Asymptotic preconditioning of linear homogeneous systems of differential equations

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Abstract

We consider the asymptotic behavior of solutions of a linear differential system
\[ x' = A(t)x, \]
where \( A \) is continuous on an interval \([a, \infty)\). We are interested in the situation where the system may not have a desirable asymptotic property such as stability, strict stability, uniform stability, or linear asymptotic equilibrium, but its solutions can be written as \( x = Pu \), where \( P \) is continuously differentiable on \([a, \infty)\) and \( u \) is a solution of a system \( u' = Bu \) that has the property in question. In this case we say that \( P \) preconditions the given system for the property in question.

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1 Introduction

In this paper \( J = [a, \infty) \) and \( \mathbb{C}^n, \mathbb{C}^{n \times n}, \mathbb{C}^n_0(J), \mathbb{C}^{n \times n}_0(J), \mathbb{C}^n_1(J), \) and \( \mathbb{C}^{n \times n}_1(J) \) are respectively the sets of \( n \)-vectors with complex entries, \( n \times n \) matrices with complex entries, continuous complex \( n \)-vector functions on \( J \), continuous complex \( n \times n \) matrix functions on \( J \), continuously differentiable \( n \)-vector functions on \( J \), and continuously differentiable \( n \times n \) complex matrix functions on \( J \). ("Complex" and "\( \mathbb{C} \)" can

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just as well be replaced by “real” and “\( \mathbb{R} \).”) If \( \xi \in \mathbb{C}^n \) and \( C \in \mathbb{C}^{n \times n} \) then \( \| \xi \| \) is a vector norm and \( \| C \| \) is the corresponding induced matrix norm; i.e., \( \| C \| = \max \{ \| C \xi \| : \| \xi \| = 1 \} \). Throughout the paper \( A \in \mathbb{C}^{n \times n}_0 \), \( S_A \) is the set of solutions of

\[
x' = A(t)x, \quad t \in J, \tag{1}
\]

\( J = \{(t, \tau) \mid a \leq \tau \leq t\} \), and \( \mathcal{R} = \{ R \in \mathbb{C}^{n \times n}(J) \mid R^{-1} \in \mathbb{C}^{n \times n}(J) \} \).

We recall that if \( X \in \mathbb{C}^{n \times n}(J) \) satisfies \( X' = A(t)X, \ t \in J \), then either \( X(t) \) is invertible for all \( t \in J \) or \( X(t) \) is noninvertible for all \( t \in J \). In the first case \( X \) is said to be a fundamental matrix for (1), and \( x \in S_A \) if and only if \( x = X(t)\xi \) for some \( \xi \) in \( \mathbb{C}^n \) or, equivalently,

\[
x(t) = X(t)X^{-1}(\tau)x(\tau) \text{ for all } t, \tau \in J.
\]

We begin with some standard definitions.

**Definition 1**

(a) Eq. (1) is stable if for each \( \tau \in J \) there is a constant \( M_\tau \) such that \( \| x(t) \| \leq M_\tau \| x(\tau) \| \) for all \( t \in J \) and \( x \in S_A \).

(b) Eq. (1) is strictly stable if there is a constant \( M \) such that \( \| x(t) \| \leq M \| x(\tau) \| \) for all \( t, \tau \in J \) and \( x \in S_A \).

(c) Eq. (1) is uniformly stable if there is a constant \( M \) such that \( \| x(t) \| \leq M \| x(\tau) \| \) for all \( (t, \tau) \in J \) and \( x \in S_A \).

(d) Eq. (1) is uniformly asymptotically stable if there are constants \( M \) and \( \nu > 0 \) such that \( \| x(t) \| \leq M \| x(\tau) \| e^{-\nu(t-\tau)} \) for all \( (t, \tau) \in J \) and \( x \in S_A \).

(e) Eq. (1) has linear asymptotic equilibrium if every nontrivial solution of (1) approaches a nonzero constant vector as \( t \to \infty \).

