Characterization and properties of matrices with k-involutory symmetries II

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Abstract

We say that a matrix $R \in \mathbb{C}^{n\times n}$ is k-involutory if its minimal polynomial is $x^k - 1$ for some $k \geq 2$, so $R^{k-1} = R^{-1}$ and the eigenvalues of R are 1, ζ, ζ^2,..., ζ^{k-1}, where $\zeta = e^{2\pi i/k}$. Let α, $\mu \in \{0, 1, \dots, k-1\}$. If $R \in \mathbb{C}^{m\times m}$, $A \in \mathbb{C}^{m\times n}$, $S \in \mathbb{C}^{n\times n}$ and R and S are k-involutory, we say that A is (R, S, α, μ)-symmetric if $RAS^{-\alpha} = \zeta^\mu A$. We show that an (R, S, α, μ)-symmetric matrix A can be usefully represented in terms of matrices $F_\ell \in \mathbb{C}^{c_{\alpha\ell+\mu}\times d_\ell}$, $0 \leq \ell \leq k-1$, where c_ℓ and d_ℓ are respectively the dimensions of the ζ^ℓ- eigenspaces of R and S. This continues a theme initiated in an earlier paper with the same title, in which we assumed that $\alpha = 1$. We say that a k-involution is equidimensional with width d if all of its eigenspaces have dimension d. We show that if R and S are equidimensional k-involutions with widths d_1 and d_2 respectively, then (R, S, α, μ)-symmetric matrices are closely related to generalized α-circulants $[\zeta^{\mu r} A_{s-\alpha r}]_{r,s=0}^{k-1}$, where A_0, A_1, ..., $A_{k-1} \in \mathbb{C}^{d_1\times d_2}$. For this case our results are new even if $\alpha = 1$. We also give an explicit formula for the Moore-Penrose inverse of a unilevel block circulant $[A_{s-\alpha r}]_{r,s=0}^{k-1}$ for any $\alpha \in \{0, 1, \dots, k-1\}$, generalizing a result previously obtained for the case where $\gcd(\alpha, k) = 1$.

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1 Introduction

Throughout this paper $\alpha > 0$, $k \ge 2$, and μ are integers, $\zeta = e^{2\pi i/k}$,

$$\mathbb{Z}_k = \{0, 1, \dots, k-1\},$$

and subscripts are to be reduced modulo k. We say that $R \in \mathbb{C}^{m\times m}$ is k-involutory if its minimal polynomial is $x^k - 1$ for some $k \ge 2$, so $R^{k-1} = R^{-1}$ and the eigenvalues of R are $1, \zeta, \dots, \zeta^{k-1}$.

If $R \in \mathbb{C}^{m\times m}$ and $S \in \mathbb{C}^{n\times n}$ are k-involutory we say that $A \in \mathbb{C}^{m\times n}$ is (R, S, α, μ)-symmetric if $RAS^{-\alpha} = \zeta^\mu A$. This work is a continuation of [15], where we studied matrices such that $RAS^{-1} = \zeta^\mu A$, which we called (R, S, μ)-symmetric. Sections 3–5 are extensions of results obtained in [15] for (R, S, μ)-symmetric matrices. However, Sections 6 and 8 are new even with $\alpha = 1$, and are also extensions of results obtained in [16]. In Section 7 we give an explicit formula for the Moore-Penrose inverse of a block circulant $[A_{s-\alpha r}]_{r,s=0}^{k-1}$ with A_0, $A_1, \dots, A_{k-1} \in \mathbb{C}^{d_1\times d_2}$. The formula is valid for any $\alpha \in \mathbb{Z}_k$ and extends a result in [16, Theorem 5] for the case where $\gcd(\alpha, k) = 1$.

This paper is motivated by and continues a line of research undertaken by many investigators; see, e.g., [2]–[4],[6], [7], [9] [10], [11, 13, 14, 18, 19], by no means a complete list.

2 Preliminaries

Let $R \in \mathbb{C}^{m\times m}$ and $S \in \mathbb{C}^{n\times n}$ be k-involutions. Let

$$c_\ell = \dim \left\{ z \,\middle|\, Rz = \zeta^\ell z \right\} \quad \text{and} \quad d_\ell = \dim \left\{ z \,\middle|\, Sz = \zeta^\ell z \right\}, \quad 0 \le \ell \le k-1.$$

Then there are matrices $P_\ell \in \mathbb{C}^{m\times c_\ell}$ and $Q_\ell \in \mathbb{C}^{n\times d_\ell}$, $0 \le \ell \le k-1$, such that

$$RP_\ell = \zeta^\ell P_\ell, \quad SQ_\ell = \zeta^\ell Q_\ell, \quad 0 \le \ell \le k-1, \tag{1}$$

$$P_\ell^* P_\ell = I_{c_\ell}, \quad \text{and} \quad Q_\ell^* Q_\ell = I_{d_\ell}, \quad 0 \le \ell \le k-1. \tag{2}$$

We note that (2) can be assumed without loss of generality, since the Gram-Schmidt procedure allows us to choose an orthonormal basis for any eigenspace.

Let

$$P = \left[\begin{array}{ccc} P_0 & P_1 \cdots & P_{k-1} \end{array}\right], \quad Q = \left[\begin{array}{ccc} Q_0 & Q_1 \cdots & Q_{k-1} \end{array}\right], \tag{3}$$

$$P^{-1} = \left[\begin{array}{c} \widehat{P}_0 \\ \widehat{P}_1 \\ \vdots \\ \widehat{P}_{k-1} \end{array}\right], \quad \text{and} \quad Q^{-1} = \left[\begin{array}{c} \widehat{Q}_0 \\ \widehat{Q}_1 \\ \vdots \\ \widehat{Q}_{k-1} \end{array}\right], \tag{4}$$

with $\widehat{P}_\ell \in \mathbb{C}^{c_\ell\times m}$ and $\widehat{Q}_\ell \in \mathbb{C}^{d_\ell\times n}$, $0 \le \ell \le k-1$; thus,

$$\widehat{P}_\ell P_m = \delta_{\ell m} I_{c_\ell} \quad \text{and} \quad \widehat{Q}_\ell Q_m = \delta_{\ell m} I_{d_\ell}, \quad 0 \le \ell, m \le k-1. \tag{5}$$

Therefore

$$R = PD_RP^{-1} \ \text{ with } D_R = \bigoplus_{\ell=0}^{k-1}\zeta^\ell I_{c_\ell} \text{ and } S = QD_SQ^{-1} \text{ with } \ D_S = \bigoplus_{\ell=0}^{k-1}\zeta^\ell I_{d_\ell}. \tag{6}$$

Since the eigenvalues of R are $1, \zeta, \dots, \zeta^{k-1}$, the first equality in (2) implies that P is unitary (i.e., $P^{-1} = P^*$ and therefore $\widehat{P_\ell} = P_\ell^*$, $1 \le \ell \le k$) if and only if R is unitary. A similar comment applies to S and Q.

We also define

$$V_{\mu,\alpha} = \left[\begin{array}{cccc} P_\mu & P_{\alpha+\mu} & \cdots & P_{\alpha(k-1)+\mu}\end{array}\right] \quad \text{and} \quad \widehat{V}_{\mu,\alpha} = \left[\begin{array}{c}\widehat{P}_\mu \\ \widehat{P}_{\alpha+\mu} \\ \vdots \\ \widehat{P}_{\alpha(k-1)+\mu}\end{array}\right]. \tag{7}$$

If $\gcd(\alpha,k) = q > 1$ and $p = k/q$ then the first p block columns of $V_{\mu,\alpha}$ are repeated q times. In any case, $\widehat{V}_{\mu,\alpha} = V_{\mu,\alpha}^*$ if R is unitary.

An explicit method for obtaining $P_0, P_1, \dots, P_{k-1}, \widehat{P}_0, \widehat{P}_1, \dots, \widehat{P}_{k-1}, Q_0, Q_1, \dots, Q_{k-1}$, and $\widehat{Q}_0, \widehat{Q}_1, \dots, \widehat{Q}_{k-1}$, was given in [15]; however, matrices denoted here by $\widehat{P}_\ell$, $\widehat{Q}_\ell$, etc., are denoted by $\widehat{P}_\ell^*$, $\widehat{Q}_\ell^*$, etc., in [15].

