} \preceq \underline{n} - \underline{1}, \tag{17}$$

with

$$A_{\underline{s}} = G_{\underline{0}\,\underline{s}}, \quad \underline{0} \preceq \underline{s} \preceq \underline{n} - \underline{1}. \tag{18}$$

PROOF. From (9) and (11), $RAS^{-\underline{\alpha}} = [G_{\underline{r}+\underline{1},\underline{s}+\underline{\alpha}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}}$. Therefore we must show that (17) is equivalent to

$$G_{\underline{r}+\underline{1},\underline{s}+\underline{\alpha}} = G_{\underline{r}\,\underline{s}}, \quad \underline{0} \preceq \underline{r}\,, \underline{s} \preceq \underline{n} - \underline{1}. \tag{19}$$

If (17) is true then

$$G_{\underline{r}+\underline{1},\underline{s}+\underline{\alpha}} = A_{(\underline{s}+\underline{\alpha})-(\underline{r}+\underline{1})\alpha} = A_{\underline{s}-\underline{\alpha r}} = G_{\underline{r}\,\underline{s}}, \quad \underline{0} \preceq \underline{r}\,, \underline{s} \preceq \underline{n} - \underline{1}.$$

For the converse we consider blocks at each level independently. Insofar as they involve level p, (17)–(19) can be rewritten as

$$\begin{aligned} G_{\cdots,(r_p,s_p),\cdots} &= A_{\cdots,(s_p-\alpha_p r_p),\cdots} \quad 0 \le r_p, s_p \le n_p - 1, \\ A_{\cdots,(s_p),\cdots} &= G_{\cdots,(0,s_p),\cdots} \quad 0 \le s_p \le n_p - 1, \end{aligned} \tag{20}$$

and

$$G_{\cdots,(r_p+1,s_p+\alpha_p),\cdots} = G_{\cdots,(r_p,s_p),\cdots} \quad 0 \le r_p, s_p \le n_p - 1. \tag{21}$$

Now suppose (20) and (21) hold and

$$G_{\cdots,(r_p,s_p),\cdots} = A_{\cdots,(s_p-\alpha_p r_p),\cdots} \quad 0 \le s_p \le n_p - 1, \tag{22}$$

for some $r_p < n_{p-1}$. Replacing s_p by $s_p - \alpha_p$ in (21) and (22) yields

$$G_{\cdots,(r_p+1,s_p),\cdots} = G_{\cdots,(r_p,s_p-\alpha_p),\cdots} \quad 0 \le r_p, s_p \le n_p - 1,$$

and

$$G_{\cdots,(r_p,s_p-\alpha_p),\cdots} = A_{\cdots,(s_p-\alpha_p(r_p+1)),\cdots} \quad 0 \le s \le n_p - 1.$$

Therefore

$$G_{\cdots,(r_p+1,s_p),\cdots} = A_{\cdots,(s_p-\alpha_p(r_p+1)),\cdots} \quad 0 \le s \le n_p - 1,$$

which is (22) with r_p replaced by $r_p + 1$. □

Remark 1 From (7), (11), and (12),

$$R^{\underline{u}} = \mathbf{R}_1^{u_1} \otimes \mathbf{R}_2^{u_2} \otimes \cdots \otimes \mathbf{R}_k^{u_k} \quad \text{and} \quad S^{\underline{v}} = \mathbf{S}_1^{v_1} \otimes \mathbf{S}_2^{v_2} \otimes \cdots \otimes \mathbf{S}_k^{v_k},$$

where

$$\mathbf{R}_j = I_{\mu_j} \otimes E_{n_j} \otimes I_{\nu_j d_1} \quad \text{and} \quad \mathbf{S}_j = I_{\mu_j} \otimes E_{n_j} \otimes I_{\nu_j d_2}.$$

(See (1)). Then, for example,

$$RAS^{-\underline{\alpha}} = A \quad \text{if and only if} \quad \mathbf{R}_j A \mathbf{S}_j^{-\alpha_j} = A, \quad 1 \le j \le k.$$

Theorem 2 *If*

$$A = [A_{\underline{s}-\underline{\alpha}\,\underline{r}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \in \mathbb{C}^{\underline{n}:d_1\times d_2} \quad \textit{and} \quad B = [B_{\underline{s}-\underline{\alpha}\,\underline{r}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \in \mathbb{C}^{\underline{n}:d_1\times d_2}$$

then (i) $AB^* = [C_{\underline{s}-\underline{r}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \in \mathbb{C}^{\underline{n}:d_1\times d_1}$ *with*

$$C_{\underline{m}} = \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} A_{\underline{\ell}} B^*_{\underline{\ell}-\underline{\alpha}\,\underline{m}}, \quad \underline{0} \preceq \underline{m} \preceq \underline{n} - \underline{1}. \tag{23}$$

(ii) *If* $\gcd(\underline{\alpha}, \underline{n}) = \underline{1}$ *then* $B^*A = [D_{\underline{s}-\underline{r}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \in \mathbb{C}^{\underline{n}:d_2\times d_2}$ *with*

$$D_{\underline{m}} = \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} B^*_{\underline{\ell}} A_{\underline{m}+\underline{\ell}}, \quad \underline{0} \preceq \underline{m} \preceq \underline{n} - \underline{1}. \tag{24}$$

PROOF. (i) From Theorem 1, $A = RAS^{-\underline{\alpha}}$ and $B = RBS^{-\underline{\alpha}}$. Therefore $AB^* = RAB^*R^{-\underline{1}}$, so Theorem 1 with $R = S$ implies that AB^* is a $\underline{1}$-circulant. Computing the first block row ($\underline{r} = \underline{0}$) of AB^* yields (23).

(ii) Also, $B^*A = S^{\underline{\alpha}}B^*AS^{-\underline{\alpha}}$, so

$$\mathbf{S}_j^{\alpha_j} B^* A \mathbf{S}_j^{-\alpha_j} = B^*A, \quad 1 \le j \le k.$$

Applying this equality β_j times where $\alpha_j \beta_j \equiv 1 \pmod{n_j}$ yields

$$\mathbf{S}_j BA\mathbf{S}_j^{-1} = B^* A, \quad 1 \le j \le k.$$

Now Theorem 1 and Remark 1 with $R = S$ imply that $B^* A$ is a $\underline{1}$-circulant. Computing the first block row of $B^* A$ yields $D_{\underline{m}} = \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} B^*_{-\underline{\alpha}\,\underline{\ell}} A_{\underline{m}-\underline{\alpha}\underline{\ell}}$. Since $\gcd(\underline{n}, \underline{k}) = \underline{1}$, $\underline{\ell} \to -\underline{\beta}\underline{\ell}$ is a permutation of $\mathcal{M}_{\underline{n}}$, so we can replace $\underline{\ell}$ by $-\underline{\beta}\underline{\ell}$ in the last sum to obtain (24). $\square$

