Asymptotic preconditioning of linear homogeneous systems of differential equations

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Abstract

We consider the asymptotic behavior of solutions of a linear differential system $x' = A(t)x$, where A is continuous on an interval $[a,\infty)$. We are interested in the situation where the system may not have a desirable asymptotic property such as stability, strict stability, uniform stability, or linear asymptotic equilibrium, but its solutions can be written as $x = Pu$, where P is continuously differentiable on $[a,\infty)$ and u is a solution of a system $u' = B(t)u$ that has the property in question. In this case we say that P preconditions the given system for the property in question.

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1 Introduction

In this paper $\mathcal{I} = [a,\infty)$ and $\mathbb{C}^n$, $\mathbb{C}^{n\times n}$, $\mathbb{C}_0^n(\mathcal{I})$, $\mathbb{C}_0^{n\times n}(\mathcal{I})$, $\mathbb{C}_1^n(\mathcal{I})$, and $\mathbb{C}_1^{n\times n}(\mathcal{I})$ are respectively the sets of n-vectors with complex entries, $n\times n$ matrices with complex entries, continuous complex n-vector functions on $\mathcal{I}$, continuous complex $n\times n$ matrix functions on $\mathcal{I}$, continuously differentiable n-vector functions on $\mathcal{I}$, and continuously differentiable $n\times n$ complex matrix functions on $\mathcal{I}$. ("Complex" and "$\mathbb{C}$" can

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just as well be replaced by "real" and "$\mathbb{R}$.") If $\xi \in \mathbb{C}^n$ and $C \in \mathbb{C}^{n\times n}$ then $\|\xi\|$ is a vector norm and $\|C\|$ is the corresponding induced matrix norm; i.e., $\|C\| = \max\left\{\|C\xi\| \,\middle|\, \|\xi\| = 1\right\}$. Throughout the paper $A \in \mathbb{C}_0^{n\times n}(\mathcal{I})$, $\mathcal{S}_A$ is the set of solutions of

$$x' = A(t)x, \quad t \in \mathcal{I}, \tag{1}$$

$$\mathcal{J} = \left\{(t,\tau) \,\middle|\, a \le \tau \le t\right\}, \text{ and } \quad \mathcal{R} = \left\{R \in \mathbb{C}_1^{n\times n}(\mathcal{I}) \,\middle|\, R^{-1} \in \mathbb{C}_1^{n\times n}(\mathcal{I})\right\}.$$

We recall that if $X \in \mathbb{C}_1^{n\times n}(\mathcal{I})$ satisfies $X' = A(t)X$, $t \in \mathcal{I}$, then either $X(t)$ is invertible for all $t \in \mathcal{I}$ or $X(t)$ is noninvertible for all $t \in \mathcal{I}$. In the first case X is said to be a fundamental matrix for (1), and $x \in \mathcal{S}_A$ if and only if $x = X(t)\xi$ for some ξ in $\mathbb{C}^n$ or, equivalently,

$$x(t) = X(t)X^{-1}(\tau)x(\tau) \text{ for all } t,\tau \in \mathcal{I}.$$

We begin with some standard definitions.

Definition 1

(a) Eq. (1) is stable if for each $\tau \in \mathcal{I}$ there is a constant M_τ such that $\|x(t)\| \le M_\tau\|x(\tau)\|$ for all $t \in \mathcal{I}$ and $x \in \mathcal{S}_A$.

(b) Eq. (1) is strictly stable if there is a constant M such that $\|x(t)\| \le M\|x(\tau)\|$ for all $t,\tau \in \mathcal{I}$ and $x \in \mathcal{S}_A$.

(c) Eq. (1) is uniformly stable if there is a constant M such that $\|x(t)\| \le M\|x(\tau)\|$ for all $(t,\tau) \in \mathcal{J}$ and $x \in \mathcal{S}_A$.

(d) Eq. (1) is uniformly asymptotically stable if there are constants M and $\nu > 0$ such that $\|x(t)\| \le M\|x(\tau)\|e^{-\nu(t-\tau)}$ for all $(t,\tau) \in \mathcal{J}$ and $x \in \mathcal{S}_A$.