We say that a k-involution R is equidimensional with width d if all of its eigenspaces are d-dimensional. For example, if $R_0 \in \mathbb{C}^{k\times k}$ is a k-involution (necessarily of width 1), then $R = R_0 \otimes I_d \in \mathbb{C}^{kd\times kd}$ is an equidimensional k-involution with width d. We show that if $m = kd_1$, $n = kd_2$, and $R \in \mathbb{C}^{m\times m}$ and $S \in \mathbb{C}^{n\times n}$ are equidimensional with widths d_1 and d_2, then (R,S,α,μ)-symmetric block matrices with $d_1 \times d_2$ blocks are closely related to generalized block circulants $[\zeta^{\mu r}A_{s-\alpha r}]_{r,s=0}^{k-1}$, where $A_0, A_1, \dots, A_{k-1} \in \mathbb{C}^{d_1\times d_2}$. A precursor of this result is the observation of Ablow and Brenner [1] that if A, $R \in \mathbb{C}^{k\times k}$ and R is a k-involution, then $RAR^{-\alpha} = A$ if and only if A is similar to an α-circulant $[a_{s-\alpha r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k\times k}$.

We let $\mathbb{C}^{k:d_1\times d_2}$ denote the set of all block $k \times k$ matrices $H = [H_{rs}]_{r,s=0}^{k-1}$ with $H_{rs} \in \mathbb{C}^{d_1\times d_2}$, $0 \le r, s \le k-1$.

3 Characterization of (R,S,α,μ)-symmetric matrices

Theorem 1 *$A \in \mathbb{C}^{m\times n}$ is (R,S,α,μ)-symmetric if and only if*

$$A = PCQ^{-1} \quad \text{with} \quad C = [C_{rs}]_{r,s=0}^{k-1}, \quad \text{where} \quad C_{rs} \in \mathbb{C}^{c_r\times d_s}, \tag{8}$$

and

$$C_{rs} = 0 \quad \text{if} \quad r \not\equiv \alpha s + \mu \pmod{k}, \tag{9}$$

in which case

$$C_{\alpha s+\mu,s} = P_{\alpha s+\mu}^* AQ_s \in \mathbb{C}^{c_{\alpha s+\mu}\times d_s}, \quad 0 \le s \le k-1. \tag{10}$$

PROOF. We can write an arbitrary $A \in \mathbb{C}^{m\times n}$ as in (8) with $C = P^{-1}AQ$, and we can partition C as in (8). Then (1), (3), and (6) imply that

$$RAS^{-\alpha} = (RP)C(Q^{-1}S^{-\alpha}) = (PD_R)C(D_S^{-\alpha}Q^{-1}) = P(D_RCD_S^{-\alpha})Q^{-1}.$$

From this and (8), $RAS^{-\alpha} = \zeta^\mu A$ if and only if $D_RCD_S^{-\alpha} = \zeta^\mu C$, i.e., if and only if

$$[\zeta^\mu C_{rs}]_{r,s=0}^{k-1} = [\zeta^{r-\alpha s} C_{rs}]_{r,s=0}^{k-1}.$$

This is equivalent to (9). From (8), $AQ = PC$; i.e.,

$$A \begin{bmatrix} Q_0 & Q_1 & \cdots & Q_{k-1} \end{bmatrix} = \begin{bmatrix} P_0 & P_1 & \cdots & P_{k-1} \end{bmatrix} C.$$

Now (9) implies that $AQ_\ell = P_{\alpha\ell+\mu}C_{\alpha\ell+\mu,\ell}$, $0 \le \ell \le k-1$. This implies (10), since $P_{\alpha\ell+\mu}^* P_{\alpha\ell+\mu} = I_{c_{\alpha\ell+\mu}}$ (see (2)). □

If $\gcd(\alpha, k) = 1$ then the substitution $\ell \to \alpha\ell + \mu \pmod{k}$ is a permutation of $\mathbb{Z}_k$. This implies the following corollary of Theorem 1.

Corollary 1 *If* $\gcd(\alpha, k) = 1$ *then any* $A \in \mathbb{C}^{m\times n}$ *can be written uniquely as* $A = \sum_{\mu=0}^{k-1} A^{(\mu)}$, *where* $A^{(\mu)}$ *is* (R, S, α, μ)*-symmetric,* $0 \le \mu \le k-1$. *Specifically, if* A *is as in* (8) *then*

$$A^{(\mu)} = P\left(\left[C_{rs}^{(\mu)}\right]_{r,s=0}^{k-1}\right)Q^{-1}$$

where

$$C_{rs}^{(\mu)} = \begin{cases} 0 & \text{if } r \not\equiv \alpha s + \mu \pmod{k}, \\ C_{\alpha r+\mu,s} & \text{if } r \equiv \alpha s + \mu \pmod{k}. \end{cases}$$

Eqns. (8)–(10) imply the next theorem, which is a convenient reformulation of Theorem 1.

Theorem 2 *A matrix* $A \in C^{m\times n}$ *is* (R, S, α, μ)*-symmetric if and only if*

$$A = V_{\mu,\alpha}\left(\bigoplus_{\ell=0}^{k-1} F_\ell\right)Q^{-1} = \sum_{\ell=0}^{k-1} P_{\alpha\ell+\mu}F_\ell\widehat{Q}_\ell, \tag{11}$$

in which case

$$F_\ell = P_{\alpha\ell+\mu}^* AQ_\ell \in \mathbb{C}^{c_{\alpha\ell+\mu}\times d_\ell}, \quad 0 \le \ell \le k-1, \tag{12}$$

where $\alpha\ell + \mu$ *is to be reduced modulo* k. *Moreover, if* S *is unitary* (*so* Q *is unitary*), *then* (11) *becomes*

$$A = V_{\mu,\alpha}\left(\bigoplus_{\ell=0}^{k-1} F_\ell\right)Q^* = \sum_{\ell=0}^{k-1} P_{\alpha\ell+\mu}F_\ell Q_\ell^*. \tag{13}$$

It may be reassuring to verify directly that A in (11) is in fact (R,S,α,μ)-symmetric. From (1) and (7), $RV_{\mu,\alpha} = \zeta^{\mu} V_{\mu,\alpha} D_R^{\alpha}$. From (6), $Q^{-1}S^{-1} = D_S^{-1}Q^{-1}$, so $Q^{-1}S^{-\alpha} = D_S^{-\alpha}Q^{-1}$. Therefore the first equality in (11) implies that $RAS^{-\alpha} = \zeta^{\mu}A$. Eqns. (4) and (7) imply the second equality.

Theorem 3 *Suppose*

$$\gcd(\alpha,k) = q > 1 \quad \textit{and} \quad p = k/q. \tag{14}$$

Let

$$\mathbf{Q}_\ell = \left[\begin{array}{cccc} Q_\ell & Q_{\ell+p} & \cdots & Q_{\ell+(q-1)p} \end{array}\right] \in \mathbb{C}^{n\times(d_\ell+d_{\ell+p}+\cdots+d_{\ell+(q-1)p})} \tag{15}$$

$0 \le \ell \le p-1$,

$$\widehat{\mathbf{Q}}_\ell = \left[\begin{array}{c} \widehat{Q}_\ell \\ \widehat{Q}_{\ell+1} \\ \vdots \\ \widehat{Q}_{\ell+(q-1)p} \end{array}\right] \in \mathbb{C}^{(d_\ell+d_{\ell+p}+\cdots+d_{\ell+(q-1)p})\times n},$$

$0 \le \ell \le p-1$. *If we define*

$$\mathcal{Q} = \left[\begin{array}{cccc} \mathbf{Q}_0 & \mathbf{Q}_1 & \ldots & \mathbf{Q}_{p-1} \end{array}\right] \quad \textit{then} \quad \mathcal{Q}^{-1} = \left[\begin{array}{c} \widehat{\mathbf{Q}}_0 \\ \widehat{\mathbf{Q}}_1 \\ \vdots \\ \widehat{\mathbf{Q}}_{p-1} \end{array}\right]. \tag{16}$$

Also, let

$$\mathcal{V}_{\mu,\alpha} = \left[\begin{array}{cccc} P_\mu & P_{\alpha+\mu} & \cdots & P_{(p-1)\alpha+\mu} \end{array}\right], \quad \widehat{\mathcal{V}}_{\mu,\alpha} = \left[\begin{array}{c} \widehat{P}_\mu \\ \widehat{P}_{\alpha+\mu} \\ \vdots \\ \widehat{P}_{(p-1)\alpha+\mu} \end{array}\right], \tag{17}$$

$$\mathbf{F}_\ell = \left[\begin{array}{cccc} F_\ell & F_{\ell+p} & \cdots & F_{\ell+(q-1)p} \end{array}\right], \quad 0 \le \ell \le p-1, \tag{18}$$

and

$$\mathcal{F} = \bigoplus_{\ell=0}^{p-1} \mathbf{F}_\ell. \tag{19}$$

Then $\mathcal{Q}$ is invertible since its columns are simply a rearrangement of the columns of Q,

$$\widehat{\mathcal{V}}_{\mu,\alpha}\mathcal{V}_{\mu,\alpha} = I_{c_\mu+c_{\alpha+\mu}+\cdots+c_{(p-1)\alpha+\mu}} \tag{20}$$

and (11) *can be rewritten as*

$$A = \sum_{\ell=0}^{p-1} P_{\alpha\ell+\mu}\mathbf{F}_\ell\widehat{\mathbf{Q}}_\ell = \mathcal{V}_{\mu,\alpha}\mathcal{F}\mathcal{Q}^{-1}. \tag{21}$$

Proof. Note that although α does not appear explicitly on the right sides of (15), (16), and (18), the matrices shown there are nevertheless uniquely determined by α. (See (14).) Moreover, (12) and (14) imply that F_ℓ, $F_{\ell+p}$, …, $F_{\ell+(q-1)p}$ have the same row dimension, since

$$\alpha(\ell+\nu p)+\mu \equiv \alpha\ell+\mu \pmod{k}$$

for any integer ν. Therefore $\mathbf{F}_0$, …, $\mathbf{F}_{p-1}$ are well defined.