Theorem 3 *If*

$$A = \left[A_{\underline{s}-\underline{\alpha}\,\underline{r}}\right]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \in \mathbb{C}^{\underline{n}:d_1 \times d_2} \quad \textit{and} \quad B = \left[B_{\underline{s}-\underline{\beta}\underline{r}}\right]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \in \mathbb{C}^{\underline{n}:d_2 \times d_3} \tag{25}$$

then

$$AB = [C_{\underline{s}-\underline{\alpha}\underline{\beta}\,\underline{r}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \in \mathbb{C}^{\underline{n}:d_1 \times d_3}, \tag{26}$$

with

$$C_{\underline{m}} = \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} A_{\underline{\ell}} B_{\underline{m}-\underline{\beta}\,\underline{\ell}}, \quad \underline{0} \preceq \underline{m} \preceq \underline{n} - \underline{1}. \tag{27}$$

PROOF. Let $R = E \otimes I_{d_1}$, $S = E \otimes I_{d_2}$, and $T = E \otimes I_{d_3}$. (See (6)). From (25) and Theorem 1,

$$\text{(a)} \quad A = RAS^{-\underline{\alpha}} \quad \text{and} \quad \text{(b)} \quad B = SBT^{-\underline{\beta}}.$$

Now write

$$T^{\underline{\beta}} = \mathbf{T}_1^{\beta_1} \otimes \mathbf{T}_2^{\beta_2} \otimes \cdots \otimes \mathbf{T}_k^{\beta_k} \quad \text{with} \quad \mathbf{T}_j = I_j \otimes E_{n_j} \otimes I_{\nu_j d_3}, \quad 1 \le j \le k.$$

From (b) $\mathbf{S}_j B\mathbf{T}_j^{-\beta_j} = B$, $1 \le j \le k$. Applying this equality α_j times yields $\mathbf{S}_j^{\alpha_j} B\mathbf{T}_j^{-\alpha_j \beta_j} = B$, $1 \le j \le k$. Therefore $S^{\underline{\alpha}} BT^{-\underline{\alpha}\underline{\beta}} = B$, by Remark 1. From this and (a), $R(AB)S^{-\underline{\alpha}\underline{\beta}} = AB$. Now Theorem 1 implies (26) with (27) obtained by computing the entries in the first block row of AB. $\square$

4 A dft characterization of multilevel $\underline{\alpha}$-circulants

Let $\left\{F_{\underline{\ell}} \,\middle|\, \underline{\ell} \in \mathcal{M}_{\underline{n}}\right\}$ and $\left\{A_{\underline{m}} \,\middle|\, \underline{m} \in \mathcal{M}_{\underline{n}}\right\} \subset \mathbb{C}^{d_1 \times d_2}$ be related by

$$\text{(a)} \quad F_{\underline{\ell}} = \sum_{\underline{m}=\underline{0}}^{\underline{n}-\underline{1}} \zeta^{\underline{\ell}\,\underline{m}} A_{\underline{m}} \quad \text{and (b)} \quad A_{\underline{m}} = \frac{1}{c(\underline{n})} \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} \zeta^{-\underline{\ell}\,\underline{m}} F_{\underline{\ell}}, \tag{28}$$

which are equivalent, since $\sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} \zeta^{\underline{\alpha}\,\underline{\ell}} = c(\underline{n})\delta_{\underline{\alpha}\,\underline{0}}$. Denote

$$\mathcal{F}_{\mathcal{A}} = \bigoplus_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} F_{\underline{\ell}}. \tag{29}$$

The set $\left\{F_{\underline{\ell}} \,\middle|\, \underline{\ell} \in \mathcal{M}_{\underline{n}}\right\}$ is the discrete Fourier transform (dft) of the set $\left\{A_{\underline{m}} \,\middle|\, \underline{m} \in \mathcal{M}_{\underline{n}}\right\}$.

Theorem 4 *A matrix* $A \in \mathbb{C}^{\underline{n}:d_1\times d_2}$ *is an* $\underline{\alpha}$*-circulant* $A = [A_{\underline{s}-\underline{\alpha}\,\underline{r}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}}$ *if and only if*

$$A = U_{\underline{\alpha}} \mathcal{F}_{\mathcal{A}} Q^* = \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{\alpha}\,\underline{\ell}} F_{\underline{\ell}} Q_{\underline{\ell}}^* \tag{30}$$

(see (15) *and* (16)*), where* $\left\{A_{\underline{m}} \,\middle|\, \underline{m} \in \mathcal{M}_{\underline{n}}\right\}$ *and* $\left\{F_{\underline{\ell}} \,\middle|\, \underline{\ell} \in \mathcal{M}_{\underline{n}}\right\}$ *are related as in* (28).

PROOF. Suppose $A = [A_{\underline{s}-\underline{\alpha}\,\underline{r}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}}$. From (28),

$$A_{\underline{s}-\underline{\alpha}\,\underline{r}} = \frac{1}{c(\underline{n})} \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} \zeta^{-\underline{\ell}(\underline{s}-\underline{\alpha}\,\underline{r})} F_{\underline{\ell}}.$$

Hence

$$A = \frac{1}{c(\underline{n})} \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} \begin{bmatrix} 1 \otimes I_{d_1} \\ \vdots \\ \zeta^{\underline{\ell}\,\underline{r}\,\underline{\alpha}} \otimes I_{d_1} \\ \vdots \\ \zeta^{\underline{\ell}(\underline{n}-\underline{1})\underline{\alpha}} \otimes I_{d_1} \end{bmatrix} F_{\underline{\ell}} \begin{bmatrix} 1 \otimes I_{d_2} \\ \vdots \\ \zeta^{\underline{\ell}\,\underline{s}} \otimes I_{d_2} \\ \vdots \\ \zeta^{\underline{\ell}\,(\underline{n}-\underline{1})} \otimes I_{d_2} \end{bmatrix}^H,$$

so (4), (12) and (15) imply (30). Conversely, suppose (30) holds. Then

$$\begin{aligned} RAS^{-\underline{\alpha}} &= \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} (RP_{\underline{\alpha}\,\underline{\ell}}) F_{\underline{\ell}} (SQ_{\underline{\ell}})^{-\underline{\alpha}} = \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} (\zeta^{\underline{\alpha}\,\underline{\ell}} P_{\underline{\alpha}\,\underline{\ell}}) F_{\underline{\ell}} (\zeta^{-\underline{\alpha}\,\underline{\ell}} Q_{\underline{\ell}}^*) \\ &= \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{\alpha}\,\underline{\ell}} F_{\underline{\ell}} Q_{\underline{\ell}}^* = A, \end{aligned}$$

where (14) implies the second equality. Now Theorem 1 implies that A is an $\underline{\alpha}$-circulant $A = [A_{\underline{s}-\underline{\alpha}\,\underline{r}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}}$, and the argument given in the first half of this proof implies that $\left\{A_{\underline{m}} \,\middle|\, \underline{m} \in \mathcal{M}_{\underline{n}}\right\}$ is as defined by (28). □

The following theorem provides a representation of A that reduces to (30) if $\gcd(\underline{\alpha}, \underline{n}) = \underline{1}$, but is more useful if $\gcd(\underline{\alpha}, \underline{n}) \succ \underline{1}$.

