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On nonautonomous linear systems of differential and difference equations with R-symmetric coefficient matrices

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Abstract

Let $\mathbb{C}^{n\times n}(\mathcal{J})$ denote the set of continuous $n\times n$ matrices on an interval $\mathcal{J}$. We say that $R\in\mathbb{C}^{n\times n}(\mathcal{J})$ is a nontrivial k-involution if $R=P\left(\bigoplus_{\ell=0}^{k-1}\zeta^{\ell}I_{d_\ell}\right)P^{-1}$ where $\zeta=e^{-2\pi i/k}$, $d_0+d_1+\cdots+d_{k-1}=n$, and $P'=P\bigoplus_{\ell=0}^{k-1}U_\ell$ with $U_\ell\in\mathbb{C}^{d_\ell\times d_\ell}(\mathcal{J})$. We say that $A\in\mathbb{C}^{n\times n}(\mathcal{J})$ is R-symmetric if $R(t)A(t)R^{-1}(t)=A(t)$, $t\in\mathcal{J}$, and we show that if A is R-symmetric then solving $x'=A(t)x$ or $x'=A(t)x+f(t)$ reduces to solving k independent $d_\ell\times d_\ell$ systems, $0\le\ell\le k-1$. We consider the asymptotic behavior of the solutions in the case where $\mathcal{J}=[t_0,\infty)$. Finally, we sketch analogous results for linear systems of difference equations.

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1 Introduction

Throughout this paper $\mathcal{J}$ is an interval on the real line and $\mathbb{C}^p$, $\mathbb{C}^p(\mathcal{J})$, $\mathbb{C}^{p\times q}$, $\mathbb{C}^{p\times q}(\mathcal{J})$, and $\mathbb{C}_1^{p\times q}(\mathcal{J})$ are respectively the following sets: complex p-vectors, continuous complex p-vector functions on $\mathcal{J}$, complex $p\times q$ matrices, continuous complex $p\times q$ matrix functions on $\mathcal{J}$, and continuously differentiable complex $p\times q$ matrix functions on $\mathcal{J}$. (“Complex" can just as well be replaced by “real.") If $z\in\mathbb{C}^p$ and $B\in\mathbb{C}^{p\times p}$ then $\|z\|$ and $\|B\|$ are respectively any norm of z and the corresponding induced norm of B; i.e., $\|B\|=\max\left\{\|Bz\|\,\middle|\,\|z\|=1\right\}$.

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We consider nonautonomous systems of linear differential equations

$$x' = A(t)x \quad \text{with} \quad A \in \mathbb{C}^{n\times n}(\mathcal{J}) \tag{1}$$

and

$$x' = A(t)x + f(t) \;\; \text{with} \;\; A \in \mathbb{C}^{n\times n}(\mathcal{J}) \;\; \text{and} \;\; f \in \mathbb{C}^{n}(\mathcal{J}), \tag{2}$$

where A has special structure that we will specify in Section 2. We will show that the structure can be exploited to expedite solving these system and, if $\mathcal{J} = [t_{0,\infty}$, to study the asymptotic behavior of their solutions. To illustrate the second point, we recall that (1) is said to have linear asymptotic equilibrium if every nontrivial solution of (1) approaches a nonzero limit as $t \to \infty$ or, equivalently, for every $c \in \mathbb{C}^n$, (1) has a unique solution x such that $\lim_{t\to\infty} x(t) = c$. The simplest condition for linear asymptotic equilibrium of (1) (attributed by Wintner [13] to Bôcher) is that $\int^\infty \|A(t)\|\, dt < \infty$. (Conti [3, 4], Sansone and Conti [5], Wintner [13], the author [6, 7], and others have weakened this condition, but in all cases there is ultimately a requirement that $\int^\infty \|B(t)\|\, dt < \infty$ for some $B \in \mathbb{C}^{n\times n}(\mathcal{J})$ derived from A.)

By way of motivation we first consider two examples. For the first we introduce the notation $\zeta = e^{-2\pi i/k}$ and

$$\Phi = \begin{bmatrix} \Phi_0 & \Phi_1 & \cdots & \Phi_{k-1} \end{bmatrix} \text{ with } \Phi_\ell = \frac{1}{\sqrt{k}} \begin{bmatrix} 1 \\ \zeta^\ell \\ \vdots \\ \zeta^{(k-1)\ell} \end{bmatrix} \otimes I_d, \tag{3}$$

$0 \le \ell \le k-1$, where $k \ge 2$ and $d \ge 1$.