It is convenient to include (c) and (d) in the following definition, which may be new. Let \( \rho \) be continuous and positive on \( J \) and suppose that

\[
\rho(t, t) = 1 \text{ and } \rho(t, \tau) \leq \rho(t, s)\rho(s, \tau), \quad a \leq \tau \leq s \leq t. \tag{2}
\]

We say that (1) is \( \rho \)-stable if there is a constant \( M \) such that

\[
\| x(t) \| \leq M \| x(\tau) \|/\rho(t, \tau) \text{ for all } (t, \tau) \in J \text{ and } x \in S_A.
\]

We consider the following problem: given a system that does not have one of the properties defined above, is it possible to analyze (1) in terms of a related system that has the property?

Henceforth \( P \) is a given member of \( \mathcal{R} \). We offer the following definition.

**Definition 2**

(a) Eq. (1) is stable relative to \( P \) if for each \( \tau \in J \) there is a constant \( M_\tau \) such that

\[
\| P^{-1}(t)x(t) \| \leq M_\tau \| P^{-1}(\tau)x(\tau) \| \text{ for all } t, \tau \in J \text{ and } x \in S_A.
\]
(b) Eq. (1) is strictly stable relative to $P$ if there is a constant $M$ such that
$$
\|P^{-1}(t)x(t)\| \leq M\|P^{-1}(\tau)x(\tau)\| \text{ for all } t, \tau \in \mathcal{J} \text{ and } x \in \mathcal{S}_A.
$$

(c) Eq. (1) is $\rho$-stable relative to $P$ if there is a constant $M$ such that
$$
\|P^{-1}(t)x(t)\| \leq M\|P^{-1}(\tau)x(\tau)\|/\rho(t, \tau) \text{ for all } (t, \tau) \in \mathcal{J} \text{ and } x \in \mathcal{S}_A.
$$

(d) Eq. (1) has linear asymptotic equilibrium relative to $P$ if $\lim_{t \to \infty} P^{-1}(t)x(t)$ exists and is nonzero for every nontrivial $x \in \mathcal{S}_A$.

Lemma 1 If $x \in \mathcal{C}^n_1(\mathcal{J})$ and $u = P^{-1}x$, then $x' = Ax$, $t \in \mathcal{J}$, if and only if
$$
u' = P^{-1}(AP - P')u, \quad t \in \mathcal{J},
$$
or, equivalently, if and only if $x = PU\xi$ where $U$ is a fundamental matrix for (3) and $\xi \in \mathcal{C}$.

**Proof.** Since $x = Pu$, $x' = Pu' + P'u$ and $Ax = APu$, so $x' = Ax$ if and only if $Pu' + P'u = APu$, which is equivalent to (3).

To illustrate the problem that we study here, we cite a theorem attributed by Wintner [8] to Bôcher, which says that (1) has linear asymptotic equilibrium if
$$
\int^{\infty}_{-\infty} \|A(t)\| \, dt < \infty.
$$
This theorem does not apply to (1) if $\int^{\infty}_{-\infty} \|A(t)\| \, dt = \infty$, but, by Lemma 1 it does imply that (1) has linear asymptotic equilibrium relative to $P$ if
$$
\int^{\infty}_{-\infty} \|P^{-1}(AP - P')\| \, dt < \infty.
$$

Adapting terminology commonly used in computational linear algebra, we will in this case refer to the transformation $u = P^{-1}x$ as asymptotic preconditioning, and we say that $P$ preconditions (1) for asymptotic equilibrium. More generally, if $\mathcal{P}$ is a given property of linear differential systems (for example, one of the properties mentioned earlier), we say that $P$ preconditions (1) for property $\mathcal{P}$ if (3) has property $\mathcal{P}$ or, equivalently, if (1) has property $\mathcal{P}$ relative to $P$.

This paper is strongly influenced by Conti’s work [2, 3, 4] on $t_\infty$-similarity of systems of differential equations and our extensions [5, 6] of his results. However, we believe that our reformulation of these results in the context of asymptotic preconditioning is new and useful. We offer the paper not as a breakthrough in the asymptotic theory of linear differential systems, but as an expository approach to what we believe is a new application of standard results on this subject.