Theorem 5 *Suppose* $\gcd(\underline{\alpha}, \underline{n}) = \underline{q}$ *and* $\underline{p} = \underline{n}/\underline{q}$. *Let*

$$\mathbf{Q}_{\underline{\ell},\underline{\alpha}} = \begin{bmatrix} Q_{\underline{\ell}} & \cdots & Q_{\underline{\ell}+\underline{\nu}\,\underline{p}} & \cdots & Q_{\underline{\ell}+(\underline{q}-\underline{1})\underline{p}} \end{bmatrix}, \quad \underline{0} \preceq \underline{\ell} \preceq \underline{p}-\underline{1}, \tag{31}$$

$$\mathcal{Q}_{\underline{\alpha}} = \begin{bmatrix} \mathbf{Q}_{\underline{0},\underline{\alpha}} & \cdots & \mathbf{Q}_{\underline{\ell},\underline{\alpha}} & \dots & \mathbf{Q}_{\underline{p}-\underline{1},\underline{\alpha}} \end{bmatrix}, \tag{32}$$

$$V_{\underline{\alpha}} = \begin{bmatrix} P_{\underline{0}} & \cdots & P_{\underline{\ell}\,\underline{\alpha}} & \cdots & P_{(\underline{p}-\underline{1})\underline{\alpha}} \end{bmatrix}, \tag{33}$$

$$\mathbf{F}_{\underline{\ell},\underline{\alpha}} = \begin{bmatrix} F_{\underline{\ell}} & \cdots & F_{\underline{\ell}+\underline{\nu}\,\underline{p}} & \cdots & F_{\underline{\ell}+(\underline{q}-\underline{1})\underline{p}} \end{bmatrix}, \quad \underline{0} \preceq \underline{\ell} \preceq \underline{p}-\underline{1}, \tag{34}$$

and

$$\mathcal{F}_{\underline{\alpha}} = \bigoplus_{\underline{\ell}=\underline{0}}^{\underline{p}-\underline{1}} \mathbf{F}_{\underline{\ell},\underline{\alpha}}. \tag{35}$$

Then $\mathcal{Q}_{\underline{\alpha}}$ *is unitary since its columns are simply a rearrangement of the columns of* Q,

$$V_{\underline{\alpha}}^* V_{\underline{\alpha}} = I_{c(\underline{p})d_1}, \tag{36}$$

and (30) *can be rewritten as*

$$A = \sum_{\underline{\ell}=\underline{0}}^{\underline{p}-\underline{1}} P_{\underline{\alpha}\,\underline{\ell}} \mathbf{F}_{\underline{\ell},\underline{\alpha}} Q_{\underline{\ell},\underline{\alpha}}^* = \mathcal{V}\mathcal{F}_{\underline{\alpha}} \mathcal{Q}_{\underline{\alpha}}^*. \tag{37}$$

PROOF. Since $\underline{\alpha}\,\underline{r} = \underline{\alpha}\,\underline{s}$ with $\underline{0} \preceq \underline{r}, \underline{s} \preceq \underline{p} - \underline{1}$ if and only if $\underline{r} = \underline{s}$, (13) implies (36). Since every $\underline{s} \in \mathcal{M}_{\underline{n}}$ can be written uniquely as $\underline{s} = \underline{\ell} + \underline{\nu}\,\underline{p}$ with $\underline{0} \preceq \underline{\ell} \preceq \underline{p} - \underline{1}$ and $\underline{0} \preceq \underline{\nu} \preceq \underline{q} - \underline{1}$, the second equality in (30) can be written as

$$A = \sum_{\underline{\ell}=\underline{0}}^{\underline{p}-\underline{1}} \sum_{\underline{\nu}=\underline{0}}^{\underline{q}-\underline{1}} P_{\underline{\alpha}(\underline{\ell}+\underline{\nu}\,\underline{p})} F_{\underline{\ell}+\underline{\nu}\,\underline{p}} Q_{\underline{\ell}+\underline{\nu}\,\underline{p}}^* = \sum_{\underline{\ell}=\underline{0}}^{\underline{p}-\underline{1}} P_{\underline{\alpha}\,\underline{\ell}} \sum_{\underline{\nu}=\underline{0}}^{\underline{q}-\underline{1}} F_{\underline{\ell}+\underline{\nu}\,\underline{p}} Q_{\underline{\ell}+\underline{\nu}\,\underline{p}}^*, \tag{38}$$

where the second equality here is valid because $\underline{p}\underline{\alpha} \equiv \underline{0} \pmod{\underline{n}}$. Therefore the first equality in (37) is valid because

$$\mathbf{F}_{\underline{\ell},\underline{\alpha}} \mathbf{Q}_{\underline{\ell},\underline{\alpha}}^* = \sum_{\underline{\nu}=\underline{0}}^{\underline{q}-\underline{1}} F_{\underline{\ell}+\underline{\nu}\,\underline{p}} Q_{\underline{\ell}+\underline{\nu}\,\underline{p}}^*, \quad \underline{0} \preceq \underline{\ell} \preceq \underline{p} - \underline{1}.$$

Now (32)–(34) imply the second equality in (37). □

5 Solution of $Az = w$ and the least squares problem

In this section $A = [A_{\underline{s}-\underline{\alpha}\,\underline{r}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}}$. If $z \in \mathbb{C}^{\underline{n}:d_2}$ and $w \in \mathbb{C}^{\underline{n}:d_1}$ we write

$$z = \sum_{\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} Q_{\underline{s}} u_{\underline{s}} \quad \text{and} \quad w = \sum_{\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{s}} v_{\underline{s}} \quad \text{with} \quad u_{\underline{s}} \in \mathbb{C}^{d_2} \quad \text{and} \quad v_{\underline{s}} \in \mathbb{C}^{d_1} \tag{39}$$

(see (15)), $\underline{0} \preceq \underline{s} \preceq \underline{n} - \underline{1}$.