Since 0, α, …, $(p-1)\alpha$ are distinct, (5) implies (20). Since every $m \in \mathbb{Z}_k$ can be written uniquely as $m = \ell + \nu p$ with $0 \le \ell \le p-1$ and $0 \le \nu \le q-1$, the second equality in (11) can be written as

$$A = \sum_{\ell=0}^{p-1}\sum_{\nu=0}^{q-1} P_{\alpha(\ell+\nu p)+\mu} F_{\ell+\nu p}\widehat{Q}_{\ell+\nu p} = \sum_{\ell=0}^{p-1} P_{\alpha\ell+\mu}\sum_{\nu=0}^{q-1} F_{\ell+\nu p}\widehat{Q}_{\ell+\nu p}, \tag{22}$$

where the second equality is valid because $p\alpha \equiv 0 \pmod{k}$. Therefore the first equality in (21) is valid because

$$\mathbf{F}_\ell\widehat{\mathbf{Q}}_\ell = \sum_{\nu=0}^{q-1} F_{\ell+\nu p}\widehat{Q}_{\ell+\nu p}, \quad 0 \le \ell \le p-1.$$

Now (16), (17), and (19) imply the second equality in (21). □

Theorem 4 *Suppose R and S are unitary,* $\gcd(\alpha,k)=1$, $\alpha\beta \equiv 1 \pmod{k}$, *and A is (R,S,α,μ)-symmetric. Then A^* is $(S,R,\beta,-\beta\mu)$-symmetric.*

Proof. Since S is unitary, (13) holds. Therefore

$$A^* = \sum_{\ell=0}^{k-1} Q_\ell F_\ell^* P_{\alpha\ell+\mu}^* \tag{23}$$

since R is unitary and therefore P is unitary. Since $(\beta,k)=1$, every integer in $\mathbb{Z}_k$ can written uniquely in the form $\beta(\ell-\mu)$ with $\ell \in \mathbb{Z}_k$. Therefore we can replace ℓ by $\beta(\ell-\mu)$ in (23) to obtain

$$A^* = \sum_{\ell=0}^{k-1} Q_{\beta(\ell-\mu)} F_{\beta(\ell-\mu)}^* P_\ell^*,$$

since $\alpha\beta \equiv 1 \pmod{k}$. Now Theorem 2 implies the conclusion. □

In the following theorem $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, $R \in \mathbb{C}^{m\times m}$ and $S \in \mathbb{C}^{n\times n}$ are the k-involutions in (6) and $T \in \mathbb{C}^{p\times p}$ is the k-involution with spectral decomposition

$$T = \begin{bmatrix} X_0 & X_1 \cdots & X_{k-1} \end{bmatrix} D_T \begin{bmatrix} \widehat{X}_0 \\ \widehat{X}_1 \\ \vdots \\ \widehat{X}_{k-1} \end{bmatrix}, \quad \text{where} \quad D_T = \bigoplus_{\ell=0}^{k-1} \zeta^\ell I_{e_\ell}.$$

Theorem 5 *Suppose $A \in \mathbb{C}^{m\times n}$ is (R,S,α,μ)-symmetric and $B \in \mathbb{C}^{n\times p}$ is (S,T,β,ν)-symmetric, so*

$$A = \sum_{\ell=0}^{k-1} P_{\alpha\ell+\mu} F_\ell \widehat{Q}_\ell \quad \text{and} \quad B = \sum_{\ell=0}^{k-1} Q_{\beta\ell+\nu} G_\ell \widehat{X}_\ell, \tag{24}$$

from Theorem 2. Then $AB \in \mathbb{C}^{m\times p}$ is $(R,T,\alpha\beta,\alpha\nu+\mu)$-symmetric. Moreover, if $\gcd(\beta,k)=1$ then

$$AB = \sum_{\ell=0}^{k-1} P_{\alpha\beta\ell+(\alpha\nu+\mu)} F_{\beta\ell+\nu} G_\ell \widehat{X}_\ell. \tag{25}$$

Proof. It is given that (a) $RAS^{-\alpha} = \zeta^\mu A$ and (b) $SBT^{-\beta} = \zeta^\nu B$. Applying (b) α times yields $S^\alpha BT^{-\alpha\beta} = \zeta^{\alpha\nu} B$. This and (a) imply that $RABT^{-\alpha\beta} = \zeta^{\alpha\nu+\mu} AB$, so AB is $(R,T,\alpha\beta,\alpha\nu+\mu)$-symmetric. If $\gcd(\beta,k)=1$ then replacing ℓ by $\beta\ell+\nu$ in the first equality in (24) merely rearranges the terms in the sum, so

$$A = \sum_{\ell=0}^{k-1} P_{\alpha\beta\ell+(\alpha\nu+\mu)} F_{\beta\ell+\nu} \widehat{Q}_{\beta\ell+\nu}. \tag{26}$$

Since $\gcd(\beta,k)=1$, $\widehat{Q}_{\beta\ell+\nu} Q_{\beta m+\nu} = \delta_{\ell m} I_{d_{\beta\ell+\nu}}$, $0 \le \ell, m \le k-1$. Therefore (26) and the second equality in (24) imply (25). ▯

Theorem 6 *Suppose R and S are unitary, A is (R,S,α,μ)-symmetric, B is (R,S,α,ν)-symmetric, $\gcd(\alpha,k)=1$, and $\alpha\beta \equiv 1 \pmod{k}$. Then AB^* is $(R,R,1,\mu-\nu)$-symmetric and B^*A is $(S,S,1,\beta(\mu-\nu))$-symmetric; specifically, if*

$$A = \sum_{\ell=0}^{k-1} P_{\alpha\ell+\mu} F_\ell Q_\ell^* \quad \text{and} \quad B = \sum_{\ell=0}^{k-1} P_{\alpha\ell+\nu} G_\ell Q_\ell^* \tag{27}$$

as implied by Theorem 2, then

$$AB^* = \sum_{\ell=0}^{k-1} P_{\ell+\mu-\nu} F_{\beta(\ell-\nu)} G^*_{\beta(\ell-\nu)} P_\ell^* \tag{28}$$

and

$$B^*A = \sum_{\ell=0}^{k-1} Q_{\ell+\beta(\mu-\nu)} G^*_{\ell+\beta(\mu-\nu)} F_\ell Q_\ell^*. \tag{29}$$

Proof. From (27),

$$AB^* = \left(\sum_{\ell=0}^{k-1} P_{\alpha\ell+\mu} F_\ell Q_\ell^*\right)\left(\sum_{m=0}^{k-1} Q_m G_m^* P^*_{\alpha m+\nu}\right) = \sum_{\ell=0}^{k-1} P_{\alpha\ell+\mu} F_\ell G_\ell^* P^*_{\alpha s+\nu}. \tag{30}$$

Since $\gcd(\beta,k)=1$, replacing ℓ by $\beta(\ell-\nu)$ in the last sum yields (28).