## 2 Preliminary considerations

The proof of most of the following lemma can be pieced together from applying various results in our references to the system (3); however, in keeping with our expository goal, we present a self-contained proof here.
Lemma 2 Let $U$ be a fundamental matrix for (3). Then:

(a) Eq. (1) is stable relative to $P$ if and only if $U$ is bounded on $\mathcal{J}$.
(b) Eq. (1) is $\rho$-stable relative to $P$ if and only if there is a constant $M$ such that

$$
\|U(t)U^{-1}(\tau)\| \leq M/\rho(t, \tau), \quad (t, \tau) \in \mathcal{J}.
$$

(c) Eq. (1) is strictly stable relative to $P$ if and only if $\|U\|$ and $\|U^{-1}\|$ are bounded on $\mathcal{J}$ or, equivalently, if and only if there is a constant $M$ such that

$$
\|U(t)U^{-1}(\tau)\| \leq M, \quad t, \tau \in \mathcal{J}.
$$

(d) Eq. (1) has linear asymptotic equilibrium relative to $P$ if and only if $\lim_{t \to \infty} U(t)$ exists and is invertible.

PROOF. From Lemma 1, it suffices to to show that the assumptions (a)–(d) are respectively equivalent to stability, $\rho$-stability, strict stability, and linear asymptotic equilibrium of (3). Since every solution of (3) can be written as $u(t) = U(t)\xi$ with $\xi \in \mathbb{C}^n$, (d) is obvious. For the rest of the proof, let $\mathcal{U}$ denote the set of all solutions of (3). Then $u \in \mathcal{U}$ if and only if

$$
u(t) = U(t)U^{-1}(\tau)u(\tau) \quad \text{for all } t, \tau \in \mathcal{J}.\tag{6}
$$

If $\tau$ is arbitrary but fixed and $K_\tau = \|U^{-1}(\tau)\|$, then (6) implies that

$$
\|u(t)\| \leq K_\tau \|U(t)\|\|u(\tau)\| \quad \text{for all } t, \tau \in \mathcal{J} \text{ and } u \in \mathcal{U}.
$$

This implies sufficiency for (a). Also from (6),

$$
\|u(t)\| \leq \|U(t)U^{-1}(\tau)\|\|u(\tau)\| \quad \text{for all } t, \tau \in \mathcal{J} \text{ and } u \in \mathcal{U}.
$$

Therefore (4) implies that

$$
\|u(t)\| \leq \|U(t)\|\|u(\tau)\|/\rho(t, \tau) \quad \text{for all } (t, \tau) \in \mathcal{J} \text{ and } u \in \mathcal{U},
$$

which implies sufficiency for (b). Moreover, (5) implies that

$$
\|u(t)\| \leq M\|u(\tau)\| \quad \text{for all } t, \tau \in \mathcal{J} \text{ and } u \in \mathcal{U}
$$

which implies sufficiency for (c).

We use contrapositive arguments to establish necessity in (a), (b), and (c). In all three cases let $M$ be an arbitrary positive constant. For (a), if $U$ is unbounded and $\tau$ is fixed in $\mathcal{J}$, then $U(t)U^{-1}(\tau)$ is also unbounded as a function of $t$ (since $U(t) = U(t)U^{-1}(\tau)U(\tau)$). Therefore there is a $t_0 \in \mathcal{J}$ and a $\xi \in \mathbb{C}^n$ such that $\|U(t_0)U^{-1}(\tau)\xi\| > M\|\xi\|$. Hence, if $u_0(t) = U(t)U^{-1}(\tau)\xi$ then $u_0 \in \mathcal{U}$ and

$$
\|u(t_0)\| = \|U(t_0)U^{-1}(\tau)\xi\| > M\|\xi\| = M\|u(\tau)\|,
$$

hence (3) is not stable.