Theorem 6 *If* $\gcd(\underline{\alpha}, \underline{n}) = \underline{1}$ *then*

$$\|Az - w\|^2 = \sum_{\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \|F_{\underline{s}} u_{\underline{s}} - v_{\underline{\alpha}\,\underline{s}}\|^2 \tag{40}$$

where $\|\cdot\|$ *is the Frobenius norm. Therefore the least squares problem for the* $c(\underline{n})d_1 \times c(\underline{n})d_2$ *matrix* A *reduces to* $c(\underline{n})$ *independent least squares problems for the* $d_1 \times d_2$ *matrices* $F_{\underline{s}}$, $\underline{0} \preceq \underline{s} \preceq \underline{n} - \underline{1}$. *Also,*

$$Az = w \quad \text{if and only if} \quad F_{\underline{s}} u_{\underline{s}} = v_{\underline{\alpha}\,\underline{s}}, \quad \underline{0} \preceq \underline{s} \preceq \underline{n} - \underline{1}. \tag{41}$$

PROOF. From (30) and (39),

$$\begin{aligned} Az - w &= \sum_{\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{\alpha}\,\underline{s}} F_{\underline{s}} u_{\underline{s}} - \sum_{\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{s}} v_{\underline{s}} = \sum_{\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{\alpha}\,\underline{s}} F_{\underline{s}} u_{\underline{s}} - \sum_{\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{\alpha}\,\underline{s}} v_{\underline{\alpha}\,\underline{s}} \\ &= \sum_{\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{\alpha}\,\underline{s}} (F_{\underline{s}} u_{\underline{s}} - v_{\underline{\alpha}\,\underline{s}}), \end{aligned} \tag{42}$$

where the second equality is valid because

$$\sum_{\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{s}} v_{\underline{s}} = \sum_{\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{\alpha}\,\underline{s}} v_{\underline{\alpha s}},$$

since $\gcd(\underline{\alpha}, \underline{n}) = \underline{1}$. Since $P^*_{\underline{\alpha}\,\underline{r}} P_{\underline{\alpha}\,\underline{s}} = \delta_{\underline{r}\,\underline{s}} I_{\underline{n}:d_1}$ (again, because $\gcd(\underline{\alpha}, \underline{n}) = \underline{1}$), (42) implies (40), which implies (41) □

Theorem 7 *Suppose* $\gcd(\underline{\alpha}, \underline{n}) = \underline{q}$ *and* $\underline{p} = \underline{n}/\underline{q}$. *Then* $Az = w$ *has no solution unless*

$$w = \sum_{\underline{\ell}=\underline{0}}^{\underline{p}-\underline{1}} P_{\underline{\alpha}\,\underline{\ell}} v_{\underline{\alpha}\,\underline{\ell}}, \tag{43}$$

in which case z *is a solution if and only* $z = \sum_{\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} Q_{\underline{s}} u_{\underline{s}}$, *where*

$$\sum_{\underline{\nu}=\underline{0}}^{\underline{q}-\underline{1}} F_{\underline{\ell}+\underline{\nu}\,\underline{p}} u_{\underline{\ell}+\underline{\nu}\,\underline{p}} = v_{\underline{\alpha}\,\underline{\ell}}, \quad \underline{0} \preceq \underline{\ell} \preceq \underline{n} - \underline{1}. \tag{44}$$

PROOF. From (38) and (39),

$$Az = \sum_{\underline{\ell}=\underline{0}}^{\underline{p}-\underline{1}} P_{\underline{\alpha},\underline{\ell}} \sum_{\underline{\nu}=\underline{0}}^{\underline{q}-\underline{1}} F_{\underline{\ell}+\underline{\nu}\underline{p}} u_{\underline{\ell}+\underline{\nu}\,\underline{p}}.$$

Since $\left\{\underline{\alpha}\,\underline{\ell} \,\middle|\, \underline{0} \preceq \underline{\ell} \preceq \underline{p} - \underline{1}\right\}$ is a set of distinct multiindices, (13) implies that $P^*_{\underline{\alpha}\,\underline{\ell}} P_{\underline{\alpha}\,\underline{m}} = \delta_{\underline{\ell}\,\underline{m}}$, $\underline{0} \preceq \underline{\ell}, \underline{m} \preceq \underline{p} - \underline{1}$. This and (41) imply that $Az = w$ has no solution unless (43) holds for some $v_{\underline{0}}, \dots, v_{\underline{\alpha}\,\underline{\ell}}, \dots, v_{\underline{\alpha}(\underline{p}-\underline{1})}$, in which case $z = \sum_{\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} F_{\underline{s}} u_{\underline{s}}$ is a solution if and only if (44) holds. □

6 Commutativity

The following theorem generalizes the well known commutativity property of 1-circulants $[a_{s-r}]_{r,s=0}^{n-1} \in \mathbb{C}^{n\times n}$.

Theorem 8 *Suppose* $d_1 = d_2$, $\gcd(\underline{\alpha}, \underline{n}) = \underline{1}$, *and* $\underline{\alpha\beta} \equiv \underline{1} \pmod{\underline{n}}$. *Let* $A = [A_{\underline{s}-\underline{\alpha r}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n-1}}$, $B = [B_{\underline{s}-\underline{\beta r}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n-1}}$,

$$F_{\underline{\ell}} = \sum_{\underline{m}=\underline{0}}^{\underline{n-1}} \zeta^{\underline{\ell m}} A_{\underline{m}} \quad \textit{and} \quad G_{\underline{\ell}} = \sum_{\underline{m}=\underline{0}}^{\underline{n-1}} \zeta^{\underline{\ell m}} B_{\underline{m}}, \quad \underline{0} \preceq \underline{\ell} \preceq \underline{n} - \underline{1}.$$

Then $AB = BA$ *if and only if*

$$F_{\underline{\beta\ell}} G_{\underline{\ell}} = G_{\underline{\alpha\ell}} F_{\underline{\ell}}, \quad \underline{0} \preceq \underline{\ell} \preceq \underline{n} - \underline{1}.$$

PROOF. Since $\gcd(\underline{\alpha}, \underline{n}) = \gcd(\underline{\beta}, \underline{n}) = \underline{1}$, we may change summation indices $\underline{\ell} \to \underline{\alpha\ell}$ and $\underline{\ell} \to \underline{\beta\ell}$. Therefore, from Theorem 4 with $Q = P$,

$$A = \sum_{\underline{\ell}=\underline{0}}^{\underline{n-1}} P_{\underline{\alpha\ell}} F_{\underline{\ell}} P_{\underline{\ell}}^* = \sum_{\underline{\ell}=\underline{0}}^{\underline{n-1}} P_{\underline{\ell}} F_{\underline{\beta\ell}} P_{\underline{\beta\ell}}^*, \quad B = \sum_{\underline{\ell}=\underline{0}}^{\underline{n-1}} P_{\underline{\beta\ell}} G_{\underline{\ell}} P_{\underline{\ell}}^* = \sum_{\underline{\ell}=\underline{0}}^{\underline{n-1}} P_{\underline{\ell}} G_{\underline{\alpha\ell}} P_{\underline{\alpha\ell}}^*,$$

$$AB = \sum_{\underline{\ell}=\underline{0}}^{\underline{n-1}} P_{\underline{\ell}} F_{\underline{\beta\ell}} G_{\underline{\ell}} P_{\underline{\ell}}^*, \quad \text{and} \quad BA = \sum_{\underline{\ell}=\underline{0}}^{\underline{n-1}} P_{\underline{\ell}} G_{\underline{\alpha\ell}} F_{\underline{\ell}} P_{\underline{\ell}}^*,$$

which implies the conclusion. □

7 The Moore-Penrose inverse of an $\underline{\alpha}$-circulant

In this section $\mathcal{A} = \left\{A_{\underline{m}} \,\middle|\, \underline{m} \in \mathcal{M}_{\underline{n}}\right\}$ and $\mathcal{F} = \left\{F_{\underline{\ell}} \,\middle|\, \underline{\ell} \in \mathcal{M}_{\underline{n}}\right\}$ are related as (28).

