Characterization and properties of (R, S_σ)-commutative matrices

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Abstract

Let $R = P\,\mathrm{diag}(\gamma_0 I_{m_0}, \gamma_1 I_{m_1}, \dots, \gamma_{k-1} I_{m_{k-1}})P^{-1} \in \mathbb{C}^{m\times m}$ and $S_\sigma = Q\,\mathrm{diag}(\gamma_{\sigma(0)} I_{n_0}, \gamma_{\sigma(1)} I_{n_1}, \dots, \gamma_{\sigma(k-1)} I_{n_{k-1}})Q^{-1} \in \mathbb{C}^{n\times n}$, where $m_0 + m_1 + \cdots + m_{k-1} = m$, $n_0 + n_1 + \cdots + n_{k-1} = n$, $\gamma_0, \gamma_1, \dots, \gamma_{k-1}$ are distinct complex numbers, and $\sigma : \mathbb{Z}_k \to \mathbb{Z}_k = \{0, 1, \dots, k-1\}$. We say that $A \in \mathbb{C}^{m\times n}$ is (R, S_σ)-commutative if $RA = AS_\sigma$. We characterize the class of (R, S_σ)-commutative matrrices and extend results obtained previously for the case where $\gamma_\ell = e^{2\pi i\ell/k}$ and $\sigma(\ell) = \alpha\ell + \mu \pmod k$, $0 \le \ell \le k-1$, with $\alpha, \mu \in \mathbb{Z}_k$. Our results are independent of $\gamma_0, \gamma_1, \dots, \gamma_{k-1}$, so long as they are distinct; i.e., if $RA = AS_\sigma$ for some choice of $\gamma_0, \gamma_1, \dots, \gamma_{k-1}$ (all distinct), then $RA = AS_\sigma$ for arbitrary of $\gamma_0, \gamma_1, \dots, \gamma_{k-1}$.

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1 Introduction

A matrix $A = [a_{rs}]_{r,s=0}^{n-1} \in \mathbb{C}^{n\times n}$ is said to be centrosymmetric if

$$a_{n-r-1,n-s-1} = a_{rs}, \quad 0 \le r, s \le n-1,$$

or centro-skewsymmetric if

$$a_{n-r-1,n-s-1} = -a_{rs}, \quad 0 \le r, s \le n-1.$$

The study of such matrices is facilitated by the observation that A is centrosymmetric (centro-skewsymmetric) if and only if $JA = AJ$ ($JA = -AJ$), where J is the flip matrix, with ones on the secondary diagonal and zeroes elsewhere. Several authors [2, 3, 4, 5, 8, 10, 13, 25] used this observation to show that centrosymmetric and centro-skewsymmetric matrices can be written as $A = PCP^{-1}$, where P diagonalizes J and C has a useful block structure. We will discuss this further in Example 3.

Following this idea, other authors [6, 11, 12, 14, 24] considered matrices satisfying $RA = AR$ or $RA = -AR$, where R is a nontrivial involution; i.e., $R = R^{-1} \ne \pm I$. We continued this line of investigaton in [15, 16, 17, 19], and extended it in [18, 20], defining $A \in \mathbb{C}^{m\times n}$ to be (R,S)-symmetric ((R,S)-skew symmetric) if $RA = AS$ ($RA = -AS$), where $R \in \mathbb{C}^{m\times m}$ and $S \in \mathbb{C}^{n\times n}$ are nontrivial involutions. We showed that a matrix A with either of these properties can be written as $A = PCQ^{-1}$, where P and Q diagonalize R and S respectively and C has a useful block form.