Example 1 If $A = [A_{s-r}]_{r,s=0}^{k-1}$ is a variable block circulant with $A_0, A_1, \ldots, A_{k-1} \in \mathbb{C}^{d\times d}(\mathcal{J})$, then $\int^\infty \|A(t)\|\, dt < \infty$ if and only if $\int^\infty \|A_r(t)\|\, dt < \infty$, $0 \le r \le k-1$. Hence, applying Bôcher's theorem directly to (1) yields no conclusion if $\int^\infty \|A_r(t)\|\, dt = \infty$ for some $r \in \{0, 1, \ldots, k-1\}$. However, it is well known (see, e.g., [12]) that

$$A = \Phi \left(\bigoplus_{\ell=0}^{k-1} F_\ell \right) \Phi^* = \sum_{\ell=0}^{k-1} \Phi_\ell F_\ell \Phi_\ell^*, \tag{4}$$

where

$$F_\ell = \sum_{\ell=0}^{k-1} \zeta^{\ell m} A_m \in \mathbb{C}^{d\times d}(\mathcal{J}), \quad 0 \le \ell \le k-1. \tag{5}$$

($\{F_0, F_1, \ldots, F_{k-1}\}$ is the discrete Fourier transform of $\{A_0, A_1, \ldots, A_{k-1}\}$.) Therefore every solution of (1) is of the form

$$x = \sum_{\ell=0}^{k-1} \Phi_\ell y_\ell \quad \text{where} \quad y_\ell' = F_\ell y_\ell, \quad 0 \le \ell \le k-1.$$

Moreover, if $\mathcal{J} = [t_0, \infty)$,

$$\mathcal{S} = \left\{ \ell \,\middle|\, \int^{\infty} \|F_\ell(t)\|\, dt < \infty \right\} \neq \emptyset, \text{ and } u_\ell \in \mathbb{C}^d, \quad \ell \in \mathcal{S},$$

then applying Bôcher's theorem separately to $y'_\ell = F_\ell(t) y_\ell$, $\ell \in \mathcal{S}$, shows that $x' = A(t)x$ has a unique solution such that

$$\lim_{t \to \infty} x(t) = \sum_{\ell \in \mathcal{S}} \Phi_\ell u_\ell.$$

As observed by Ablow and Brenner [1] for the case where $d = 1$, the decomposition (4) is possible because the block circulant A commutes with

$$R = [\delta_{r,s-1(\text{mod k})}]_{r,s=0}^{k-1} = \Phi \left(\bigoplus_{\ell=0}^{k-1} \zeta^\ell I_d \right) \Phi^*.$$

We explored this idea more generally in [11, 12].

Example 2 Suppose $R \in \mathbb{C}^{n \times n}$, $R \neq \pm I$, and $R^2 = I$, so

$$R = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & -I_s \end{bmatrix} \begin{bmatrix} \widehat{P} \\ \widehat{Q} \end{bmatrix},$$

where

$$\widehat{P} P = I_r, \quad \widehat{Q} Q = I_s, \quad \widehat{P} Q = 0, \quad \text{and} \quad \widehat{Q} P = 0.$$

In [10] we defined a matrix A to be R-symmetric if it commutes with A and showed that A is R-symmetric if and only if

$$A = \begin{bmatrix} P & Q \end{bmatrix} \begin{bmatrix} A_P & 0 \\ 0 & A_Q \end{bmatrix} \begin{bmatrix} \widehat{P} \\ \widehat{Q} \end{bmatrix} \text{ with } A_P \in \mathbb{C}^{r \times r} \text{ and } A_Q \in \mathbb{C}^{s \times s}.$$

Therefore, if $A \in \mathbb{C}^{n \times n}(\mathcal{J})$ is R-symmetric for all $t \in \mathcal{J}$, then the solutions of (1) are of the form $x = Pu + Qv$ where $u' = A_P u$ and $v' = A_Q v$. Moroever, Bôcher's theorem implies that if $\int^\infty \|A_P(t)\|\, dt < \infty$ and $u_0 \in \mathbb{C}^r$ then (1) has a unique solution $y_P = Pu$ such that $\lim_{t \to \infty} y_P(t) = Pu_0$. Also, if $\int^\infty \|A_Q(t)\|\, dt < \infty$ and $v_0 \in \mathbb{C}^s$ then (1) has a unique solution $y_Q = Qv$ such that $\lim_{t \to \infty} y_Q(t) = Qv_0$.