Recall that the Moore-Penrose inverse of a matrix $G \in \mathbb{C}^{r \times s}$ is the unique matrix $G^\dagger \in \mathbb{C}^{s \times r}$ that satisfies the Penrose conditions

$$(GG^\dagger)^* = GG^\dagger, \quad (G^\dagger G)^* = G^\dagger G, \quad GG^\dagger G = G \quad \text{and} \quad G^\dagger GG^\dagger = G^\dagger.$$

We need the following lemma.

Lemma 1 *Suppose* $L \in \mathbb{C}^{r \times p}$, $M \in \mathbb{C}^{s \times q}$, $L^*L = I_r$, $M^*M = I_s$, $G = LCM^*$, *and* $H = MC^\dagger L^*$. *Then* $H = G^\dagger$.

PROOF. (i) The following computations are straightforward:

$$GH = LCC^\dagger L^* = L(CC^\dagger)^* L = (GH)^*,$$

$$HG = MC^\dagger CM^* = M(C^\dagger C)^* M^* = (HG)^*,$$

$$GHG = LCC^\dagger CM^* = LCM^* = G$$

and

$$HGH = MG^\dagger GG^\dagger L^* = MG^\dagger L^* = H,$$

so G and H satisfy the Penrose conditions. □

For clarity we first consider the case where $\gcd(\underline{\alpha}, \underline{n}) = \underline{1}$.

Theorem 9 *If* $A = [A_{\underline{s}-\underline{\alpha}\,\underline{r}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}}$ *and* $\gcd(\underline{\alpha},\underline{n}) = \underline{1}$, *then*

$$A^{\dagger} = [B_{\underline{r}-\underline{\alpha}\,\underline{s}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \quad \text{with} \quad B_{\underline{m}} = \frac{1}{c(\underline{n})}\sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} \zeta^{\underline{\ell}\,\underline{m}} F_{\underline{\ell}}^{\dagger}, \quad \underline{0} \preceq \underline{m} \preceq \underline{n}-\underline{1}. \tag{45}$$

PROOF. From Theorem 4, $A = U_{\alpha}\mathcal{F}_{\mathcal{A}}P^*$ (see (15), (16), and (29)), which is written in expanded form in (30). As noted following (16), $U_{\underline{\alpha}}$ is unitary because $(\underline{\alpha},\underline{n}) = \underline{1}$. Since P is unitary in any case, Lemma 1 with $L = U_{\underline{\alpha}}$, $M = P$, and $C = \mathcal{F}_A$ implies that

$$\begin{aligned} A^{\dagger} &= P\mathcal{F}_{\mathcal{A}}^{\dagger}U_{\alpha}^{*} = \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} P_{\ell}F_{\ell}^{\dagger}P_{\underline{\alpha}\,\underline{\ell}} \\ &= \frac{1}{c(\underline{n})}\sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} \begin{bmatrix} 1\otimes I_{d_2} \\ \vdots \\ \zeta^{\underline{\ell}\,\underline{r}}\otimes I_{d_2} \\ \vdots \\ \zeta^{\underline{\ell}(\underline{n}-\underline{1})}\otimes I_{d_2} \end{bmatrix} F_{\underline{\ell}}^{\dagger} \begin{bmatrix} 1\otimes I_{d_1} \\ \vdots \\ \zeta^{\underline{\ell}\,\underline{\alpha}\,\underline{s}}\otimes I_{d_1} \\ \vdots \\ \zeta^{\underline{\ell}\,\underline{\alpha}\,(\underline{n}-\underline{1})}\otimes I_{d_1} \end{bmatrix}^{H} \\ &= \frac{1}{c(\underline{n})}\left[\sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} \zeta^{\underline{\ell}(\underline{r}-\underline{\alpha}\,\underline{s})}F_{\ell}^{\dagger}\right]_{\underline{r},\underline{s}=0}^{\underline{n}-\underline{1}} = [B_{\underline{r}-\underline{\alpha}\,\underline{s}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}}, \end{aligned} \tag{46}$$

from (45).

Theorem 10 *Let* $A = [A_{\underline{s}-\underline{\alpha}\,\underline{r}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}}$, $\gcd(\underline{\alpha},\underline{n}) = \underline{q}$, *and* $\underline{p} = \underline{k}/\underline{q}$. *Let* $\mathbf{F}_{\underline{\ell},\underline{\alpha}}$ *be as in* (34) *and partition* $\mathbf{F}_{\underline{\ell},\underline{\alpha}}^{\dagger}$ *as*

$$\mathbf{F}_{\underline{\ell},\underline{\alpha}}^{\dagger} = \begin{bmatrix} G_{\underline{\ell},\underline{\alpha}} \\ \vdots \\ G_{\underline{\ell}+\underline{\nu}\,\underline{p},\underline{\alpha}} \\ \vdots \\ G_{\underline{\ell}+(\underline{q}-\underline{1})\underline{p},\underline{\alpha}} \end{bmatrix}, \quad \underline{0} \preceq \underline{\ell} \preceq \underline{p}-\underline{1}, \tag{47}$$

where $G_{\underline{\ell},\underline{\alpha}} \in \mathbb{C}^{d_2\times d_1}$, $\underline{0} \preceq \underline{\ell} \preceq \underline{n}-\underline{1}$. *Then*

$$A^{\dagger} = [B_{\underline{r}-\underline{\alpha}\,\underline{s}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \quad \text{with} \quad B_{\underline{m}} = \frac{1}{c(\underline{n})}\sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} \zeta^{\underline{\ell}\,\underline{m}} G_{\underline{\ell},\underline{\alpha}}, \quad \underline{0} \preceq \underline{m} \preceq \underline{n}-\underline{1}. \tag{48}$$