Also from (27),

$$B^*A = \left(\sum_{\ell=0}^{k-1} Q_\ell G_\ell^* P_{\alpha\ell+\nu}^*\right)\left(\sum_{m=0}^{k-1} P_{\alpha m+\mu} F_m Q_m^*\right)$$

Replacing ℓ by $\ell + \beta(\mu - \nu)$ in the first sum yields

$$B^*A = \left(\sum_{\ell=0}^{k-1} Q_{\ell+\beta(\mu-\nu)} G_{\ell+\beta(\mu-\nu)}^* P_{\alpha\ell+\mu}^*\right)\left(\sum_{m=0}^{k-1} P_{\alpha m+\mu} F_m Q_m^*\right),$$

which implies (29), since $P_{\alpha\ell+\mu}^* P_{\alpha m+\mu} = \delta_{\ell m} I_{c_{\alpha\ell+\mu}}$, $0 \le \ell, m \le k-1$. $\square$

Remark 1 If R and S are unitary, A is (R, S, α, μ)-symmetric, and B is (R, S, α, ν)-symmetric, then

$$RAB^*R^{-1} = (RAS^{-\alpha})(S^\alpha B^* R^{-1}) = (\zeta^\mu A)(\zeta^{-\nu} B^*) = \zeta^{\mu-\nu} AB^*.$$

Hence, AB^* is $(R, R, 1, \mu - \nu)$-symmetric even if $\gcd(\alpha, k) \ne 1$; moreover, (30) is valid.

4 Generalized inverses and SVD

If $A \in \mathbb{C}^{m\times n}$ then A^- is a reflexive inverse of A if $AA^-A = A$ and $A^-AA^- = A^-$ [5, p. 51], and the Moore-Penrose inverse $A^\dagger$ of A is the unique matrix that satisfies the Penrose conditions

$$(AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A, \quad AA^\dagger A = A, \quad \text{and} \quad A^\dagger AA^\dagger = A^\dagger.$$

If $A \in \mathbb{C}^{n\times n}$ and there is a matrix $A^\#$ such that $AA^\#A = A$, $A^\#AA^\# = A^\#$, and $AA^\# = A^\#A$ then $A^\#$ is called the group inverse of A [5, p.156]. A matrix may fail to have a group inverse, but if one exists it is unique.

Theorem 7 (i) *If A^- is a reflexive inverse of an (R, S, α, μ)-symmetric matrix A then $B = \zeta^\mu S^\alpha A^- R^{-1}$ is a reflexive inverse of A.* (ii) *If $A \in \mathbb{C}^{n\times n}$ is $(R, R, 1, \mu)$-symmetric and has a group inverse $A^\#$, then $A^\#$ is $(R, R, 1, -\mu)$-symmetric.*

Proof. (i) Since $A = \zeta^{-\mu} RAS^{-\alpha}$,

$$AB = RAA^-R^{-1}, \quad BA = S^\alpha A^- AS^{-\alpha},$$

so

$$ABA = \zeta^{-\mu} RAA^-AS^{-\alpha} = \zeta^{-\mu} RAS^{-\alpha} = A$$

and

$$BAB = \zeta^\mu S^\alpha A^-AA^-R^{-1} = \zeta^\mu S^\alpha A^- R^{-1} = B.$$

(ii) It is given that $A = \zeta^{-\mu} RAR^{-1}$. Let $B = \zeta^{\mu} RA^{\#}R^{-1}$. Then $AB = RAA^{\#}R^{-1}$ and $BA = RA^{\#}AR^{-1}$. Therefore $AB = BA$, since $AA^{\#} = A^{\#}A$. Also,

$$ABA = \zeta^{-\mu} RAA^{\#}AR^{-1} = \zeta^{-\mu} RAR^{-1} = A$$

and

$$BAB = \zeta^{\mu} RA^{\#}AA^{\#}R^{-1} = \zeta^{\mu} RA^{\#}R^{-1} = A^{\#}.$$

Hence B is a group inverse of A. Since A can have only one group inverse, it follows that $A^{\#} = B = \zeta^{\mu} RA^{\#}R^{-1}$, which is $(R, R, 1, -\mu)$-symmetric. □

For convenience of notation, denote $\mathbf{F} = \bigoplus_{\ell=0}^{k-1} F_\ell$. It is straightforward to verify that $\mathbf{F}$ and $\bigoplus_{\ell=0}^{k-1} F_\ell^\dagger$ satisfy the Penrose conditions, so $\mathbf{F}^\dagger = \bigoplus_{\ell=0}^{k-1} F_\ell^\dagger$.

Theorem 8 *Suppose that A is (R, S, α, μ)-symmetric, so*

$$A = V_{\mu,\alpha}\mathbf{F}Q^{-1} = \sum_{\ell=0}^{k-1} P_{\alpha\ell+\mu} F_\ell \widehat{Q}_\ell, \tag{31}$$

by Theorem 2. *Suppose also that* $\gcd(\alpha, k) = 1$ *and* $\alpha\beta \equiv 1 \pmod{k}$. *Let*

$$B = Q\mathbf{F}^\dagger V_{\mu,\alpha}^{-1} = \sum_{\ell=0}^{k-1} Q_\ell F_\ell^\dagger \widehat{P}_{\alpha\ell+\mu}. \tag{32}$$

Then B is a reflexive inverse of A. Moreover, if R and S are unitary then $B = A^\dagger$, i.e.,

$$A^\dagger = Q\mathbf{F}^\dagger V_{\mu,\alpha}^* = \sum_{\ell=0}^{k-1} Q_\ell \mathbf{F}_\ell^\dagger P_{\alpha\ell+\mu}^*. \tag{33}$$

Finally, $A^\dagger$ is $(S, R, \beta, -\beta\mu)$-symmetric.

Proof. From (2), (31), and (32),

$$AB = V_{\mu,\alpha}\mathbf{F}\mathbf{F}^\dagger V_{\mu,\alpha}^{-1}, = V_{\mu,\alpha}(\mathbf{F}\mathbf{F}^\dagger)^* V_{\mu,\alpha}^{-1}, \tag{34}$$

$$BA = Q\mathbf{F}^\dagger\mathbf{F}Q^{-1} = Q(\mathbf{F}^\dagger\mathbf{F})^*Q^{-1}, \tag{35}$$

$$ABA = V_{\mu,\alpha}\mathbf{F}\mathbf{F}^\dagger\mathbf{F}Q^{-1} = V_{\mu,\alpha}\mathbf{F}Q^{-1} = A, \tag{36}$$

and

$$BAB = Q\mathbf{F}^\dagger\mathbf{F}\mathbf{F}^\dagger V_{\mu,\alpha}^{-1} = Q\mathbf{F}^\dagger V_{\mu,\alpha}^{-1} = B. \tag{37}$$

From (36) and (37), B is a reflexive inverse of A. If R and S are unitary then $Q^{-1} = Q^*$ and $V_{\mu,\alpha}^{-1} = V_{\mu,\alpha}^*$, so (34) and (35) imply that $(AB)^* = AB$ and $(BA)^* = BA$. Therefore A and B satisfy the Penrose conditions, so $B = A^\dagger$, which implies (33). Finally, replacing ℓ by $\beta(\ell - \mu)$ in (33) yields

$$A^\dagger = \sum_{\ell=0}^{k-1} Q_{\beta(\ell-\mu)} F_{\beta(\ell-\mu)}^\dagger P_\ell^*$$

so $A^\dagger$ is $(S, R, \beta, -\beta\mu)$-symmetric by Theorem 2. □

Theorem 9 *If* (14) *holds then the matrix*

$$B = \mathcal{Q}\mathcal{F}^{\dagger}\widehat{\mathcal{V}}_{\mu,\alpha} = \sum_{\ell=0}^{p-1} \mathbf{Q}_{\ell}\mathbf{F}_{\ell}^{\dagger}\widehat{P}_{\alpha\ell+\mu} \tag{38}$$

is a reflexive inverse of A. (*See* (21).) *If in addition R and S are unitary, then*

$$A^{\dagger} = \mathcal{Q}\mathcal{F}^{\dagger}\mathcal{V}_{\mu,\alpha}^{*} = \sum_{\ell=0}^{p-1} \mathbf{Q}_{\ell}\mathbf{F}_{\ell}^{\dagger}P_{\alpha\ell+\mu}^{*}. \tag{39}$$

Moreover, if we partition $\mathbf{F}_{\ell}^{\dagger}$ (see (18)) *as*

$$\mathbf{F}_{\ell}^{\dagger} = \begin{bmatrix} G_{\ell} \\ G_{\ell+p} \\ \vdots \\ G_{\ell+(q-1)p} \end{bmatrix}, \quad 0 \le \ell \le p-1,$$

with $G_{\ell} \in \mathbb{C}^{d_{\ell}\times c_{\alpha\ell+\mu}}$, $0 \le \ell \le k-1$ (see (12)), *then* (38) *and* (39) *can be written as*

$$B = \sum_{\ell=0}^{k-1} Q_{\ell}G_{\ell}\widehat{P}_{\alpha\ell+\mu} \quad \textit{and} \quad A^{\dagger} = \sum_{\ell=0}^{k-1} Q_{\ell}G_{\ell}P_{\alpha\ell+\mu}^{*} \tag{40}$$

respectively.