(e) Eq. (1) has linear asymptotic equilibrium if every nontrivial solution of (1) approaches a nonzero constant vector as $t \to \infty$.

It is convenient to include **(c)** and **(d)** in the following definition, which may be new. Let ρ be continuous and positive on $\mathcal{J}$ and suppose that

$$\rho(t,t) = 1 \text{ and } \rho(t,\tau) \le \rho(t,s)\rho(s,\tau), \quad a \le \tau \le s \le t. \tag{2}$$

We say that (1) is ρ-stable if there is a constant M such that

$$\|x(t)\| \le M\|x(\tau)\|/\rho(t,\tau) \text{ for all } (t,\tau) \in \mathcal{J} \text{ and } x \in \mathcal{S}_A.$$

We consider the following problem: given a system that does not have one of the properties defined above, is it possible to analyze (1) in terms of a related system that has the property?

Henceforth P is a given member of $\mathcal{R}$. We offer the following definition.

Definition 2

(a) Eq. (1) is stable relative to P if for each $\tau \in \mathcal{I}$ there is a constant M_τ such that

$$\|P^{-1}(t)x(t)\| \le M_\tau\|P^{-1}(\tau)x(\tau)\| \text{ for all } t,\tau \in \mathcal{I} \text{ and } x \in \mathcal{S}_A.$$

(b) Eq. (1) is strictly stable relative to P if there is a constant M such that

$$\|P^{-1}(t)x(t)\| \le M\|P^{-1}(\tau)x(\tau)\| \text{ for all } t,\tau \in \mathcal{J} \text{ and } x \in \mathcal{S}_A.$$

(c) Eq. (1) is ρ-stable relative to P if there is a constant M such that

$$\|P^{-1}(t)x(t)\| \le M\|P^{-1}(\tau)x(\tau)\|/\rho(t,\tau) \text{ for all } (t,\tau) \in \mathcal{J} \text{ and } x \in \mathcal{S}_A.$$

(d) Eq. (1) has linear asymptotic equilibrium relative to P if $\lim_{t\to\infty} P^{-1}(t)x(t)$ exists and is nonzero for every nontrivial $x \in \mathcal{S}_A$.

Lemma 1 *If $x \in \mathbb{C}_1^n(\mathcal{J})$ and $u = P^{-1}x$, then $x' = Ax$, $t \in \mathcal{J}$, if and only if*

$$u' = P^{-1}(AP - P')u, \quad t \in \mathcal{J}, \tag{3}$$

or, equivalently, if and only if $x = PU\xi$ where U is a fundamental matrix for (3) *and $\xi \in \mathbb{C}$.*

PROOF. Since $x = Pu$, $x' = Pu' + P'u$ and $Ax = APu$, so $x' = Ax$ if and only if $Pu' + P'u = APu$, which is equivalent to (3). □

To illustrate the problem that we study here, we cite a theorem attributed by Wintner [8] to Bôcher, which says that (1) has linear asymptotic equilibrium if $\int^\infty \|A(t)\|\,dt < \infty$. This theorem does not apply to (1) if $\int^\infty \|A(t)\|\,dt = \infty$, but, by Lemma 1 it does imply that (1) has linear asymptotic equilibrium relative to P if

$$\int^\infty \|P^{-1}(AP - P')\|\,dt < \infty.$$

Adapting terminology commonly used in computational linear algebra, we will in this case refer to the transformation $u = P^{-1}x$ as asymptotic preconditioning, and we say that P preconditions (1) for asymptotic equilibrium. More generally, if $\mathcal{P}$ is a given property of linear differential systems (for example, one of the properties mentioned earlier), we say that P preconditions (1) for property $\mathcal{P}$ if (3) has property $\mathcal{P}$ or, equivalently, if (1) has property $\mathcal{P}$ relative to P.

This paper is strongly influenced by Conti's work [2, 3, 4] on t_∞-similarity of systems of differential equations and our extensions [5, 6] of his results. However, we believe that our reformulation of these results in the context of asymptotic preconditioning is new and useful. We offer the paper not as a breakthrough in the asymptotic theory of linear differential systems, but as an expository approach to what we believe is a new application of standard results on this subject.