PROOF. From Theorem 5 (specifically, (37)), $A = \mathcal{V}_{\underline{\alpha}}\mathcal{F}_{\underline{\alpha}}\mathcal{Q}_{\underline{\alpha}}^{*}$. Recalling (36), Lemma 1 with $L = \mathcal{V}_{\underline{\alpha}}$, $M = \mathcal{Q}_{\underline{\alpha}}$, and $C = \mathcal{F}_{\underline{\alpha}}$ implies that

$$A^{\dagger} = \mathcal{Q}_{\underline{\alpha}}\mathcal{F}_{\underline{\alpha}}^{\dagger}\mathcal{V}_{\underline{\alpha}}^{*} = \sum_{\underline{\ell}=\underline{0}}^{\underline{p}-\underline{1}} \mathbf{Q}_{\underline{\ell},\underline{\alpha}}\mathbf{F}_{\underline{\ell},\underline{\alpha}}^{\dagger}P_{\underline{\alpha}\,\underline{\ell}}^{*} = \sum_{\underline{\ell}=\underline{0}}^{\underline{p}-\underline{1}}\left(\sum_{\underline{\nu}=\underline{0}}^{\underline{q}-\underline{1}} Q_{\underline{\ell}+\underline{\nu}\,\underline{p}}G_{\underline{\ell}+\underline{\nu}\,\underline{p},\underline{\alpha}}\right)P_{\underline{\alpha}\,\underline{\ell}}^{*},$$

where the second equality follows from (32), (33), and (35) and the third equality follows from (31) and (47). Since $P_{\alpha(\underline{\ell}+\underline{\nu}\,\underline{p})} = P_{\alpha\,\underline{\ell}}$, $\underline{0} \preceq \underline{\nu} \preceq \underline{q}-\underline{1}$, we can now write

$$A^{\dagger} = \sum_{\underline{\ell}=\underline{0}}^{\underline{p}-\underline{1}} \sum_{\underline{\nu}=\underline{0}}^{\underline{q}-\underline{1}} Q_{\underline{\ell}+\underline{\nu}\,\underline{p}} G_{\underline{\ell}+\underline{\nu}\underline{p},\underline{\alpha}} P^{*}_{\underline{\alpha}(\underline{\ell}+\underline{\nu}\underline{p})} = \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} Q_{\underline{\ell}} G_{\underline{\ell},\underline{\alpha}} P^{*}_{\underline{\alpha}\,\underline{\ell}}.$$

Now (4) and (12) imply that

$$\begin{aligned} A^{\dagger} &= \frac{1}{c(\underline{n})} \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} \begin{bmatrix} 1 \otimes I_{d_2} \\ \vdots \\ \zeta^{\underline{\ell}\,\underline{r}} \otimes I_{d_2} \\ \vdots \\ \zeta^{\underline{\ell}\,(\underline{n}-\underline{1})} \otimes I_{d_2} \end{bmatrix} G_{\underline{\ell},\underline{\alpha}} \begin{bmatrix} 1 \otimes I_{d_1} \\ \vdots \\ \zeta^{\underline{\ell}\,\underline{\alpha}\,\underline{s}} \otimes I_{d_1} \\ \vdots \\ \zeta^{\underline{\ell}\,\underline{\alpha}\,(\underline{n}-\underline{1})} \otimes I_{d_1} \end{bmatrix}^{H} \\ &= \frac{1}{c(\underline{n})} \left[\sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} \zeta^{\underline{\ell}(\underline{r}-\underline{\alpha}\,\underline{s})} G_{\underline{\ell},\underline{\alpha}} \right]_{\underline{r},\underline{s}=0}^{\underline{n}-\underline{1}} = [B_{\underline{r}-\underline{\alpha}\,\underline{s}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \end{aligned}$$

with $B_{\underline{0}}, \ldots, B_{\underline{m}}, \ldots, B_{\underline{n}-\underline{1}}$ as in (48). □

Remark 2 If $A = [a_{\underline{s}-\underline{\alpha}\underline{r}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \in \mathbb{C}^{\underline{n}:1\times 1}$ then (28) and (34) reduce to

$$f_{\underline{\ell}} = \sum_{\underline{m}=\underline{0}}^{\underline{n}-\underline{1}} a_{\underline{m}} \zeta^{\underline{\ell}\,\underline{m}} \quad \text{and} \quad \mathbf{f}_{\underline{\ell},\underline{\alpha}} = \begin{bmatrix} f_{\underline{\ell}} & f_{\underline{\ell}+\underline{p}} & \cdots & f_{\underline{\ell}+(\underline{q}-\underline{1})\underline{p}} \end{bmatrix}, \quad \underline{0} \preceq \underline{\ell} \preceq \underline{p}-\underline{1}.$$

Since

$$\mathbf{f}^{\dagger}_{\underline{\ell},\underline{\alpha}} = \frac{1}{\|\mathbf{f}_{\underline{\ell},\underline{\alpha}}\|^2} \begin{bmatrix} \overline{f}_{\underline{\ell}} \\ \vdots \\ \overline{f}_{\underline{\ell}+\underline{\nu}\,\underline{p}} \\ \vdots \\ \overline{f}_{\underline{\ell}+(\underline{q}-\underline{1})\underline{p}} \end{bmatrix} \quad \text{if} \quad \mathbf{f}_{\underline{\ell},\underline{\alpha}} \neq 0 \quad \text{or} \quad \mathbf{f}^{\dagger}_{\underline{\ell},\underline{\alpha}} = 0 \quad \text{if} \quad \mathbf{f}_{\underline{\ell},\underline{\alpha}} = 0,$$

it follows that

$$g_{\underline{\ell}+\underline{\nu}\underline{p},\underline{\alpha}} = \begin{cases} \overline{f}_{\underline{\ell}+\underline{\nu}\,\underline{p}} / |\mathbf{f}_{\underline{\ell},\underline{\alpha}}|^2 & \text{if} \quad \mathbf{f}_{\underline{\ell},\underline{\alpha}} \neq 0, \\ 0 & \text{if} \quad \mathbf{f}_{\underline{\ell},\underline{\alpha}} = \underline{0}, \end{cases} \quad \underline{0} \preceq \underline{\ell} \leq \underline{p}-\underline{1}, \quad \underline{0} \preceq \underline{\nu} \preceq \underline{q}-\underline{1}.$$

Hence

$$A^{\dagger} = [b_{\underline{r}-\underline{\alpha}\,\underline{s}}]_{r,\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \quad \text{where} \quad b_m = \frac{1}{c(\underline{n})} \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} g_{\underline{\ell},\underline{\alpha}} \zeta^{\underline{\ell}\,\underline{m}}.$$