Chen [7] and Fasino [9] studied matrices $A \in \mathbb{C}^{n\times n}$ such that $RAR^* = \zeta^\mu A$, where R is a unitary matrix that satisfies $R^k = I$ for some $k \le n$ and $\zeta = e^{2\pi i/k}$. In [21] we studied matrices $A \in \mathbb{C}^{m\times n}$ such that $RA = \zeta^\mu AS$, where

$$R = P \operatorname{diag}\left(I_{m_0}, \zeta I_{m_1}, \dots, \zeta^{k-1} I_{m_{k-1}}\right) P^{-1}, \tag{1}$$

$$S = Q \operatorname{diag}\left(I_{n_0}, \zeta I_{n_1}, \dots, \zeta^{k-1} I_{n_{k-1}}\right) Q^{-1}, \tag{2}$$

$$m_0 + m_1 + \cdots + m_{k-1} = m, \quad n_0 + n_1 + \cdots + n_{k-1} = n, \tag{3}$$

and

$$\alpha, \mu \in \mathbb{Z}_k = \{0, 1, \dots, k-1\}.$$

Finally, motivated by a problem concerning unilevel block circulants [22], in [23] we considered matrices $A \in \mathbb{C}^{m\times n}$ such that $RA = \zeta^\mu AS^\alpha$, with $\alpha, \mu \in \mathbb{Z}_k$. We called such matrices (R,S,α,μ)-symmetric, and showed that A has this property if and only if

$$A = \sum_{\ell=0}^{k-1} P_{\alpha\ell+\mu (\bmod k)} F_\ell \widehat{Q}_\ell \quad \text{with} \quad F_\ell \in \mathbb{C}^{\alpha\ell+\mu (\bmod k)\times n_\ell}, \quad 0 \le \ell \le k-1, \tag{4}$$

which has useful computational and theoretical applications. ($P_0, \dots, P_{k-1}$ and $\widehat{Q}_0, \dots, \widehat{Q}_{k-1}$ are defined in Section 2, specifically, (7)–(10).) The class of (R,S,α,μ)-symmetric matrices includes, for example, centrosymmetric, skew-centrosymmetric,

R-symmetric, R-skew symmetric, (R, S)-symmetric, and (R, S)-skew symmetric matrices, and block circulants $[A_{s-\alpha r}]_{r,s=0}$.

Having said this, we now propose that all the papers in our bibliography – including our own – are based on an unnecessarily restrictive assumption; namely, that the spectra of the matrices R and S that are used to define the symmetries consist of a set (usually the complete set) of k-th roots of unity for some $k \geq 2$. In this paper we point out that this assumption is irrelevant and present an alternative approach that eliminates this requirement and exposes a wider class of generalized symmetries if $k > 2$. We extend our results in [21] and [23] to this larger class of matrices.

2 Preliminary considerations

Throughout the rest of this paper,

$$R = P\,\mathrm{diag}(\gamma_0 I_{m_0}, \gamma_1 I_{m_1}, \ldots, \gamma_{k-1} I_{m_{k-1}})P^{-1} \in \mathbb{C}^{m\times m} \tag{5}$$

and

$$S = Q\,\mathrm{diag}(\gamma_0 I_{n_0}, \gamma_1 I_{n_1}, \ldots, \gamma_{k-1} I_{n_{k-1}})Q^{-1} \in \mathbb{C}^{n\times n}, \tag{6}$$

where $\gamma_0, \gamma_1, \ldots, \gamma_{k-1}$ are distinct complex numbers, except when there is an explicit statement to the contrary. We define

$$R_\sigma = P\,\mathrm{diag}\left(\gamma_{\sigma(0)} I_{m_0}, \gamma_{\sigma(1)} I_{m_1}, \ldots, \gamma_{\sigma(k-1)} I_{m_{k-1}}\right) P^{-1}$$

and

$$S_\sigma = Q\,\mathrm{diag}(\gamma_{\sigma(0)} I_{n_0}, \gamma_{\sigma(1)} I_{n_1}, \ldots, \gamma_{\sigma(k-1)} I_{n_{k-1}})Q^{-1},$$

where $\sigma : \mathbb{Z}_k \to \mathbb{Z}_k$.