2 Preliminary considerations

The proof of most of the following lemma can be pieced together from applying various results in our references to the system (3); however, in keeping with our expository goal, we present a self-contained proof here.

Lemma 2 *Let U be a fundamental matrix for* (3). *Then*:

(a) *Eq.* (1) *is stable relative to P if and only if U is bounded on $\mathcal{I}$.*

(b) *Eq.* (1) *is ρ-stable relative to P if and only if there is a constant M such that*

$$\|U(t)U^{-1}(\tau)\| \le M/\rho(t,\tau), \quad (t,\tau) \in \mathcal{J}. \tag{4}$$

(c) *Eq.* (1) *is strictly stable relative to P if and only if $\|U\|$ and $\|U^{-1}\|$ are bounded on $\mathcal{I}$ or, equivalently, if and only if there is a constant M such that*

$$\|U(t)U^{-1}(\tau)\| \le M, \quad t,\tau \in \mathcal{I}. \tag{5}$$

(d) *Eq.* (1) *has linear asymptotic equilibrium relative to P if and only if* $\lim_{t\to\infty} U(t)$ *exists and is invertible.*

PROOF. From Lemma 1, it suffices to to show that the assumptions **(a)**–**(d)** are respectively equivalent to stability, ρ-stability, strict stability, and linear asymptotic equilibrium of (3). Since every solution of (3) can be written as $u(t) = U(t)\xi$ with $\xi \in \mathbb{C}^n$, **(d)** is obvious. For the rest of the proof, let $\mathcal{U}$ denote the set of all solutions of (3). Then $u \in \mathcal{U}$ if and only if

$$u(t) = U(t)U^{-1}(\tau)u(\tau) \text{ for all } t,\tau \in \mathcal{I}. \tag{6}$$

If τ is arbitrary but fixed and $K_\tau = \|U^{-1}(\tau)\|$, then (6) implies that

$$\|u(t)\| \le K_\tau \|U(t)\| \|u(\tau)\| \text{ for all } t,\tau \in \mathcal{I} \text{ and } u \in \mathcal{U}.$$

This implies sufficiency for **(a)**. Also from (6),

$$\|u(t)\| \le \|U(t)U^{-1}(\tau)\| \|u(\tau)\| \text{ for all } t,\tau \in \mathcal{I} \text{ and } u \in \mathcal{U}.$$

Therefore (4) implies that

$$\|u(t)\| \le M\|u(\tau)\|/\rho(t,\tau) \text{ for all } (t,\tau) \in \mathcal{J} \text{ and } u \in \mathcal{U},$$

which implies sufficiency for **(b)**. Moreover, (5) implies that

$$\|u(t)\| \le M\|u(\tau)\| \text{ for all } t,\tau \in \mathcal{I} \text{ and } u \in \mathcal{U}$$

which implies sufficiency for **(c)**.

We use contrapositive arguments to establish necessity in **(a)**, **(b)**, and **(c)**. In all three cases let M be an arbitrary positive constant. For **(a)**, if U is unbounded and τ is fixed in $\mathcal{I}$, then $U(t)U^{-1}(\tau)$ is also unbounded as a function of t (since $U(t) = U(t)U^{-1}(\tau)U(\tau)$). Therefore there is a $t_0 \in \mathcal{I}$ and a $\xi \in \mathbb{C}^n$ such that $\|U(t_0)U^{-1}(\tau)\xi\| > M\|\xi\|$. Hence, if $u_0(t) = U(t)U^{-1}(\tau)\xi$ then $u_0 \in \mathcal{U}$ and

$$\|u(t_0)\| = \|U(t_0)U^{-1}(\tau)\xi\| > M\|\xi\| = M\|u(\tau)\|;$$

hence (3) is not stable.