8 The case where $\gcd(\underline{\alpha}, \underline{n}) = \underline{1}$ and $d_1 = d_2$

In this section we assume that $\gcd(\underline{\alpha}, \underline{n}) = \underline{1}$ and $d_1 = d_2 = d$, so (30) becomes

$$A = U_{\underline{\alpha}} \mathcal{F}_{\mathcal{A}} P^*. \tag{49}$$

Theorem 11 *$A = [A_{\underline{s}-\underline{\alpha}\,\underline{r}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}}$ is invertible if and only if* $\gcd(\underline{\alpha}, \underline{n}) = \underline{1}$ *and $F_{\underline{0}}, \dots, F_{\underline{m}}, \dots, F_{\underline{n}-\underline{1}}$ are all invertible, in which case*

$$A^{-1} = [B_{\underline{r}-\underline{\alpha}\,\underline{s}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} \quad \textit{with} \quad B_{\underline{m}} = \frac{1}{c(\underline{n})} \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} \zeta^{\underline{\ell}\,\underline{m}} F_{\underline{\ell}}^{-1}, \quad \underline{0} \preceq \underline{m} \preceq \underline{n} - \underline{1}, \tag{50}$$

and the solution of $Az = w$ is $z = \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{\ell}} F_{\underline{\ell}}^{-1} v_{\underline{\alpha}\,\underline{\ell}}$.

PROOF. If A is invertible then $U_{\underline{\alpha}}$ must be invertible, which is true if and only if $\gcd(\underline{\alpha}, \underline{n}) = \underline{1}$. Hence this is a necessary condition for A to be invertible. If $\gcd(\underline{\alpha}, \underline{n}) = \underline{1}$ then (41) implies that A is invertible if and only if $F_{\underline{0}}, \dots, F_{\underline{s}}, \dots, F_{\underline{n}-\underline{1}}$ are all invertible or, equivalently, $F_{\underline{s}}^{\dagger} = F_{\underline{s}}^{-1}$, $\underline{0} \preceq \underline{s} \preceq \underline{n} - \underline{1}$. Now Theorem 9 implies (50) which, with (39) and (41), implies the final conclusion. □

Theorem 12 *Suppose A is as in* (49) *and $\underline{\alpha}\underline{\beta} \equiv \underline{1} \pmod{\underline{n}}$. Then*:

(i) *A is Hermitian if and only if $P_{\underline{\beta}\underline{\ell}} F_{\underline{\beta}\underline{\ell}}^* = P_{\underline{\alpha}\,\underline{\ell}} F_{\underline{\ell}}$, $\underline{0} \preceq \underline{\ell} \preceq \underline{n} - \underline{1}$*

(ii) *A is normal if and only if $F_{\underline{\beta}\underline{\ell}} F_{\underline{\beta}\underline{\ell}}^* = F_{\underline{\ell}}^* F_{\underline{\ell}}$, $\underline{0} \preceq \underline{\ell} \preceq \underline{n} - \underline{1}$*

(iii) *A is EP (i.e., $A^{\dagger} A = A A^{\dagger}$) if and only if $F_{\underline{\ell}}^{\dagger} F_{\underline{\ell}} = F_{\underline{\beta}\underline{\ell}} F_{\underline{\beta}\underline{\ell}}^{\dagger}$, $\underline{0} \preceq \underline{\ell} \preceq \underline{n} - \underline{1}$.*

PROOF. From (49) and (46) with $\underline{\alpha} = \underline{1}$,

$$A = \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{\alpha}\,\underline{\ell}} F_{\underline{\ell}} P_{\underline{\ell}}^*, \quad A^* = \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{\ell}} F_{\underline{\ell}}^* P_{\underline{\alpha}\,\underline{\ell}}^*, \quad \text{and} \quad A^{\dagger} = \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{\ell}} F_{\underline{\ell}}^{\dagger} P_{\underline{\alpha}\,\underline{\ell}}^*. \tag{51}$$

(i) Since $\underline{\alpha}\underline{\beta} \equiv 1 \pmod{\underline{n}}$, replacing $\underline{\ell}$ by $\underline{\beta}\,\underline{\ell}$ in the second sum in (51) yields $A^* = \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{\beta}\underline{\ell}} F_{\underline{\beta}\,\underline{\ell}}^* P_{\underline{\ell}}^*$, and comparing this with the first sum in (51) yields (i).

(ii) From (51),

$$AA^* = \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{\alpha}\,\underline{\ell}} F_{\underline{\ell}} F_{\underline{\ell}}^* P_{\underline{\alpha}\,\underline{\ell}}^* = \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{\ell}} F_{\underline{\beta}\,\underline{\ell}} F_{\underline{\beta}\,\underline{\ell}}^* P_{\underline{\ell}}^* \quad \text{and} \quad A^*A = \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{\ell}} F_{\underline{\ell}}^* F_{\underline{\ell}} P_{\underline{\ell}}^*,$$

which implies (ii).

(iii) From (51),

$$AA^{\dagger} = \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{\alpha}\,\underline{\ell}} F_{\underline{\ell}} F_{\underline{\ell}}^{\dagger} P_{\underline{\alpha}\,\underline{\ell}}^* = \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{\ell}} F_{\underline{\beta}\underline{\ell}} F_{\underline{\beta}\,\underline{\ell}}^{\dagger} P_{\underline{\ell}}^* \quad \text{and} \quad A^{\dagger}A = \sum_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} P_{\underline{\ell}} F_{\underline{\ell}}^{\dagger} F_{\underline{\ell}} P_{\underline{\ell}}^*,$$

which implies (iii). □

Remark 3 If A is a square matrix and there is a matrix B such that $ABA = A$, $BAB = B$, and $AB = BA$, then B is unique and is called the group inverse of A, denoted by $B = A^{\#}$. Theorem 12(iii) implies that $A^{\dagger} = A^{\#}$ if and only if $F_{\underline{\ell}}^{\dagger} F_{\underline{\ell}} = F_{\underline{\beta\ell}} F_{\underline{\beta\ell}}^{\dagger}$, $\underline{0} \preceq \underline{\ell} \preceq \underline{n} - \underline{1}$.

9 The eigenvalue problem with $\underline{\alpha} = \underline{1}$

Here we assume that $\underline{\alpha} = \underline{1}$ and $d_1 = d_2 = d$, so (6) and (12) reduce to

$$R = S = E_1 \otimes E_2 \otimes \cdots \otimes E_k \otimes I_d = ([\delta_{\underline{r},\underline{s}-\underline{1}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}}) \otimes I_d,$$

$$P_{\underline{s}} = Q_{\underline{s}} = \phi_{\underline{s}} \otimes I_d, \quad \underline{0} \preceq \underline{s} \preceq \underline{n} - \underline{1},$$

and (30) reduces to

$$A = P \mathcal{F}_{\mathcal{A}} P^* = \sum_{\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} P_s F_s P_s^*.$$

The following theorem and its proof are motivated by [2, Theorem 2].