In these two examples R is a constant matrix. Here we extend these ideas to include the possibility that A in (1) commutes with a variable matrix function R which is k-involutory (defined precisely in the next section) for all t. We will show how to solve such systems efficiently and indicate how their asymptotic behavior can be analyzed by appropriate application of theorems such as Bôcher's.

In Section 4 we sketch an analogous approach to linear systems of difference equations with coefficient matrices that commute with appropriately defined variable k-involutory matrix functions.

2 Preliminaries

Henceforth n and k are integers such that $2 \le k \le n$ and $d_0, d_1, \dots, d_{k-1}$ are positive integers such that $\sum_{\ell=0}^{k-1} d_\ell = n$. In [10, 11] we defined a nontrivial k-involution $R \in \mathbb{C}^{n\times n}$ to be a constant matrix of the form

$$R = P\mathbf{D}P^{-1} \quad \text{where} \quad \mathbf{D} = \bigoplus_{\ell=0}^{k-1} \zeta^\ell I_{d_\ell}, \tag{6}$$

and showed that if $A \in \mathbb{C}^{n\times n}$ commutes with R then all computational problems associated with A reduce to the corresponding problems for k independent systems with coefficient matrices in $\mathbb{C}^{d_\ell \times d_\ell}$, $0 \le \ell \le k-1$. Here we define a nontrivial k-involution to be a matrix function $R \in \mathbb{C}^{n\times n}(\mathcal{J})$ of the form (6), where P is a fundamental (i.e., invertible solution) matrix for the system

$$P' = P\mathbf{U} \;\text{ with }\; \mathbf{U} = \bigoplus_{\ell=0}^{k-1} U_\ell \;\text{ and }\; U_\ell \in \mathbb{C}^{d_\ell\times d_\ell}(\mathcal{J}),\; 0 \le \ell \le k-1. \tag{7}$$

As we will see, (7) is a technical assumption that seems to be crucial for the construction of a useful theory of R-symmetric differential systems. (Except for $\mathbf{D}$ and $\mathbf{D}_0$ as in (6) and (9) below, a boldface symbol always denotes a direct sum of this form in which the identifiers of the direct sum - $\mathbf{U}$ in this case - and the summands - U_0, U_1, $\dots$, U_{k-1} in this case - are related as they are here.) We say that R is equidimensional with width d if $n = kd$ and $d_0 = d_1 = \cdots = d_{k-1} = d$. Note that $R^k(t) = I$ for all $t \in \mathcal{J}$ and $R^m(t_0) \ne I$ for any $t_0 \in \mathcal{J}$ if $m < k$.

We define $A \in \mathbb{C}^{n\times n}(\mathcal{J})$ to be R-symmetric if $RAR^{-1} = A$. If A is R-symmetric we say that the systems (1) and (2) are R-symmetric. We show that solving an R-symmetric system of linear differential equations reduces to solving k independent systems with coefficient matrices in $\mathbb{C}^{d_\ell\times d_\ell}(\mathcal{J})$, $0 \le \ell \le k-1$. We also consider the asymptotic behavior of solutions of R-symmetric systems in the case where $\mathcal{J} = [t_0, \infty)$. In Section 4 we sketch an analogous theory of nonautonomous R-symmetric systems of difference equations.

We write

$$P = \begin{bmatrix} P_0 & P_1 & \cdots & P_{k-1} \end{bmatrix} \;\text{ with }\; P_\ell \in \mathbb{C}^{n\times d_\ell}(\mathcal{J}), \quad 0 \le \ell \le k-1,$$

and

$$P^{-1} = \begin{bmatrix} \widehat{P}_0 \\ \widehat{P}_1 \\ \vdots \\ \widehat{P}_{k-1} \end{bmatrix} \;\text{ with }\; \widehat{P}_\ell \in \mathbb{C}^{d_\ell\times n}(\mathcal{J}), \;\text{ so }\; \widehat{P}_\ell P_m = \delta_{\ell m} I_{d_\ell}, \tag{8}$$