For **(b)**, if there is a $(t_0, \tau_0) \in \mathcal{J}$ such that

$$\|U(t_0, \tau_0)\| > M/\rho(t_0, \tau_0),$$

then

$$\|U(t_0, \tau_0)\xi\| > M\|\xi\|/\rho(t_0, \tau_0)$$

for some $\xi \in \mathbb{C}^n$. If $u(t) = U(t)U^{-1}(\tau_0)\xi$ then

$$\|u(t_0)\| = \|U(t_0)U^{-1}(\tau_0)\xi\| > M\|\xi\|/\rho(t_0, \tau_0) = M\|u(\tau_0)\|/\rho((t_0, \tau_0)),$$

so (3) is not ρ-stable. A similar argument shows that if (3) is strictly stable, then (5) holds for some M.

Eq. (5) obviously holds for some M if U and U^{-1} are bounded on $\mathcal{I}$. It remains to show that (5) implies that U and U^{-1} are bounded on $\mathcal{I}$. If $\tau \in \mathcal{I}$ is fixed and t is arbitrary, then (5) implies that

$$\|U(t)\| = \|U(t)U^{-1}(\tau)U(\tau)\| \le \|U(t)U^{-1}(\tau)\|\|U(\tau)\| \le M\|U(\tau)\|,$$

so U is bounded on I. To complete the proof, we must show that if U^{-1} is unbounded then (5) is false for every M. Let $t_0 \in \mathcal{I}$ be fixed and let $\sigma = \min\left\{\|U(t_0)\eta\| \;\middle|\; \|\eta\| = 1\right\}$, which is positive, since $U(t_0)$ is invertible. If U^{-1} is unbounded on $\mathcal{I}$ there is a $\tau \in \mathcal{I}$ and $\xi \in \mathbb{C}^n$ such that $\|\xi\| = 1$ and $\|U^{-1}(\tau)\xi\| > M/\sigma$. Then

$$\|U(t_0)U^{-1}(\tau)\xi\| > \sigma\|U^{-1}(\tau)\xi\| > M\|\xi\|, \quad \square$$

so $\|U(t_0)U^{-1}(\tau)\| > M$. $\square$

Lemma 3 *Suppose that $R, Q \in \mathcal{R}$ and let*

$$F = R' - Q'Q^{-1}R + RP^{-1}(P' - AP). \tag{7}$$

Then $X = PU \in \mathbb{C}^{n\times n}(\mathcal{I})$ satisfies $X' = AX$, $t \in \mathcal{I}$, if and only if

$$(Q^{-1}RU)' = Q^{-1}FU, \quad t \in \mathcal{I}. \tag{8}$$

PROOF. From (7),

$$\begin{aligned}(Q^{-1}RU)' &= Q^{-1}(R'U - Q'Q^{-1}RU + RU') \\ &= Q^{-1}FU + Q^{-1}R\left(U' - P^{-1}(P' - AP)U\right),\end{aligned}$$

so Lemma 1 implies the conclusion. $\square$

This lemma provides an infinite family of linear differential systems, all with the same solutions; namely, u is a solution of (3) (and consequently $x = Pu$ is a solution of (1)) if and only if u is a solution of *every* system of the form (8). Therefore, if (8) has a given property $\mathcal{P}$ for *some suitably chosen* R and Q in $\mathcal{R}$, then P preconditions (1) for $\mathcal{P}$.

3 Main results

Theorem 1 *Suppose that there are $R, Q \in \mathcal{R}$ such that R and R^{-1} are bounded on $\mathcal{J}$ and*

$$\int^{\infty} \|F(s)\|\, ds < \infty. \tag{9}$$

Then:

(a) *P preconditions Eq. (1) for ρ-stability if there is a constant M such that*

$$\|Q(t)Q^{-1}(\tau)\| \le M/\rho(t,\tau), \quad a \le \tau \le t. \tag{10}$$

(b) *P preconditions Eq. (1) for strict stability if Q and Q^{-1} are bounded on $\mathcal{J}$.*

PROOF. Integrating (8) yields

$$U(t) = R^{-1}(t)Q(t)\left(Q^{-1}(\tau)R(\tau)U(\tau) + \int_{\tau}^{t} Q^{-1}(s)F(s)U(s)\, ds\right), \tag{11}$$