We can partition

$$P = \left[\begin{array}{cccc} P_0 & P_1 & \cdots & P_{k-1} \end{array}\right], \quad Q = \left[\begin{array}{cccc} Q_0 & Q_1 & \cdots & Q_{k-1} \end{array}\right], \tag{7}$$

$$P^{-1} = \left[\begin{array}{c} \widehat{P}_0 \\ \widehat{P}_1 \\ \vdots \\ \widehat{P}_{k-1}, \end{array}\right] \quad \text{and} \quad Q^{-1} = \left[\begin{array}{c} \widehat{Q}_0 \\ \widehat{Q}_1 \\ \vdots \\ \widehat{Q}_{k-1}, \end{array}\right], \tag{8}$$

where

$$P_r \in \mathbb{C}^{m\times m_r}, \quad \widehat{P}_r \in \mathbb{C}^{m_r\times m}, \quad \widehat{P}_r P_s = \delta_{rs} I_{m_r}, \quad 0 \leq r, s \leq k-1, \tag{9}$$

$$Q_r \in \mathbb{C}^{n\times n_r}, \quad \widehat{Q}_r \in \mathbb{C}^{n_r\times n}, \quad \text{and} \quad \widehat{Q}_r Q_s = \delta_{rs} I_{n_r}, \quad 0 \leq r, s \leq k-1. \tag{10}$$

We can now write

$$R = \sum_{\ell=0}^{k-1} \gamma_\ell P_\ell \widehat{P}_\ell, \quad R_\sigma = \sum_{\ell=0}^{k-1} \gamma_{\sigma(\ell)} P_\ell \widehat{P}_\ell, \tag{11}$$

$$S = \sum_{\ell=0}^{k-1} \gamma_\ell Q_\ell \widehat{Q}_\ell, \quad \text{and} \quad S_\sigma = \sum_{\ell=0}^{k-1} \gamma_{\sigma(\ell)} Q_\ell \widehat{Q}_\ell. \tag{12}$$

Definition 1 In general, if $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$, and $A \in \mathbb{C}^{m\times n}$, we say that A is (U,V)-commutative if $UA = AV$. In particular, we say that $A \in \mathbb{C}^{m\times n}$ is (R,S_σ)-commutative if $RA = AS_\sigma$. If σ is the identity (i.e., $RA = AS$), we say that A is (R,S)-commutative. If A, $R \in \mathbb{C}^{n\times n}$ and $RA = AR$, we say – as usual – that A commutes with R.

3 Necessary and sufficient conditions for (R, S_σ)-commutativity

Theorem 1 *$A \in \mathbb{C}^{m\times n}$ is (R,S_σ)-commutative if and only if*

$$A = P\left([C_{rs}]_{r,s=0}^{k-1}\right)Q^{-1}, \quad \textit{where} \quad C_{rs} \in \mathbb{C}^{m_r\times n_s} \tag{13}$$

and

$$C_{rs} = 0 \quad \textit{if} \quad r \neq \sigma(s), \quad 0 \le r,s \le k-1. \tag{14}$$

PROOF. Any $A \in \mathbb{C}^{m\times n}$ can be written as in (13) with $C = P^{-1}AQ$ partitioned as indicated. If

$$D = \operatorname{diag}\left(\gamma_0 I_{m_0}, \gamma_1 I_{m_1}, \dots, \gamma_{k-1} I_{m_{k-1}}\right)$$

and

$$D_\sigma = \operatorname{diag}\left(\gamma_{\sigma(0)} I_{n_0}, \gamma_{\sigma(1)} I_{n_1}, \dots, \gamma_{\sigma(k-1)} I_{n_{k-1}}\right),$$

then

$$RA = (PDP^{-1})(PCQ^{-1}) = PDCQ^{-1} = P\left([\gamma_r C_{rs}]_{r,s=0}^{k-1}\right)Q^{-1}$$

and

$$AS_\sigma = (PCQ^{-1})(QD_\sigma Q^{-1}) = PCD_\sigma Q^{-1} = P\left([\gamma_{\sigma(s)} C_{rs}]_{r,s=0}^{k-1}\right)Q^{-1}.$$

Therefore $RA = AS_\sigma$ if and only if $(\gamma_r - \gamma_{\sigma(s)})C_{rs} = 0$, $0 \le r,s \le k-1$, which is equivalent to (14), since $\gamma_0, \gamma_1, \dots, \gamma_{k-1}$ are distinct. □

The following theorem is a convenient reformulation of Theorem 1.