$0 \le \ell, m \le k-1$. From (7),

$$P'_\ell = P_\ell U_\ell, \quad 0 \le \ell \le k-1.$$

Since R-symmetry is particularly transparent if R is equidimensional, we consider this case separately. To this end, let $E = [\delta_{r,s-1}]_{r,s=0}^{k-1} \otimes I_d$, and $B = [B_{rs}]_{r,s=0}^{k-1}$, where $B_{rs} \in \mathbb{C}^{d\times d}(\mathcal{J})$, $0 \le r, s \le k-1$, and subscripts are to be reduced modulo k. Then

$$EBE^{-1} = [B_{r+1,s+1}]_{r,s=0}^{k-1} \text{ and } E\Phi = \Phi\mathbf{D_0} \text{ where } \mathbf{D_0} = \bigoplus_{\ell=0}^{k-1} \zeta^\ell I_d \tag{9}$$

and Φ is as in (3). Therefore

$$\text{(a) } E = \Phi\mathbf{D}_0\Phi^* \text{ and } \quad \text{(b)} \quad EBE^{-1} = B \text{ if and only if } B = [A_{s-r}]_{r,s=0}^{k-1}, \tag{10}$$

with $A_0, A_1, \dots, A_{k-1} \in \mathbb{C}^{d\times d}(\mathcal{J})$. (Conclusion (b) is a special case of [12, Theorem 1], an elementary extension of [1, Theorem 2.1].)

Theorem 1 *If R is as in (6) and $d_0 = d_1 = \cdots = d_{k-1} = d$ then $RAR^{-1} = A$ if and only if*

$$\Phi P^{-1} AP\Phi^* = [A_{s-r}]_{r,s=0}^{k-1} \text{ with } A_0, A_1, \dots, A_{k-1} \in \mathbb{C}^{d\times d}(\mathcal{J}) \tag{11}$$

and Φ as in (3). In this case

$$A = P\mathbf{F}P^{-1} = \sum_{\ell=0}^{k-1} P_\ell F_\ell \widehat{P}_\ell \tag{12}$$

with $F_0, F_1, \dots, F_{k-1}$ as in (5).

PROOF. Since $R = P\mathbf{D}_0 P^{-1}$ and $E = \Phi\mathbf{D}_0\Phi^*$, we can write

$$R = P\Phi^*(\Phi\mathbf{D}_0\Phi^*)\Phi P^{-1} = P\Phi^* E\Phi P^{-1}.$$

Then

$$RAR^{-1} = P\Phi^* E\Phi P^{-1} AP\Phi^* E^{-1}\Phi P^{-1},$$

so $RAR^{-1} = A$ if and only if

$$E\Phi P^{-1} AP\Phi^* E^{-1} = \Phi P^{-1} AP\Phi^*.$$

Therefore, (10)(b) implies (11). Since (5) is equivalent to

$$A_m = \frac{1}{k}\sum_{\ell=0}^{k-1} F_\ell \zeta^{-\ell m}, \quad 0 \le m \le k-1,$$

we can write

$$[A_{s-r}]_{r,s=0}^{k-1} = \frac{1}{k}\left[\sum_{\ell=0}^{k-1} \zeta^{-\ell(s-r)} F_\ell\right]_{r,s=0}^{k-1} = \sum_{\ell=0}^{k-1} \Phi_\ell F_\ell \Phi_\ell^* = \Phi\mathbf{F}\Phi^*,$$

so (11) implies that $P^{-1}AP = \mathbf{F}$, which implies (12). □

The following theorem characterizes matrices A such that $RAR^{-1} = A$, where $d_0, d_1, \dots, d_{k-1}$ are not necessarily equal. It extends [11, Theorem 2], where R is constant.

Theorem 2 *If R is as in (6) and $A \in \mathbb{C}^{n\times n}(\mathcal{J})$ then $RAR^{-1} = A$ if and only (12) holds with*

$$F_\ell = \widehat{P}_\ell A P_\ell \in \mathbb{C}^{d_\ell \times d_\ell}(\mathcal{J}), \quad 0 \le \ell \le k-1. \tag{13}$$

PROOF. From (6), $RAR^{-1} = A$ if and only if

$$P\mathbf{D}P^{-1}A\,P\mathbf{D}^{-1}P^{-1} = A \text{ or, equivalently, } \mathbf{D}(P^{-1}A\,P)\mathbf{D}^{-1} = P^{-1}AP. \tag{14}$$

If we write $P^{-1}A\,P = [C_{rs}]_{r,s=0}^{k-1}$ with $C_{rs} \in \mathbb{C}^{d_r\times d_s}(\mathcal{J})$, $0 \le r,s \le k-1$, then the second equality in (14) is equivalent to $\zeta^{r-s}C_{rs} = C_{rs}$, $0 \le r,s \le k-1$, which is equivalent to $C_{rs} = 0$ if $r \ne s$, $0 \le r,s \le k-1$. This is equivalent to (12) with $F_\ell = C_{\ell\ell}$, so $AP_\ell = P_\ell F_\ell$, $0 \le \ell \le k-1$, and (8) implies (13). $\square$

Note that the proofs of Theorems 1 and 2 did not require (7), which does not come into play until we consider the differential equations (1) and (2).