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For (b), if there is a \((t_0, \tau_0) \in J\) such that

\[ \|U(t_0, \tau_0)\| > M/\rho(t_0, \tau_0). \]

then

\[ \|U(t_0, \tau_0)\| > M \|\xi\|/\rho(t_0, \tau_0) \]

for some \(\xi \in \mathbb{C}^n\). If \(u(t) = U(t)U^{-1}(t_0)\xi\) then

\[ \|u(t_0)\| = \|U(t_0)U^{-1}(t_0)\| > M \|\xi\|/\rho(t_0, \tau_0) = M \|u(t_0)\|/\rho((t_0, \tau_0)). \]

so (3) is not \(\rho\)-stable. A similar argument shows that if (3) is strictly stable, then (5) holds for some \(M\).

Eq. (5) obviously holds for some \(M\) if \(U\) and \(U^{-1}\) are bounded on \(J\). It remains to show that (5) implies that \(U\) and \(U^{-1}\) are bounded on \(J\). If \(\tau \in J\) is fixed and \(t\) is arbitrary, then (5) implies that

\[ \|U(t)\| = \|U(t)U^{-1}(\tau)U(\tau)\| \leq \|U(t)U^{-1}(\tau)\|\|U(\tau)\| \leq M \|U(\tau)\|. \]

so \(U\) is bounded on \(J\). To complete the proof, we must show that if \(U^{-1}\) is unbounded then (5) is false for every \(M\). Let \(t_0 \in J\) be fixed and let \(\sigma = \min \{\|U(t_0)\eta\| : \|\eta\| = 1\}\), which is positive, since \(U(t_0)\) is invertible. If \(U^{-1}\) is unbounded on \(J\) there is a \(\tau \in J\) and \(\xi \in \mathbb{C}^n\) such that \(\|\xi\| = 1\) and \(\|U^{-1}(\tau)\| > M/\sigma\). Then

\[ \|U(t_0)U^{-1}(\tau)\| > \sigma \|U^{-1}(\tau)\| > M \|\xi\|. \]

so \(\|U(t_0)U^{-1}(\tau)\| > M\).

Lemma 3 Suppose that \(R, Q \in \mathcal{R}\) and let

\[ F = R' - Q'Q^{-1}R + RP^{-1}(P' - AP). \]  

Then \(X = PU \in \mathbb{C}^{n \times n}(J)\) satisfies \(X' = AX, t \in J\), if and only if

\[ (Q^{-1}RU)' = Q^{-1}FU, \quad t \in J. \]  

Proof. From (7),

\[ (Q^{-1}RU)' = Q^{-1}(R'U - Q'Q^{-1}RU + RU') \]

\[ = Q^{-1}FU + Q^{-1}R(U' - P^{-1}(P' - AP)U). \]

so Lemma 1 implies the conclusion.

This lemma provides an infinite family of linear differential systems, all with the same solutions; namely, \(u\) is a solution of (3) (and consequently \(x = Pu\) is a solution of (1)) if and only if \(u\) is a solution of every system of the form (8). Therefore, if (8) has a given property \(\mathcal{P}\) for some suitably chosen \(R\) and \(Q\) in \(\mathcal{R}\), then \(P\) preconditions (1) for \(\mathcal{P}\).
3 Main results

Theorem 1  Suppose that there are $R, Q \in \mathbb{R}$ such that $R$ and $R^{-1}$ are bounded on $\mathcal{J}$ and
\[
\int_0^\infty \|F(s)\| \, ds < \infty.
\]  

Then:

(a) $P$ preconditions Eq. (1) for $p$-stability if there is a constant $M$ such that
\[
\|Q(t)Q^{-1}(\tau)\| \leq M/\rho(t, \tau), \quad a \leq \tau \leq t.
\]  

(b) $P$ preconditions Eq. (1) for strict stability if $Q$ and $Q^{-1}$ are bounded on $\mathcal{J}$.