Proof. From (20), (21), and (38),

$$AB = \mathcal{V}_{\mu,\alpha}\mathcal{F}\mathcal{F}^{\dagger}\widehat{\mathcal{V}}_{\mu,\alpha} = \mathcal{V}_{\mu,\alpha}(\mathcal{F}\mathcal{F}^{\dagger})^{*}\widehat{\mathcal{V}}_{\mu,\alpha}, \tag{41}$$

$$BA = \mathcal{Q}\mathcal{F}^{\dagger}\mathcal{F}\mathcal{Q}^{-1} = \mathcal{Q}(\mathcal{F}^{\dagger}\mathcal{F})^{*}\mathcal{Q}^{-1}, \tag{42}$$

$$ABA = \mathcal{V}_{\mu,\alpha}\mathcal{F}\mathcal{F}^{\dagger}\mathcal{F}\mathcal{Q}^{-1} = \mathcal{V}_{\mu,\alpha}\mathcal{F}\mathcal{Q}^{-1} = A, \tag{43}$$

and

$$BAB = \mathcal{Q}\mathcal{F}^{\dagger}\mathcal{F}\mathcal{F}^{\dagger}\widehat{\mathcal{V}}_{\mu,\alpha} = \mathcal{Q}\mathcal{F}^{\dagger}\widehat{\mathcal{V}}_{\mu,\alpha} = B. \tag{44}$$

From (43) and (44), B is a reflexive inverse of A. If R and S are unitary then $\mathcal{Q}^{-1} = \mathcal{Q}^{*}$ and $\widehat{\mathcal{V}}_{\mu,\alpha} = \mathcal{V}_{\mu,\alpha}^{*}$, so (41) and (42) imply that $(AB)^{*} = AB$ and $(BA)^{*} = BA$. Therefore A and B satisfy the Penrose conditions, so $B = A^{\dagger}$. ▯

Theorem 2 and (43) imply the following corollary.

Corollary 2 *If A is (R,S,α,μ)-symmetric and R and S are unitary then $(A^{\dagger})^{*}$ is (R,S,α,μ)-symmetric.*

Theorem 10 *Suppose* $\gcd(\alpha,k) = q$, $p = k/q$, *A is (R,S,α,μ)-symmetric and $\mathbf{F}_{\ell} = \Omega_{\ell}\Sigma_{\ell}\Phi_{\ell}^{*}$ (see* (18)) *is a singular value decomposition of $\mathbf{F}_{\ell}$, $0 \le \ell \le p-1$. Let*

$$\Omega = \begin{bmatrix} P_{\mu}\Omega_0 & P_{\alpha+\mu}\Omega_1 & \cdots & P_{(p-1)\alpha+\mu}\Omega_{p-1} \end{bmatrix}$$

and

$$\Gamma = \left[\begin{array}{cccc} \mathbf{Q}_0\Gamma_0 & \mathbf{Q}_1\Gamma_1 & \cdots & \mathbf{Q}_{p-1}\Gamma_{p-1} \end{array} \right].$$

(*See* (15).) *Then*

$$A = \Omega\left(\bigoplus_{\ell=0}^{p-1}\Sigma_\ell\right)\Gamma^{-1}. \tag{45}$$

Moreover, if R and S are unitary then Ω and Γ are unitary, so (45) *is a singular value decomposition of A, except that the singular values are not neccesarily arranged in nonincreasing order.*

5 Solution of $Az = w$ and the least squares problem

In this section we assume that A is (R, S, α, μ)-symmetric and can therefore be written as in (11). If $z \in \mathbb{C}^n$ and $w \in \mathbb{C}^m$ we write

$$z = Qu = \sum_{\ell=0}^{k-1} Q_\ell u_\ell \quad \text{and} \quad w = Pv = \sum_{\ell=0}^{k-1} P_\ell v_\ell, \tag{46}$$

with $u_\ell \in \mathbb{C}^{d_\ell}$ and $v_\ell \in \mathbb{C}^{c_\ell}$, $0 \le \ell \le k-1$.

Theorem 11 *If* $\gcd(\alpha, k) = 1$ *then*

$$\text{(a)}\quad Az = w \quad \textit{if and only if} \quad \text{(b)}\quad F_\ell u_\ell = v_{\alpha\ell+\mu}, \quad 0 \le \ell \le k-1. \tag{47}$$

Moreover, if R is unitary then

$$\|Az - w\|^2 = \sum_{\ell=0}^{k-1} \|F_\ell u_\ell - v_{\alpha\ell+\mu}\|^2, \tag{48}$$

so the least squares problem for A reduces to k independent least squares problems for F_0, F_1, $\ldots$, F_{k-1}.

Proof. From (11) and (46),

$$\begin{aligned} Az - w &= \sum_{\ell=0}^{k-1} P_{\alpha\ell+\mu}F_\ell u_\ell - \sum_{\ell=0}^{k-1} P_\ell v_\ell = \sum_{\ell=0}^{k-1} P_{\alpha\ell+\mu}F_\ell u_\ell - \sum_{\ell=0}^{k-1} P_{\alpha\ell+\mu} v_{\alpha\ell+\mu} \\ &= \sum_{\ell=0}^{k-1} P_{\alpha\ell+\mu}(F_\ell u_\ell - v_{\alpha\ell+\mu}), \end{aligned} \tag{49}$$

where $\sum_{\ell=0}^{k-1} P_\ell v_\ell = \sum_{\ell=0}^{k-1} P_{\alpha\ell+\mu} v_{\alpha\ell+\mu}$ because $\gcd(\alpha, k) = 1$, the substitution $s \to \alpha\ell + \mu \pmod k$ is a permutation of $\mathbb{Z}_k$. Therefore (47)(b) and (49) imply (47)(a). Since $V_{\mu,\alpha}$ (see (7)) is invertible (again, because $\gcd(\alpha, k) = 1$), (47)(a) and (49) imply (47)(b). Finally, if R is unitary then $P^*_{\alpha\ell+\mu}P_{\alpha m+\mu} = \delta_{\ell m} I_{c_{\alpha\ell+\mu}}$, $0 \le \ell, m \le k-1$, so (49) implies (48). □

Theorems 2 and 11 imply the following theorem.

Theorem 12 *If A is (R,S,α,μ)-symmetric then A is invertible if and only if* $\gcd(\alpha,k)=1$,

$$c_{\alpha\ell+\mu}=d_\ell, \quad 0\le \ell\le k-1, \tag{50}$$

and F_0, F_1, …, F_{k-1} are all invertible, in which case

$$A^{-1}=Q\left(\bigoplus_{\ell=0}^{k-1}F_\ell^{-1}\right)V_{\mu,\alpha}^{-1}=\sum_{\ell=0}^{k-1}Q_\ell F_\ell^{-1}\widehat{P}_{\alpha\ell+\mu} \tag{51}$$

and the solution of $Az=w$ is

$$z=\sum_{\ell=0}^{k-1}Q_\ell F_\ell^{-1}v_{\alpha\ell+\mu}. \tag{52}$$

Proof. From Theorem 2, $A=V_{\mu,\alpha}\left(\bigoplus_{\ell=0}^{k-1}F_\ell\right)Q^{-1}$. If A is invertible then $V_{\mu,\alpha}$ is invertible, which is true if and only if $\gcd(\alpha,k)=1$. Hence, this is a necessary condition for A to be invertible, so assume that it holds. From Theorem 11, $Az=w$ has a solution for every z if and only (47)(b) has a solution for every $\{v_0,v_1,\dots,v_{k-1}\}$. Since $F_\ell\in\mathbb{C}^{c_{\alpha\ell+\mu}\times d_\ell}$, this is true if and only if (50) holds and F_0, F_1, …, F_{k-1} are all invertible, in which case (11) implies (51). Finally, (46) and (51) imply (52). □

Remark 2 If R and S are unitary, and therefore Q and $V_{\mu,\alpha}$ are unitary, then (51) implies that

$$(A^{-1})^*=V_{\mu,\alpha}\left(\bigoplus_{\ell=0}^{k-1}(F_\ell^{-1})^*\right)Q^*,$$

so $(A^{-1})^*$ is (R,S,α,μ)-symmetric, by Theorem 2.