3 Solution of R-symmetric systems of linear differential equations

Recall that if $A, X \in \mathbb{C}^{n\times n}(\mathcal{J})$, $X' = A(t)X$, and $X(t_0)$ is invertible for some $t_0 \in \mathcal{J}$, then $X(t)$ is invertible for all $t \in \mathcal{J}$ and every solution of (1) can be written as $x(t) = X(t)c$, where $c \in \mathbb{C}^n$. In this case we say that X is a fundamental matrix for (1) and $x = Xc$ is the general solution of (1).

Now suppose A is R-symmetric; thus, from Theorem 2 (specifically, (12)) and (7), $A = P\mathbf{F}P^{-1}$ and $P' = P\mathbf{U}$. If we write $x = Py$ then $x' = P'y + Py' = P(\mathbf{U}y + y')$ and $Ax = P\mathbf{F}y$, so $x' = Ax$ if and only if $y' = \mathbf{G}y$ where $\mathbf{G} = \mathbf{F} - \mathbf{U}$. This last condition is equivalent to

$$y_\ell' = G_\ell(t)y_\ell \quad \text{with} \quad G_\ell = F_\ell - U_\ell \in \mathbb{C}^{d_\ell\times d_\ell}(\mathcal{J}),\ 0 \le \ell \le k-1, \tag{15}$$

and

$$y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{k-1} \end{bmatrix}.$$

This implies the following theorem.

Theorem 3 *If A is R-symmetric and $\mathbf{Y} = \bigoplus_{\ell=0}^{k-1} Y_\ell$ where $Y_0, Y_1, \dots, Y_{k-1}$ are fundamental matrices for the systems in (15), then*

$$X = P\mathbf{Y} = \begin{bmatrix} P_0Y_0 & P_1y_1 & \cdots & P_{k-1}Y_{k-1} \end{bmatrix}$$

is a fundamental matrix for (1). Hence, if $t_0 \in \mathcal{J}$ and $x_0 \in \mathbb{C}^{n\times n}$ then the solution of the initial value problem $x' = A(t)x$, $x(t_0) = x_0$, is

$$x(t) = \sum_{\ell=0}^{k-1} P_\ell(t)Y_\ell(t)Y_\ell^{-1}(t_0)y_{0\ell} \quad \text{where} \quad x_0 = \sum_{\ell=0}^{k-1} P_\ell(t_0)y_{0\ell}. \tag{16}$$

The general solution of (1) *is*

$$x(t) = \sum_{\ell=0}^{k-1} P_\ell(t)Y_\ell(t)c_\ell \quad \textit{where} \quad c_\ell \in \mathbb{C}^{d_\ell}, \quad 0 \le \ell \le k-1.$$

Corollary 1 *If A is R-symmetric then the general solution of $x' = A(t)x$ is $x = \sum_{\ell=0}^{k-1} P_\ell c_\ell$ with $c_\ell \in \mathbb{C}^{d_\ell}$, $0 \le \ell \le k-1$, if and only if $F_\ell = U_\ell$, $0 \le \ell \le k-1$.*

The following theorem is motivated by a theorem of Andrew [2] concerning the eigenvectors of constant centrosymmetric matrices. We extended Andrew's theorem to constant R-symmetric matrices in [10, Theorem 7] for $k = 2$ and in [11, Theorem 13] for $k \ge 2$.

Theorem 4 *Suppose $A, R \in \mathbb{C}^{n\times n}(\mathcal{I})$ and R is a nontrivial k-involution. Let*

$$\mathcal{S}_A = \left\{x \in \mathbb{C}_1^{n\times n}(\mathcal{I}) \,\middle|\, x'(t) = A(t)x(t),\ t \in \mathcal{I}\right\}$$

and

$$\mathcal{E}_R = \bigcup_{\ell=0}^{k-1} \left\{x \in \mathbb{C}_1^{n\times n}(\mathcal{I}) \,\middle|\, R(t)x(t) = \zeta^\ell x(t),\ t \in \mathcal{I}\right\}.$$

Then A is R-symmetric if and only if $\mathcal{S}_A$ has a basis in $\mathcal{E}_R$.