PROOF. Integrating (8) yields
\[
U(t) = R^{-1}(t)Q(t) \left( Q^{-1}(\tau)R(\tau)U(\tau) + \int_\tau^t Q^{-1}(s)F(s)U(s) \, ds \right),
\]  

\[ t, \tau \in \mathcal{J}. \]  

Therefore
\[
U(t)U^{-1}(\tau) = R^{-1}(t)Q(t) \left( Q^{-1}(\tau)R(\tau) + \int_\tau^t Q^{-1}(s)F(s)U(s)U^{-1}(\tau) \, ds \right).
\]  

To prove (a), let
\[
g(t, s) = \|Q(t)Q^{-1}(s)\|\rho(t, s) \quad \text{and} \quad h(s, \tau) = \|U(s)U^{-1}(\tau)\|\rho(s, \tau).
\]  

By Lemma 2(b), we must show that $h(t, \tau)$ is bounded for $(t, \tau) \in \mathcal{J}$. If $\tau \leq s \leq t$ then (2) implies that
\[
\rho(t, \tau)\|Q(t)Q^{-1}(s)F(s)U(s)U^{-1}(\tau)\| \leq g(t, s)\|F(s)\|h(s, \tau).
\]  

Since $R$ and $R^{-1}$ are bounded, multiplying both sides of (12) by $\rho(t, \tau)$ yields the inequality
\[
h(t, \tau) \leq c_1 g(t, \tau) + c_2 \int_\tau^t g(t, s)\|F(s)\|h(s, \tau) \, ds, \quad a \leq \tau \leq t,
\]  

for suitable constants $c_1$ and $c_2$. Now (10) and (13) imply that
\[
h(t, \tau) \leq M \left[ c_1 + c_2 \int_\tau^t \|F(s)\|h(s, \tau) \, ds \right], \quad a \leq \tau \leq t.
\]  

Therefore
\[
\frac{c_2h(t, \tau)\|F(t)\|}{c_1 + c_2 \int_\tau^t \|F(s)\|h(s, \tau) \, ds} \leq M c_2 \|F(\tau)\| \quad a \leq \tau \leq r.
\]  

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Integrating this with respect to \( t \) yields

\[
\log\left(c_1 + c_2 \int_\tau^t \|F(s)\| h(s, \tau) \, ds\right) - \log c_1 \leq M c_2 \int_\tau^t \|F(s)\| \, ds.
\] (16)

This and (14) imply that

\[
\sup_{(t, \tau) \in J} \{\|h(t, \tau)\| \mid (t, \tau) \in J\} \leq M c_1 \exp\left(M \int_a^\infty \|F(s)\| \, ds\right) < \infty,
\] (17)

from (9). This completes the proof of (a).

To prove (b), replace (13) by

\[
g(t, s) = \|Q(t)Q^{-1}(s)\| \quad \text{and} \quad h(s, \tau) = \|U(s)U^{-1}(\tau)\|.
\]

Now we must show that \( h(t, \tau) \) is bounded for all \( t, \tau \in J \). If (1) is strictly stable then there is a constant \( M \) such that \( g(t, s) \leq M \) for \( s, t \geq a \). This, (12) and the boundedness of \( R \) and \( R^{-1} \) imply that

\[
h(t, \tau) \leq M \left[ c_1 + c_2 \int_\tau^t \|F(s)\| h(s, \tau) \, ds\right], \quad t, \tau \geq a,
\]

for suitable positive constants \( c_1 \) and \( c_2 \). Now the argument used in the proof of (a) again implies (17). If \( a \leq t \leq \tau \) then (14)–(17) all hold with \( t \) and \( \tau \) interchanged, which completes the proof of (b).

**Remark 1** The use of logarithmic integration that produced (16) was motivated by the proof of Gronwall’s inequality [1, p. 35], a standard tool for studying the asymptotic behavior of solutions of differential equations.

**Theorem 2** In addition to the assumptions of Theorem 1(b), suppose that

\[
\lim_{t \to \infty} R^{-1}(t)Q(t) = J
\]

is invertible. Then \( P \) preconditions (1) for linear asymptotic equilibrium.