Theorem 13 *If A is (R,S,α,μ)-symmetric*, $\gcd(\alpha,k)=q$, *and $p=k/q$, then $Az=w$ has no solution unless $w=\sum_{\ell=0}^{p-1}P_{\alpha\ell+\mu}v_{\alpha\ell+\mu}$, in which case z is a solution if and only $z=\sum_{\ell=0}^{k-1}Q_\ell u_\ell$, where*

$$\sum_{\nu=0}^{q-1}F_{\ell+\nu p}u_{\ell+\nu p}=v_{\alpha\ell+\mu}, \quad 0\le\ell\le p-1.$$

Proof. Since our assumptions imply (22),

$$Az=\sum_{\ell=0}^{p-1}P_{\alpha\ell+\mu}\sum_{\nu=0}^{q-1}F_{\ell+\nu p}u_{\ell+\nu p}$$

if $z=\sum_{\ell=0}^{k-1}Q_\ell u_\ell$. This implies the conclusion. □

6 Equidimensional block permutation matrices

We begin with two lemmas. It is straightforward to verify the first by direct matrix multiplication, bearing in mind that subscripts are to be reduced modulo k.

Lemma 1 *If ω_1 and ω_2 are permutations of $\mathbb{Z}_k$ and $H = [H_{rs}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_1\times d_2}$, then*

$$([\delta_{r,\omega_1^{-1}(s)}]_{r,s=0}^{k-1} \otimes I_{d_1})H([\delta_{r,\omega_2^{-1}(s)}]_{r,s=0}^{k-1} \otimes I_{d_2})^{-\alpha} = [H_{\omega_1(r),\omega_2^{\alpha}(s)}]_{r,s=0}^{k-1}. \quad (53)$$

In particular, letting $\omega_1(s) = \omega_2(s) = s+1 \pmod{k}$ yields

$$([\delta_{r,s-1}]_{r,s=0}^{k-1} \otimes I_{d_1}) \left([H_{rs}]_{r,s=0}^{k-1}\right) ([\delta_{r,s-1}]_{r,s=0}^{k-1} \otimes I_{d_2})^{-\alpha} = [H_{r+1,s+\alpha}]_{r,s=0}^{k-1}. \quad (54)$$

Lemma 2 *Let σ be a permutation of $\mathbb{Z}_k$ and $\sigma(\kappa) = 0$. Let ρ be the unique cyclic permutation of $\mathbb{Z}_k$ such that $\sigma(\rho^r(\kappa)) = r$, $0 \le r \le k-1$. Then*

$$\sigma(\rho^\alpha(r)) \equiv \sigma(r) + \alpha \pmod{k}. \quad (55)$$

Proof. Since $\sigma(\rho^r(\kappa)) = r$, $\rho^r(\kappa) = \sigma^{-1}(r)$. Replacing r by $\sigma(r)$ yields $\rho^{\sigma(r)}(\kappa) = r$. Now replacing r by $\rho^\alpha(r)$ yields

$$\rho^{\sigma(\rho^\alpha(r))}(\kappa) = \rho^\alpha(r) = \rho^\alpha(\rho^{\sigma(r)}(\kappa)) = \rho^{\sigma(r)+\alpha}(\kappa),$$

which implies (55). □

In the rest of this paper σ_i and ρ_i, $i = 1, 2, 3$ are related as σ and ρ are related in Lemma 2.

For future reference,

$$f_\ell = \frac{1}{\sqrt{k}} \begin{bmatrix} 1 \\ \zeta^\ell \\ \zeta^{2\ell} \\ \vdots \\ \zeta^{(k-1)\ell} \end{bmatrix}, \quad 0 \le \ell \le k-1, \quad (56)$$

$$\Phi_\ell = f_\ell \otimes I_{d_1}, \quad \Psi_\ell = f_\ell \otimes I_{d_2}, \quad 0 \le \ell \le k-1, \quad (57)$$

$$\mathbf{\Phi} = \begin{bmatrix} \Phi_0 & \Phi_1 & \cdots & \Phi_{k-1} \end{bmatrix}, \quad \text{and} \quad \mathbf{\Psi} = \begin{bmatrix} \Psi_0 & \Psi_1 & \cdots & \Psi_{k-1} \end{bmatrix}. \quad (58)$$

Let

$$E = [\delta_{r,s-1}]_{r,s=0}^{k-1}, \quad R_0 = E \otimes I_{d_1}, \quad S_0 = E \otimes I_{d_2}, \quad T_0 = E \otimes I_{d_3}, \quad (59)$$

$$L_i = [\delta_{r,\sigma_i^{-1}(s)}]_{r,s=0}^{k-1} \otimes I_{d_i}, \quad \text{and} \quad R_i = [\delta_{r,\rho_i^{-1}(s)}]_{r,s=0}^{k-1} \otimes I_{d_i} \quad i = 1,2,3. \quad (60)$$

From (54) with $\alpha = 0$ and (56)–(58),

$$R_0\mathbf{\Phi} = \mathbf{\Phi} D_1 \quad \text{and} \quad S_0\mathbf{\Psi} = \mathbf{\Psi} D_2 \quad \text{with} \quad D_i = \bigoplus_{\ell=0}^{k-1} \zeta^\ell I_{d_i}, \quad i = 1,2, \quad (61)$$

so

$$R_0 = \mathbf{\Phi} D_1 \mathbf{\Phi}^* \quad \text{and} \quad S_0 = \mathbf{\Psi} D_2 \mathbf{\Psi}^*.$$

Theorem 14 *A matrix $A \in \mathbb{C}^{k:d_1\times d_2}$ is (R_1,R_2,α,μ)-symmetric if and only if*

$$A = [\zeta^{\mu\sigma_1(r)} A_{\sigma_2(s)-\alpha\sigma_1(r)}]_{r,s=0}^{k-1} \tag{62}$$

for some $A_0,\ A_1,\ \ldots,\ A_{k-1} \in \mathbb{C}^{d_1\times d_2}$.

Proof. For now we write $A = [B_{rs}]_{r,s=0}^{k-1}$. From (60) and (53) with $\omega_1 = \rho_1$ and $\omega_2 = \rho_2$,

$$R_1 A R_2^{-\alpha} = [B_{\rho_1(r),\rho_2^\alpha(s)}]_{r,s=0}^{k-1} = \zeta^\mu A$$

if and only if

$$B_{\rho_1(r),\rho_2^\alpha(s)} = \zeta^\mu B_{rs}, \quad 0 \le r,s \le k-1. \tag{63}$$

This holds if

$$B_{rs} = \zeta^{\mu\sigma_1(r)} A_{\sigma_2(s)-\alpha\sigma_1(r)}, \quad 0 \le r,s \le k-1, \tag{64}$$

since (55) implies that $\sigma_1(\rho_1(r)) \equiv \sigma_1(r)+1 \pmod k$ and

$$\sigma_2(\rho_2^\alpha(s)) - \alpha\sigma_1(\rho_1(r)) \equiv (\sigma_2(s)+\alpha) - \alpha(\sigma_1(r)+1) \equiv \sigma_2(s) - \alpha\sigma_1(r) \pmod k.$$

For the converse we will show that (63) implies (64) with

$$A_{\sigma_2(s)} = B_{\kappa_1,s} \quad \text{or, equivalently,} \quad A_\ell = B_{\kappa_1,\sigma_2^{-1}(s)}, \quad 0 \le \ell \le k-1. \tag{65}$$

Replacing r by $\rho_1^r(\kappa_1)$ in (64) and noting from (55) that $\sigma_1(\rho_1^r(\kappa_1)) = r$ shows that (64) is equivalent to

$$B_{\rho_1^r(\kappa_1),s} = \zeta^{\mu r} A_{\sigma_2(s)-\alpha r}, \quad 0 \le r,s \le k-1. \tag{66}$$

We will prove this by finite induction on r. Eqn. (65) implies (66) for $r = 0$. Suppose (66) holds for a given r. Replacing r by $\rho_1^r(\kappa_1)$ and s by $\rho_2^{-\alpha s}$ in (63) yields

$$B_{\rho_1^{r+1}(\kappa_1),s} = \zeta^\mu B_{\rho_1^r(\kappa_1),\rho_2^{-\alpha}(s)}.$$

Therefore, from (55) and our induction assumption (66),

$$B_{\rho_1^{r+1}(\kappa_1),s} = \zeta^{\mu(r+1)} A_{\sigma_2(\rho_2^{-\alpha}(s))-\alpha r} = \zeta^{\mu(r+1)} A_{\sigma_2(s)-\alpha(r+1)},$$

which completes the induction. □

Corollary 3 *A matrix $A \in \mathbb{C}^{k:d_1\times d_2}$ is (R_1,R_2,α,μ)-symmetric if and only if*

$$A = L_1\left([\zeta^{\mu r} A_{s-\alpha r}]_{r,s=0}^{k-1}\right) L_2^{-1} \tag{67}$$

(see (60)) with $A_0,\ A_1,\ \ldots,\ A_{k-1} \in \mathbb{C}^{d_1\times d_2}$.