Proof. Since $RP_\ell = \zeta^\ell P_\ell$, $0 \le \ell \le k-1$, Theorem 3 (specifically, (16)) implies necessity. For sufficiency, if $\mathcal{S}_A$ has a basis in $\mathcal{E}_R$ then (1) has a fundamental matrix of the form

$$X = P\mathbf{Y} \quad \text{where} \quad \mathbf{Y} = \bigoplus_{\ell=0}^{k-1} Y_\ell \ \text{ with } \ Y_\ell \ \text{ and } \ Y_\ell^{-1} \in \mathbb{C}_1^{d_\ell\times d_\ell}(\mathcal{I}), \quad 0 \le \ell \le k-1.$$

Therefore $AP\mathbf{Y} = (P\mathbf{Y})' = P'\mathbf{Y} + P\mathbf{Y}'$, so

$$\begin{aligned} A &= (P'\mathbf{Y} + P\mathbf{Y}')\mathbf{Y}^{-1}P^{-1} = P'P^{-1} + P(\mathbf{Y}'\mathbf{Y}^{-1})P^{-1} \\ &= P(P^{-1}P')P^{-1} + P(\mathbf{Y}'\mathbf{Y}^{-1})P^{-1} \\ &= P(\mathbf{U} + \mathbf{Y}'\mathbf{Y}^{-1})P^{-1} = P\mathbf{F}P^{-1} \end{aligned}$$

(see (7)), with

$$\mathbf{F} = \mathbf{U} + \mathbf{Y}'\mathbf{Y}^{-1} = \bigoplus_{\ell=0}^{k-1}(U_\ell + Y_\ell' Y_\ell^{-1}).$$

Hence A is R-symmetric, by Theorem 2 and (7). □

Theorem 5 *Suppose $A \in \mathbb{C}^{n\times n}(\mathcal{I})$ is R-symmetric, $f \in \mathbb{C}^n(\mathcal{I})$, and $t_0 \in \mathcal{I}$. Let Y_0, $Y_1, \dots, Y_{k-1}$ be fundamental matrices for the systems in* (15) *and write*

$$x_0 = \sum_{\ell=0}^{k-1} P_\ell y_{0\ell} \ \textit{ with } \ y_{0\ell} \in \mathbb{C}^{d_\ell} \ \textit{ and } \ f = \sum_{\ell=0}^{k-1} P_\ell h_\ell \ \textit{ with } \ h_\ell \in \mathbb{C}^{d_\ell}(\mathcal{I}),$$

$0 \le \ell \le k-1$. *Then the solution of*

$$x' = A(t)x + f(t), \quad x(t_0) = x_0, \tag{17}$$

is

$$x(t) = \sum_{\ell=0}^{k-1} P_\ell(t) Y_\ell(t) \left(Y_\ell^{-1}(t_0) y_{0\ell} + \int_{t_0}^{t} Y_\ell^{-1}(\tau) h_\ell(\tau)\, d\tau \right). \tag{18}$$

PROOF. Apply the method of variation of parameters to each of the independent systems $y_\ell' = G_\ell(t) y_\ell + h_\ell$, $y_\ell(t_0) = y_{0\ell}$, $0 \le \ell \le k-1$. ☐

In [11] we defined a constant vector x to be (R, ℓ)-symmetric if R is a constant nontrivial k-involution and $Rx = \zeta^\ell x$. This extended a definition in [10] for $k = 2$. Andrew [2] originated this idea in connection with centrosymmetric matrices by defining x to be symmetric (skew-symmetric) if $Jx = x$ ($Jx = -x$), where J is the flip matrix with ones on the secondary diagonal and zeros elsewhwere. Here we say that a vector function $x = x(t) \in \mathbb{C}^{n\times n}(\mathcal{I})$ is (R, ℓ)-symmetric if $R(t)x(t) = \zeta^\ell x(t)$, $t \in \mathcal{I}$. Any $x \in \mathbb{C}^n(\mathcal{I})$ can be written uniquely as $x = \sum_{\ell=0}^{k-1} P_\ell y_\ell$ with $y_\ell \in \mathbb{C}^{d_\ell}(\mathcal{I})$, or equivalently, as $x = x_0 + x_1 + \cdots + x_{k-1}$ where $x_\ell = P_\ell y_\ell$ is (R, ℓ)-symmetric. We will call x_ℓ the (R, ℓ)-symmetric component of x. Thus, (18) exhibits the solution of (17) as the sum of its (R, ℓ)-symmetric components, $0 \le \ell \le k-1$. ☐

Now Bôcher's theorem implies the following result.