$t, \tau \in \mathcal{J}$. Therefore

$$U(t)U^{-1}(\tau) = R^{-1}(t)Q(t)\left(Q^{-1}(\tau)R(\tau) + \int_{\tau}^{t} Q^{-1}(s)F(s)U(s)U^{-1}(\tau)\, ds\right). \tag{12}$$

To prove **(a)**, let

$$g(t,s) = \|Q(t)Q^{-1}(s)\|\rho(t,s) \quad \text{and} \quad h(s,\tau) = \|U(s)U^{-1}(\tau)\|\rho(s,\tau). \tag{13}$$

By Lemma 2**(b)**, we must show that $h(t,\tau)$ is bounded for $(t,\tau) \in \mathcal{G}$. If $\tau \le s \le t$ then (2) implies that

$$\rho(t,\tau)\|Q(t)Q^{-1}(s)F(s)U(s)U^{-1}(\tau)\| \le g(t,s)\|F(s)\|h(s,\tau).$$

Since R and R^{-1} are bounded, multiplying both sides of (12) by $\rho(t,\tau)$ yields the inequality

$$h(t,\tau) \le c_1 g(t,\tau) + c_2 \int_{\tau}^{t} g(t,s)\|F(s)\|h(s,\tau)\, ds, \quad a \le \tau \le t,$$

for suitable constants c_1 and c_2. Now (10) and (13) imply that

$$h(t,\tau) \le M\left[c_1 + c_2 \int_{\tau}^{t} \|F(s)\|h(s,\tau)\, ds\right], \quad a \le \tau \le t. \tag{14}$$

Therefore

$$\frac{c_2 h(t,\tau)\|F(t)\|}{c_1 + c_2 \displaystyle\int_{\tau}^{t} \|F(s)\|h(s,\tau)\, ds} \le Mc_2\|F(r)\| \quad a \le \tau \le r. \tag{15}$$

Integrating this with respect to t yields

$$\log\left(c_1 + c_2 \int_\tau^t \|F(s)\| h(s,\tau)\, ds\right) - \log c_1 \le M c_2 \int_\tau^t \|F(s)\|\, ds. \qquad (16)$$

This and (14) imply that

$$\sup\left\{\|h(t,\tau)\| \,\middle|\, (t,\tau) \in \mathcal{J}\right\} \le M c_1 \exp\left(M \int_a^\infty \|F(s)\|\, ds\right) < \infty, \qquad (17)$$

from (9). This completes the proof of **(a)**.

To prove **(b)**, replace (13) by

$$g(t,s) = \|Q(t)Q^{-1}(s)\| \quad \text{and} \quad h(s,\tau) = \|U(s)U^{-1}(\tau)\|.$$

Now we must show that $h(t,\tau)$ is bounded for all $t,\, \tau \in \mathcal{J}$. If (1) is strictly stable then there is a constant M such that $g(t,s) \le M$ for $s,\, t \ge a$. This, (12) and the boundedness of R and R^{-1} imply that

$$h(t,\tau) \le M\left[c_1 + c_2 \left|\int_\tau^t \|F(s)\| h(s,\tau)\, ds\right|\right], \quad t,\tau \ge a,$$

for suitable positive constants c_1 and c_2. Now the argument used in the proof of **(a)** again inplies (17). If $a \le t \le \tau$ then (14)–(17) all hold with t and τ interchanged, which completes the proof of **(b)**. $\square$

Remark 1 The use of logarithmic integration that produced (16) was motivated by the proof of Gronwall's inequality [1, p. 35], a standard tool for studying the asymptotic behavior of solutions of differential equations.