Theorem 13 *Let*

$$\mathcal{S}_R = \bigcup_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} \left\{ z \in \mathbb{C}^{\underline{n}:d} \,\middle|\, Rz = \zeta^{\underline{\ell}} z \right\}.$$

If λ is an eigenvalue of A let $\mathcal{E}_A(\lambda)$ be the λ-eigenspace of A; i.e,

$$\mathcal{E}_A(\lambda) = \left\{ z \,\middle|\, Az = \lambda z \right\}.$$

Then:

(i) *If λ is an eigenvalue of $A = [A_{\underline{s}-\underline{r}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}}$ then $\mathcal{E}_A(\lambda)$ has a basis in $\mathcal{S}_R$.*

(ii) *If $A \in \mathbb{C}^{\underline{n}:d \times d}$ and has $c(\underline{n})d$ linearly independent eigenvectors in $\mathcal{S}_R$, then A is a $\underline{1}$-circulant.*

Proof. (i) From (41) with $w = \lambda z$ and $\underline{\alpha} = \underline{1}$, $z = \sum_{\underline{\ell}=\underline{0}}^{\underline{k}-\underline{1}} P_{\underline{\ell}} u_{\underline{\ell}} \in \mathcal{E}_A(\lambda)$ if and only if $F_{\underline{\ell}} u_{\underline{\ell}} = \lambda u_{\underline{\ell}}$, $\underline{0} \preceq \underline{\ell} \preceq \underline{n} - \underline{1}$. Therefore λ is an eigenvalue of A if and only if it is an eigenvalue of $F_{\underline{\ell}}$ for some $\underline{\ell} \in \mathcal{M}_{\underline{n}}$. Let $\mathcal{T}_\lambda$ be the subset of $\mathcal{M}_{\underline{n}}$ for which this is true. Then $\mathcal{E}_A(\lambda)$ consists of linear combinations of the vectors of the form $P_{\underline{\ell}} u_{\underline{\ell}}$ with $\underline{\ell} \in \mathcal{T}_\lambda$ and $(\lambda, u_{\underline{\ell}})$ an eigenpair of $F_{\underline{\ell}}$. Since $RP_{\underline{\ell}} = \zeta^{\underline{\ell}} P_{\underline{\ell}}$ (see (6)), this completes the proof of (i).

(ii) From Theorem 1 with $R = S$ and $\underline{\alpha} = \underline{1}$, we must show that $RA = AR$. If $Az = \lambda z$ and $Rz = \zeta^{\underline{s}} z$ then $RAz = \lambda Rz = \lambda \zeta^{\underline{s}} z$ and $ARz = \zeta^{\underline{s}} Az = \zeta^{\underline{s}} \lambda z$. Hence $ARz = RAz$ for all z in a basis for $\mathbb{C}^{\underline{n}:d}$, so $AR = RA$. □

Theorem 14 *Suppose $\left\{ F_{\underline{\ell}} \,\middle|\, \underline{\ell} \in \mathcal{M}_{\underline{n}} \right\}$ and $\left\{ A_{\underline{m}} \,\middle|\, \underline{m} \in \mathcal{M}_{\underline{n}} \right\}$ are related as in* (28)*, and $F_{\underline{\ell}} = \Psi_{\underline{\ell}} J_{\underline{\ell}} \Psi_{\underline{\ell}}^{-1}$ is the Jordan canonical form of $F_{\underline{\ell}}$, $\underline{0} \preceq \underline{\ell} \preceq \underline{n} - \underline{1}$. Let*

$$\Gamma = \left[\begin{array}{ccccc} P_{\underline{0}} \Psi_{\underline{0}} & \cdots & P_{\underline{\ell}} \Psi_{\underline{\ell}} & \cdots & P_{\underline{n}} \Psi_{\underline{n}} \end{array} \right].$$

Then

$$[A_{\underline{s}-\underline{r}}]_{\underline{r},\underline{s}=\underline{0}}^{\underline{n}-\underline{1}} = \Gamma \left(\bigoplus_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} J_{\underline{\ell}} \right) \Gamma^{-1}.$$

In particular, suppose that $F_{\underline{0}}, \dots, F_{\underline{\ell}}, \dots, F_{\underline{n}-\underline{1}}$ *are all diagonalizable with spectral decompositions*

$$F_{\underline{\ell}} = T_{\underline{\ell}} D_{\underline{\ell}} T_{\underline{\ell}}^{*}, \quad \underline{0} \preceq \ell \preceq \underline{n} - \underline{1},$$

and

$$\Delta = \left[\begin{array}{ccccc} P_{\underline{0}} D_{\underline{0}} & \cdots & P_{\underline{\ell}} D_{\underline{\ell}} & \dots & P_{\underline{n}-\underline{1}} D_{\underline{n}-\underline{1}} \end{array}\right].$$

Then

$$A = \Delta \left(\bigoplus_{\underline{\ell}=\underline{0}}^{\underline{n}-\underline{1}} D_{\underline{\ell}} \right) \Delta^{-1}.$$

References

[1] C. M. Ablow, J. L. Brenner, Roots and canonical forms for circulant matrices, Trans. Amer. Math. Soc. 107 (1963) 360–376.

[2] A. L. Andrew, Eigenvectors of certain matrices, Linear Algebra Appl. 7 (1973) 151–162.

[3] V. Olshevsky, I. Oseledets and E. Tyrtyshnikov, Tensor properties of multilevel Toeplitz and related matrices, Lin. Alg. Appl. 412 (2006) 1–21

[4] S. Serra Capizzano, A Korovkin-type theory for finite Toeplitz operators via matrix algebras, Numerische Mathematik 82 (1999) 117–142.

[5] S. Serra Capizzano, A Korovkin based approximation of multilevel Toeplitz matrices (with rectangular unstructured blocks) via multilevel trigonometric matrix spaces, SIAM Journal on Numerical Analysis 36 (1999) 1831–1857.

[6] S. Serra Capizzano and E. Tyrtyshnikov, Any circulant-like preconditioner for multilevel matrices is not superlinear, SIAM J. Matrix Anal. Appl. 21-2 (1999) 431Ű-439.

[7] S. Serra-Capizzano, Stefano and C. Tablino-Possio, Multigrid methods for multilevel circulant matrices SIAM J. Sci. Comput. 26 (2004) 55-Ű85.

[8] W. F. Trench, Properties of unilevel block circulants, Linear Algebra Appl. 430 (2009) 2012Ű-2025.

[9] E. Tyrtyshnikov, Optimal and superoptimal circulant preconditioners, SIAM J. Matrix Anal. Appl. 13 (1992) 459-473.

[10] E. E. Tyrtyshnikov, Toeplitz matrices, some analogs and applications (in Russian), Inst. of Numer. Math., Rus. Acad. of Sci. (1989).

[11] V. V. Voevodin, E. E. Tyrtyshnikov, Computer Processes with Toeplitz Matrices, Nauka, Moscow, 1987 (in Russian),