Theorem 6 *If Y_ℓ is a fundamental matrix for the system $y_\ell' = G_\ell(t) y_\ell$ (see (15)) on $[t_0, \infty)$ and $\int^\infty \|G_\ell(t)\|\, dt < \infty$ for some $\ell \in \{0, 1, \dots, k-1\}$, then $Y_\ell(\infty) = \lim_{t\to\infty} Y_\ell(t)$ exists and is invertible. Therefore the (R, ℓ)-symmetric component of any solution of $x' = A(t)x$ can be written uniquely as $x_\ell = P_\ell y_\ell$, where $y_\ell(\infty) = \lim_{t\to\infty} y_\ell(t)$ exists and is nonzero if $y_\ell(t_0) \neq 0$. Moreover, if $\lim_{t\to\infty} P_\ell(t)$ exists and has rank d_ℓ then $x_\ell(\infty) = \lim_{t\to\infty} x_\ell(t)$ exists and is nonzero if $x_\ell(t_0) \neq 0$.*

At the risk of making a sweeping statement, it seems reasonable to say that many theorems concerning the asymptotic behavior of solutions of arbitrary linear systems can be adapted in this way to (R, ℓ)-symmetric systems.

4 R-symmetric systems of linear difference equations

In this section $\mathbb{Z}_+$ is the set of positive integers and $\mathbb{C}^p(\mathbb{Z}_+)$ and $\mathbb{C}^{p\times q}(\mathbb{Z}_+)$ are respectively the sets of complex p-vector functions on $\mathbb{Z}_+$ and complex $p \times q$ matrix functions on $\mathbb{Z}_+$. (Again, “complex" can just as well be replaced by “real.") We briefly consider linear systems of difference equations

$$x_{t+1} = (I + A_t)x_t, \quad t \in \mathbb{Z}_+, \quad x_0 = \xi, \tag{19}$$

with $\{A_t \mid t \in \mathbb{Z}_+\} \subset \mathbb{C}^{n\times n}(\mathbb{Z}_+)$ We assume throughout that $I + A_t$ is invertible for all $t \in \mathbb{Z}_+$. Let

$$\mathbb{P}_t = \begin{bmatrix} P_{0t} & P_{1t} & \cdots & P_{k-1,t} \end{bmatrix} \text{ with } \mathbb{P}_t^{-1} = \begin{bmatrix} \widehat{P}_{0t} \\ \widehat{P}_{1t} \\ \vdots \\ \widehat{P}_{k-1,t} \end{bmatrix},$$

where

$$P_{\ell t} \in \mathbb{C}^{d_\ell \times n}(\mathbb{Z}_+),\ \widehat{P}_{\ell t} \in \mathbb{C}^{n\times d_\ell}(\mathbb{Z}_+),\ \text{ and } \widehat{P}_{\ell t}P_{mt} = \delta_{\ell m} I_{d_\ell},\ 0 \le \ell, m \le k-1,\ t \in \mathbb{Z}_+.$$

Let

$$R_t = \mathbb{P}_t \mathbf{D}_0 \mathbb{P}_t^{-1} \quad (see (9)) \quad and \quad \mathbb{P}_{t+1} = \mathbb{P}_t(\mathbf{I} + \mathbf{U}_t), \tag{20}$$

where

$$\mathbf{U}_t = \bigoplus_{\ell=0}^{k-1} U_{\ell t} \quad \text{with} \quad U_{\ell t} \in \mathbb{C}^{d_\ell \times d_\ell}(\mathbb{Z}_+), \quad 0 \le \ell \le k-1,$$

and $I + \mathbf{U}_t$ is invertible for all $t \in \mathbb{Z}_+$. Finally, denote $\mathcal{A} = \{A_t \mid t \in \mathbb{Z}_+\}$ and $\mathcal{R} = \{R_t \mid t \in \mathbb{Z}_+\}$. We say that $\mathcal{R}$ is a nontrivial k-involution (again, equidimensional if $n = kd$ and $d_0 = d_1 = \cdots = d_{k-1} = d$) and that $\mathcal{A}$ is $\mathcal{R}$-symmetric if $R_t A_t R_t^{-1} = A_t$, $t \in \mathbb{Z}_+$.