**Proof.** From (11) and (18), \( \lim_{t \to \infty} U(t) = V \), where

\[
V = J \left(Q^{-1}(\tau)R(\tau)U(\tau) + \int_\tau^\infty Q^{-1}(s)F(s)U(s) \, ds\right)
\]

and the integral converges because of (9), the boundedness of \( Q^{-1} \) (assumed) and \( U \) (from Theorem 1(b)). Now we must show that \( V \) is inverible. Since Theorem 1(b) implies that (1) is strictly stable relative to \( P \), there is a constant \( K \) such that \( \|U^{-1}\| < K \), \( t \in J \). If \( \xi \in \mathbb{C}^n \) then

\[
\|\xi\| = \|U^{-1}(t)U(t)\| \leq \|U^{-1}(t)\| \|U(t)\| \leq K \|U(t)\| \|\xi\|, \quad t \leq a,
\]

so

\[
\|\xi\| \leq K \lim_{t \to \infty} \|U(t)\| = K \|V\|.
\]

Therefore \( V\xi = 0 \) if and only if \( \xi = 0 \), so \( V \) is invertible.
Theorem 3 If there are $Q$ and $R$ in $\mathcal{R}$ such that $R^{-1}Q$ is bounded and
\[
\int_{-}\infty^{\infty} \|Q^{-1}(s)F(s)\|\ ds < \infty,
\]
then $P$ preconditions (1) for stability; moreover, if (18) holds then $P$ preconditions (1) for linear asymptotic equilibrium.

PROOF. Our assumptions imply that if $0 < \rho < 1$ then there is a $\tau \geq a$ such that
\[
\|R^{-1}(t)Q(t)\|\int_{\tau}^{\infty} \|Q^{-1}(s)F(s)\|\ ds \leq \rho, \quad t \geq \tau.
\]
Let $\mathcal{B}$ be the Banach space of bounded continuous $n \times n$ vector functions on $J = [\tau, \infty)$ with norm $\|U\|_{\mathcal{B}} = \sup_{t \in J} \|U(t)\|$, and define $T : \mathcal{B} \to \mathcal{B}$ by
\[
(TU)(t) = R^{-1}(t)Q(t) \left( C - \int_{t}^{\infty} Q^{-1}(s)F(s)U(s)\ ds \right)
\]
where $C \in \mathbb{C}^{n \times n}$ is invertible. If $U_1, U_2 \in \mathcal{B}$ then
\[
(TU_1)(t) - (TU_2)(t) \leq \|R^{-1}(t)Q(t)\|\int_{t}^{\infty} \|Q^{-1}(s)F(s)\|\|U_1(s) - U_2(s)\|\ ds,
\]
so $\|TU_1 - TU_2\|_{\mathcal{B}} \leq \rho\|U_1 - U_2\|_{\mathcal{B}}$. Therefore, by the contraction mapping principal [7, p. 545], there is a $U \in \mathcal{B}$ such that
\[
U(t) = R^{-1}(t)Q(t) \left( C - \int_{t}^{\infty} Q^{-1}(s)F(s)U(s)\ ds \right).
\]
Since $U$ satisfies (8), Theorem 1 implies that $X = PU$ satisfies (1). Therefore $P$ preconditions (1) for stability. Finally, if (18) holds then $\lim_{t \to \infty} U(t) = JC$ is invertible, so $P$ preconditions (1) for linear asymptotic equilibrium. \qed

Remark 2 Strictly speaking, our proof of Theorem 3 defines $U$ only on the interval $[\tau, \infty)$, which has the appearance of leaving a gap if $\tau > a$. However, in this case we appeal to the elementary theory of linear differential systems, which guarantees that $U$ can extended uniquely over $J$ as an invertible solution of $U' = P^{-1}(AP - P')U$.

From (7),
\[
Q^{-1}F = (Q^{-1}R)' + (Q^{-1}R)P^{-1}(P' - AP).
\]
Therefore we can reformulate Theorem 3 as follows.

Theorem 4 If there is a $T \in \mathcal{R}$ such that $T^{-1}$ is bounded and
\[
\int_{-}\infty^{\infty} \|T' + TP^{-1}(P' - AP)\|\ ds < \infty,
\]
then $P$ preconditions (1) for stability; moreover, if $\lim_{t \to \infty} T(t)$ exists and is invertible then $P$ preconditions (1) for linear asymptotic equilibrium.

The assertion concerning linear asymptotic equilibrium can also be proved by applying a theorem of Conti [3] to (3).
References