Theorem 2 *In addition to the assumptions of Theorem 1*(**b**), *suppose that*

$$\lim_{t\to\infty} R^{-1}(t)Q(t) = J \quad \textit{is invertible.} \qquad (18)$$

Then P preconditions (1) *for linear asymptotic equilibrium.*

PROOF. From (11) and (18), $\lim_{t\to\infty} U(t) = V$, where

$$V = J\left(Q^{-1}(\tau)R(\tau)U(\tau) + \int_\tau^\infty Q^{-1}(s)F(s)U(s)\, ds\right)$$

and the integral converges because of (9), the boundedness of Q^{-1} (assumed) and U (from Theorem 1(**b**)). Now we must show that V is inverible. Since Theorem 1(**b**) implies that (1) is strictly stable relative to P, there is a constant K such that $\|U^{-1}\| < K$, $t \in \mathcal{J}$. If $\xi \in \mathbb{C}^n$ then

$$\|\xi\| = \|U^{-1}(t)U(t)\xi\| \le \|U^{-1}(t)\|\|U(t)\xi\| \le K\|U(t)\xi\|, \quad t \le a,$$

so

$$\|\xi\| \le K \lim_{t\to\infty} \|U(t)\xi\| = K\|V\xi\|.$$

Therefore $V\xi = 0$ if and only if $\xi = 0$, so V is invertible. $\square$

Theorem 3 *If there are Q and R in $\mathcal{R}$ such that $R^{-1}Q$ is bounded and*

$$\int^{\infty} \|Q^{-1}(s)F(s)\|\, ds < \infty,$$

then P preconditions (1) *for stability; moreover, if* (18) *holds then P preconditions* (1) *for linear asymptotic equilibrium.*

PROOF. Our assumptions imply that if $0 < \rho < 1$ then there is a $\tau \geq a$ such that

$$\|R^{-1}(t)Q(t)\| \int_{\tau}^{\infty} \|Q^{-1}(s)F(s)\|\, ds \leq \rho, \quad t \geq \tau.$$

Let $\mathcal{B}$ be the Banach space of bounded continuous $n \times n$ vector functions on $\mathcal{J} = [\tau, \infty)$ with norm $\|U\|_{\mathcal{B}} = \sup_{t \in \mathcal{J}} \|U(t)\|$, and define $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ by

$$(\mathcal{T}U)(t) = R^{-1}(t)Q(t)\left(C - \int_{t}^{\infty} Q^{-1}(s)F(s)U(s)\, ds\right)$$

where $C \in \mathbb{C}^{n \times n}$ is invertible. If $U_1, U_2 \in \mathcal{B}$ then

$$(\mathcal{T}U_1)(t) - (\mathcal{T}U_2)(t) \leq \|R^{-1}(t)Q(t)\| \int_{t}^{\infty} \|Q^{-1}(s)F(s)\|\|U_1(s) - U_2(s)\|\, ds,$$

so $\|\mathcal{T}U_1 - \mathcal{T}U_2\|_{\mathcal{B}} \leq \rho\|U_1 - U_2\|_{\mathcal{B}}$. Therefore, by the contraction mapping principal [7, p. 545], there is a $U \in \mathcal{B}$ such that

$$U(t) = R^{-1}(t)Q(t)\left(C - \int_{t}^{\infty} Q^{-1}(s)F(s)U(s)\, ds\right).$$

Since U satisfies (8), Theorem 1 implies that $X = PU$ satisfies (1). Therefore P preconditions (1) for stability. Finally, if (18) holds then $\lim_{t\to\infty} U(t) = JC$ is invertible, so P preconditions (1) for linear asymptotic equilibrium. $\square$

Remark 2 Strictly speaking, our proof of Theorem 3 defines U only on the interval $[\tau, \infty)$, which has the appearance of leaving a gap if $\tau > a$. However, in this case we appeal to the elementary theory of linear differential systems, which guarantees that U can extended uniquely over $\mathcal{J}$ as an invertible solution of $U' = P^{-1}(AP - P')U$.

From (7),

$$Q^{-1}F = (Q^{-1}R)' + (Q^{-1}R)P^{-1}(P' - AP).$$

Therefore we can reformulate Theorem 3 as follows.

Theorem 4 *If there is a $T \in \mathcal{R}$ such that T^{-1} is bounded and*

$$\int^{\infty} \|T' + TP^{-1}(P' - AP)\|\, ds < \infty,$$

then P preconditions (1) *for stability; moreover, if $\lim_{t\to\infty} T(t)$ exists and is invertible then P preconditions* (1) *for linear asymptotic equilibrium.*

The assertion concerning linear asymptotic equilbrium can also be proved by applying a theorem of Conti [3] to (3).

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