Theorem 7 *Let Φ be as in (3). If $\mathcal{R}$ is equidimensional with width d then*

$$R_t A_t R_t^{-1} = A_t, \quad t \in \mathbb{Z}_+, \tag{21}$$

if and only if

$$\Phi \mathbb{P}_t^{-1} A_t \mathbb{P}_t \Phi^* = [A_{s-r,t}]_{r,s=0}^{k-1} \text{ with } A_{0t}, A_{1t}, \dots, A_{k-1,t} \in \mathbb{C}^{d\times d}(\mathbb{Z}_+).$$

In this case

$$A_t = \sum_{\ell=0}^{k-1} P_{\ell t} F_{\ell t} \widehat{P}_{\ell t} = \mathbb{P}_t \mathbf{F}_t \mathbb{P}_t^{-1} \tag{22}$$

with

$$F_{\ell t} = \sum_{m=0}^{k-1} \zeta^{\ell m} A_{mt}, \quad 0 \le \ell \le k-1, \quad t \in \mathbb{Z}_+.$$

PROOF. See the proof of Theorem 1. □

Dropping the assumption that $\mathcal{R}$ is equidimensional leaves the following theorem.

Theorem 8 *Eqn. (21) holds if and only (22) holds with*

$$F_{\ell t} = \widehat{P}_{\ell t} A_t P_{\ell t} \in \mathbb{C}^{d_\ell \times d_\ell}(\mathbb{Z}_+), \quad 0 \le \ell \le k-1, \quad t \in \mathbb{Z}_+.$$

PROOF. See the proof of Theorem 2. □

Theorem 9 *Suppose $\mathcal{A}$ is $\mathcal{R}$-symmetric and let*

$$Q_{\ell t} = \mathbb{P}_{\ell t} \prod_{j=1}^{t-1} (I + U_{\ell j})^{-1}(I + F_{\ell j}), \quad t \in \mathbb{Z}_+, \quad Q_{\ell 0} = I_{d_\ell}, \quad 0 \le \ell \le k-1.$$

Then

$$X_t = \mathbb{P}_t \prod_{j=1}^{t-1} (I + \mathbf{U}_j)^{-1}(I + \mathbf{F}_j) = \begin{bmatrix} Q_{0t} & Q_{1t} & \cdots & Q_{k-1,t} \end{bmatrix}, \; t \in \mathbb{Z}_+, \; X_0 = I,$$

is a fundamemental matrix for (19).

Proof. Write $X_t = \mathbb{P}_t \mathbf{Y}_t$. Since $\mathbb{P}_{t+1} = \mathbb{P}_t (I + \mathbf{U}_t)$ (see (20)) and $I + A_t = \mathbb{P}_t (I + \mathbf{F}_t) \mathbb{P}_t^{-1}$ (see (22)), $X_{t+1} = (I + A_t) X_t$ is equivalent to

$$\mathbf{Y}_{t+1} = (I + \mathbf{U}_t)^{-1}(I + \mathbf{F}_t)\mathbf{Y}_t, \quad t \in \mathbb{Z}_+,$$

or, equivalently,

$$Y_{\ell,t+1} = (I + U_{\ell t})^{-1}(I + F_{\ell t}) Y_{\ell t}, \quad 0 \le \ell \le k-1, \quad t \in \mathbb{Z}_+.$$

This implies the conclusion. □

As we saw earlier in connection with differential equations, it may be useful to study the asymptotic behavior of the distinct $(\mathcal{R}, \ell)$-symmetric components of (19). The analog of Bôcher's states that if $I + A_t$ is invertible for all $t \in \mathbb{Z}_+$ and $\sum_{t=0}^{k-1} \|A_t\| < \infty$, then (19) has linear asymptotic equilibrium. This result can adapted to an $\mathcal{R}$-symmetric linear difference system as follows.

Theorem 10 *If $\mathcal{A}$ is R-symmetric and*

$$\sum_{t=0}^{k-1} \|(I + U_{\ell t})^{-1}(I + F_{\ell t}) - I\| < \infty,$$

then the $(\mathcal{R}, \ell)$-symmetric component of any solution of (19) *can be written uniquely as $x_{\ell t} = \mathbb{P}_{\ell t} y_{\ell t}$ where $y_{\ell,\infty} = \lim_{t\to\infty} y_{\ell t}$ exists and is nonzero if $y_{\ell,0} \ne 0$. Moreover, if $\lim_{t\to\infty} \mathbb{P}_{\ell t}$ exists and has rank d_ℓ then $x_{\ell,\infty} = \lim_{t\to\infty} x_{\ell t}$ exists and is nonzero if $x_{\ell,0} \ne 0$.*

It seems reasonable to expect that results like those in [8] and [9] can be extended in this way for systems of the form (19).

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