Proof. From (60) and (67), applying (53) with $\omega_1 = \sigma_1$, $\omega_2 = \sigma_2$, $\alpha = 1$, and $H = [\zeta^{\mu r} A_{s-\alpha r}]_{r,s=0}^{k-1}$ yields (62). □

Corollary 4 *A matrix* $A \in \mathbb{C}^{k:d_1\times d_2}$ *is* (R_0, S_0, α, μ)*-symmetric (see* (59)*) if and only if*

$$A = [\zeta^{\mu r} A_{s-\alpha r}]_{r,s=0}^{k-1} \tag{68}$$

for some $A_0,\ A_1,\ \dots,\ A_{k-1} \in \mathbb{C}^{d_1\times d_2}$.

Proof. Setting $\sigma_1(r) = \sigma_2(r) = r+1 \pmod{k}$ in (62) yields

$$A = [\zeta^{\mu(r+1)} A_{(1-\alpha)+s-\alpha r}]_{r,s=0}^{k-1}$$

for some $A_0,\ A_1,\ \dots,\ A_{k-1} \in \mathbb{C}^{d_1\times d_2}$. Redefining A (i.e., replacing $\zeta^{\mu} A_{(1-\alpha)+m}$ with A_m yields (68). □

7 Moore-Penrose inversion of $[A_{s-\alpha r}]_{r,s=0}^{k-1}$

The following theorem is an extension of [15, Theorem 5], where we assumed that $\gcd(\alpha, k) = 1$.

Theorem 15 *Suppose* $A = [A_{s-\alpha r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k:d_1\times d_2}$ *and*

$$F_\ell = \sum_{m=0}^{k-1} \zeta^{\ell m} A_m, \quad 0 \le \ell \le k-1. \tag{69}$$

Suppose also that $\gcd(\alpha, k) = q$ *and* $p = k/q$. *Let*

$$\mathbf{F}_\ell = \left[\begin{array}{cccc} F_\ell & F_{\ell+p} & \cdots & F_{\ell+(q-1)p} \end{array}\right] \tag{70}$$

and partition $\mathbf{F}_\ell^\dagger$ *as*

$$\mathbf{F}_\ell^\dagger = \left[\begin{array}{c} G_\ell \\ G_{\ell+p} \\ \vdots \\ G_{\ell+(q-1)p} \end{array}\right], \quad 0 \le \ell \le p-1,$$

where $G_0,\ G_1,\ \dots,\ G_{k-1} \in \mathbb{C}^{d_2\times d_1}$. *Then*

$$A^\dagger = [B_{r-\alpha s}]_{r,s=0}^{k-1} \quad \textit{where} \quad B_m = \frac{1}{k}\sum_{\ell=0}^{k-1} \zeta^{\ell m} G_\ell, \quad 0 \le m \le k-1. \tag{71}$$

Proof. First, note that (69) is equivalent to

$$A_m = \frac{1}{k}\sum_{\ell=0}^{k-1} \zeta^{-\ell m} F_\ell, \quad 0 \le m \le k-1, \quad \text{so} \quad A = \sum_{\ell=0}^{k-1} P_{\alpha\ell} F_\ell Q_\ell^*$$

where

$$P_{\alpha\ell} = \frac{1}{\sqrt{k}} \begin{bmatrix} 1 \otimes I_{d_1} \\ \zeta^{\alpha\ell} \otimes I_{d_1} \\ \vdots \\ \zeta^{(k-1)\alpha\ell} \otimes I_{d_1} \end{bmatrix} \quad \text{and} \quad Q_\ell = \frac{1}{\sqrt{k}} \begin{bmatrix} 1 \otimes I_{d_2} \\ \zeta^{\ell} \otimes I_{d_2} \\ \vdots \\ \zeta^{(k-1)\ell} \otimes I_{d_2} \end{bmatrix},$$

$0 \le \ell \le k-1$. From Theorem 9 (specifically, (40) with $\mu = 0$),

$$A^\dagger = \sum_{\ell=0}^{k-1} Q_\ell G_\ell P_{\alpha\ell}^* = \frac{1}{k} \left[\sum_{\ell=0}^{k-1} \zeta^{\ell(s-\alpha r)} G_\ell \right]_{r,s=0}^{k-1},$$

which implies (71). □

Remark 3 Theorem 15 is extended to multilevel circulants in [17], which was submitted for publication after this paper was submitted.

Remark 4 The set $\mathcal{F} = \{F_0, F_1, \dots, F_{k-1}\}$ is often called the discrete Fourier transform (dft) of the set $\mathcal{A} = \{A_0, A_1, \dots, A_{k-1}\}$.

Remark 5 If $\gcd(\alpha, k) = 1$ (so $q = 1$ and $p = k$), then (70) reduces to $\mathbf{F}_\ell = G_\ell = F_\ell^\dagger$. Hence, the second equality in (71) reduces to

$$B_m = \frac{1}{k} \sum_{\ell=0}^{k-1} \zeta^{\ell m} F_\ell^\dagger, \quad 0 \le m \le k-1,$$

as we showed in [16, Theorem 5].

Remark 6 Suppose $A = [a_{s-\alpha r}]_{r,s=0}^{k-1} \in \mathbb{C}^{k\times k}$. Then (69) and (70) reduce to

$$f_\ell = \sum_{m=0}^{k-1} a_m \zeta^{\ell m} \quad \text{and} \quad \mathbf{f}_\ell = \begin{bmatrix} f_\ell & f_{\ell+p} & \dots & f_{\ell+(q-1)p} \end{bmatrix}, \quad 0 \le \ell \le p-1.$$

Since

$$\mathbf{f}_\ell^\dagger = \frac{1}{\|\mathbf{f}_\ell\|^2} \begin{bmatrix} \overline{f}_\ell \\ \overline{f}_{\ell+p} \\ \vdots \\ \overline{f}_{\ell+(q-1)p} \end{bmatrix} \quad \text{if} \quad \mathbf{f}_\ell \ne 0 \quad \text{or} \quad \mathbf{f}_\ell^\dagger = 0 \quad \text{if} \quad \mathbf{f}_\ell = 0,$$

it follows that

$$g_{\ell+\nu p} = \begin{cases} \overline{f}_{\ell+\nu p}/|\mathbf{f}_\ell|^2 & \text{if} \quad \mathbf{f}_\ell \ne 0, \\ 0 & \text{if} \quad \mathbf{f}_\ell = 0, \end{cases} \quad 0 \le \ell \le p-1, \quad 0 \le \nu \le q-1.$$

Hence $A^\dagger = [b_{r-\alpha s}]_{r,s=0}^{k-1}$ where $b_m = \frac{1}{k}\sum_{\ell=0}^{k-1} g_\ell \zeta^{\ell m}$. This is a direct generalization of the result of Davis [8], who showed that that if $A = [a_{s-r}]_{r,s=0}^{k-1}$ then $A^\dagger = [b_{r-s}]_{r,s=0}^{k-1}$, where $b_\ell = \frac{1}{k}\sum_{m=0}^{k-1} f_m^\dagger \zeta^{\ell m}$.

Corollary 5 *If* $d_1 = d_2$ *then* $A = [A_{s-\alpha r}]_{r,s=0}^{k-1}$ *is invertible if and only if* $\gcd(\alpha, k) = 1$ *and* $F_0, F_1, \ldots, F_{k-1}$ *are all invertible, in which case*

$$A^{-1} = [B_{r-\alpha s}]_{r,s=0}^{k-1} \quad \textit{where} \quad B_m = \frac{1}{k}\sum_{\ell=0}^{k-1} \zeta^{\ell m} F_\ell^{-1}, \quad 0 \le m \le k-1.$$

8 Arbitrary equidimensional k-involutions

For the rest of this paper $R \in \mathbb{C}^{m\times m}$ and $S \in \mathbb{C}^{n\times n}$ are arbitrary equidimensional k-involutions with widths d_1 and d_2 respectively. Since all equidimensional k-involutions of a given order have the same spectrum, we can write

$$R = XR_0X^{-1} \quad \text{and} \quad S = YS_0Y^{-1} \tag{72}$$

(see (59)) for suitable $X \in \mathbb{C}^{m\times m}$ and $Y \in \mathbb{C}^{n\times n}$.

Theorem 16 *A matrix* $A \in \mathbb{C}^{k:d_1\times d_2}$ *is* (R, S, α, μ)*-symmetric if and only if*

$$A = X\left([\zeta^{\mu r} A_{s-\alpha r}]_{r,s=0}^{k-1}\right) Y^{-1} \tag{73}$$

for some $A_0, A_1, \ldots, A_{k-1} \in \mathbb{C}^{d_1\times d_2}$.

Proof. From (72), A is (R, S, α, μ)-symmetric if and only if

$$(XR_0X^{-1})A(YS_0^{-\alpha}Y^{-1}) = \zeta^\mu A,$$

which is equivalent to

$$R_0(X^{-1}AY)S_0^{-\alpha} = \zeta^\mu(X^{-1}AY).$$

This is equivalent to (73), by Corollary 4. □

Remark 7 We can rewrite (73) as $A = XD_1^\mu CY^{-1}$ with $C = [A_{s-\alpha r}]_{r,s=0}^{k-1}$ and D_1 as in (61). It is straightforward to verify that $B = YC^\dagger D_1^{-\mu}X^{-1}$ is a reflexive inverse of A, and that $B = A^\dagger$ if R and S are unitary.

Remark 8 Eqn. (73) must reduce to (67) when $R = R_1$ and $S = S_1$. To verify this explicitly, we note that from (53) with $\omega_1 = \sigma_1$, $\omega_2 = \sigma_2$, $\alpha = 1$, and $H_{rs} = \delta_{rs}$,

$$\begin{aligned}\left([\delta_{r,\sigma_i^{-1}(s)}]_{r,s=0}^{k-1}\right)\left([\delta_{r,s-1}]_{r,s=0}^{k-1}\right)\left([\delta_{r,\sigma_i^{-1}(s)}]_{r,s=0}^{k-1}\right)^{-1} &= [\delta_{\sigma_i(r),\sigma_i(s)-1}]_{r,s=0}^{k-1} \\ &= [\delta_{r,\rho_i^{-1}(s)}]_{r,s=0}^{k-1},\end{aligned}$$

where the last equality is valid because (55) with $\alpha = -1$ implies that $\sigma_i(r) = \sigma_i(s)-1$ if and only if $r = \rho_i^{-1}(s)$. Therefore, from (59) and (60), $R_1 = L_1R_0L_1^{-1}$ and $R_2 = L_2S_0L_2^{-1}$. Hence, if $R = R_1$ and $S = R_2$ then $X = L_1$ and $Y = L_2$ in (73), which is consistent with (67).

Remark 9 The conclusion of Theorem 5 can made more explicit if R, S are as in (72) and $T = ZT_0Z^{-1}$. (See (59).) If $A \in \mathbb{C}^{k:d_1\times d_2}$ is (R, S, α, μ)-symmetric and $B \in \mathbb{C}^{k:d_2\times d_3}$ is (S, T, β, ν)-symmetric, then Theorem 16 implies that

$$A = X\left([\zeta^{\mu r} A_{s-\alpha r}]_{r,s=0}^{k-1}\right) Y^{-1} \quad \text{and} \quad B = Y\left([\zeta^{\nu r} B_{s-\beta r}]_{r,s=0}^{k-1}\right) Z^{-1}$$

for some $A_0, A_1, \ldots, A_{k-1} \in \mathbb{C}^{d_1\times d_2}$ and $B_0, B_1, \ldots, B_{k-1} \in \mathbb{C}^{d_2\times d_3}$. Therefore

$$AB = X\left([\zeta^{\mu r} A_{s-\alpha r}]_{r,s=0}^{k-1}\right)\left([\zeta^{\nu r} B_{s-\beta r}]_{r,s=0}^{k-1}\right) Z^{-1}. \tag{74}$$

On the other hand, Theorem 5 implies that AB is $(R, T, \alpha\beta, \alpha\mu+\nu)$-symmetric, so Theorem 16 implies that

$$AB = X\left([\zeta^{(\alpha\mu+\nu)r} C_{s-\alpha\beta r}]_{r,s=0}^{k-1}\right) Z^{-1}$$

for suitable $C_0, C_1, \ldots, C_{k-1} \in \mathbb{C}^{d_1\times d_3}$. Computing the first row ($r = 0$) of the product between X and Z^{-1} in (74) yields

$$C_m = \sum_{\ell=0}^{k-1} \zeta^{\nu\ell} A_\ell B_{m-\beta\ell}, \quad 0 \le m \le k-1.$$

This extends [16, Theorem 2], which in turn extended [1, Theorem 3.1]. Note that the assumption that $\gcd(\beta, k) = 1$, which we imposed to obtain (25), is no longer required.

Remark 10 Letting $X = I_{nd_1}$, $Y = I_{nd_2}$, $Z = I_{nd_3}$, and $\mu = \nu = 0$, we see from Remark 9 that if $A = [A_{s-\alpha r}]_{r,s=0}^{k-1}$ and $B = [B_{s-\beta r}]_{r,s=0}^{k-1}$ with $\alpha\beta \equiv 1 \pmod k$, then

$$AB = [C_{s-r}]_{r,s=0}^{k-1} \quad \text{with} \quad C_m = \sum_{\ell=0}^{k-1} A_\ell B_{m-\beta\ell} \quad 0 \le m \le k-1.$$

This generalizes a well known result; namely, the product of 1-circulants is a 1-circulant.

Remark 11 The conclusions of Theorem 6 can also be made more explicit if R and S are as in (72) and unitary. If $A \in \mathbb{C}^{k:d_1\times d_2}$ is (R, S, α, μ)-symmetric and $B \in \mathbb{C}^{k:d_1\times d_2}$ is (R, S, α, ν)-symmetric, then Theorem 16 implies that

$$A = X\left([\zeta^{\mu r} A_{s-\alpha r}]_{r,s=0}^{k-1}\right) Y^* \quad \text{and} \quad B = X\left([\zeta^{\nu r} B_{s-\alpha r}]_{r,s=0}^{k-1}\right) Y^*. \tag{75}$$

Therefore

$$AB^* = X\left([\zeta^{\mu r} A_{s-\alpha r}]_{r,s=0}^{k-1}\right)\left([\zeta^{-\nu s} B^*_{r-\alpha s}]_{r,s=0}^{k-1}\right) X^* \tag{76}$$

On the other hand, Theorem 6 and Remark 1 imply that AB^* is $(R, R, 1, \mu-\nu)$-symmetric. Hence, Theorem 16 implies that

$$AB^* = X\left([\zeta^{(\mu-\nu)r} C_{s-r}]_{r,s=0}^{k-1}\right) X^*$$

with $C_0,\ C_1,\ \ldots,\ C_{k-1}\in\mathbb{C}^{d_1\times d_1}$. Computing the first row of the product between X and X^* in (76) yields

$$C_m=\zeta^{-\nu m}\sum_{\ell=0}^{k-1}A_\ell B^*_{\ell-\alpha m},\quad 0\le m\le k-1.$$

As noted in Remark 1, we did not need to assume that $\gcd(\alpha,k)=1$ in this argument.

From (75),

$$B^*A=Y\left([\zeta^{-\nu s}B^*_{r-\alpha s}]_{r,s=0}^{k-1}\right)\left([\zeta^{\mu r}A_{s-\alpha r}]_{r,s=0}^{k-1}\right)Y^*. \tag{77}$$

Now suppose $\gcd(\alpha,k)=1$ and $\alpha\beta\equiv 1 \pmod{k}$. Then Theorem 6 implies that B^*A is $(S,1,\beta(\mu-\nu))$-symmetric. Hence, Theorem 16 implies that

$$B^*A=Y\left([\zeta^{\beta(\mu-\nu)r}D_{s-r}]_{r,s=0}^{k-1}\right)Y^*$$

with $D_0,\ D_1,\ \ldots,\ D_{k-1}\in\mathbb{C}^{d_2\times d_2}$. Computing the first row of the product between Y and Y^* in (77) yields

$$D_m=\sum_{\ell=0}^{k-1}\zeta^{\ell(\mu-\nu)}B^*_{-\alpha\ell}A_{m-\alpha\ell},\quad 0\le m\le k-1.$$

Replacing ℓ by $-\beta\ell$ simplifies this to

$$D_m=\sum_{\ell=0}^{k-1}\zeta^{-\beta\ell(\mu-\nu)}B^*_\ell A_{m+\ell},\quad 0\le m\le k-1.$$

This extends [16, Corollary 2], which in turn extended [12, Corollary 1].

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