Theorem 2 *$A \in \mathbb{C}^{m\times n}$ is (R,S_σ)-commutative if and only if*

$$A = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F_\ell \widehat{Q}_\ell \quad \textit{with} \quad F_\ell \in \mathbb{C}^{m_{\sigma(\ell)}\times n_\ell}, \quad 0 \le \ell \le k-1, \tag{15}$$

in which case

$$F_\ell = \widehat{P}_{\sigma(\ell)} A Q_\ell, \quad 0 \le \ell \le k-1, \tag{16}$$

and

$$RA = AS_\sigma = \sum_{\ell=0}^{k-1} \gamma_{\sigma(\ell)} P_{\sigma(\ell)} F_\ell \widehat{Q}_\ell \tag{17}$$

for arbitrary $\gamma_0, \gamma_1, \dots, \gamma_{k-1}$.

PROOF. From (13), an arbitrary $A \in \mathbb{C}^{m\times n}$ can be written as

$$A = \sum_{s=0}^{k-1}\sum_{r=0}^{k-1} P_r C_{rs} \widehat{Q}_\ell. \tag{18}$$

From Theorem 1, A is (R, S_σ)-commutative if and and only if $C_{rs} = 0$ if $r \neq \sigma(s)$, in which case (18) reduces to (15) with $F_\ell = C_{\sigma(\ell),\ell} \in \mathbb{C}^{m_{\sigma(\ell)}\times n_\ell}$. From (10) and (15), $AQ_\ell = P_{\sigma(\ell)}F_\ell$, $0 \le \ell \le k-1$, so (9) with $r = \sigma(\ell)$ implies (16). Eqns. (9)–(12) and (15) imply (17). □

Example 1 If σ is the permutation

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 0 & 2 & 5 \end{pmatrix} = (0,1,3)(2,4)(5),$$

then (15) becomes

$$A = P_1F_0\widehat{Q}_0 + P_3F_1\widehat{Q}_1 + P_4F_2\widehat{Q}_2 + P_0F_3\widehat{Q}_3 + P_2F_4\widehat{Q}_4 + P_5F_5\widehat{Q}_5,$$

with

$$\begin{aligned} &F_0 \in \mathbb{C}^{m_1\times n_0}, \quad F_1 \in \mathbb{C}^{m_3\times n_1}, \quad F_2 \in \mathbb{C}^{m_4\times n_2}, \\ &F_3 \in \mathbb{C}^{m_0\times n_3}, \quad F_4 \in \mathbb{C}^{m_2\times n_4}, \quad F_5 \in \mathbb{C}^{m_5\times n_5}, \end{aligned}$$

and

$$RA = AS_\sigma = \gamma_1P_1F_0\widehat{Q}_0+\gamma_3P_3F_1\widehat{Q}_1+\gamma_4P_4F_2\widehat{Q}_2+\gamma_0P_0F_3\widehat{Q}_3+\gamma_2P_2F_4\widehat{Q}_4+\gamma_5P_5F_5\widehat{Q}_5$$

for arbitrary $\gamma_0, \dots, \gamma_5$.

Example 2 If

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 0 & 1 & 2 & 0 \end{pmatrix}$$

(which is not a permutation), then (15) becomes

$$A = P_2F_0\widehat{Q}_0 + P_1F_1\widehat{Q}_1 + P_0F_2\widehat{Q}_2 + P_1F_3\widehat{Q}_3 + P_2F_4\widehat{Q}_4 + P_0F_5\widehat{Q}_5,$$

with

$$\begin{aligned} &F_0 \in \mathbb{C}^{m_2\times n_0}, \quad F_1 \in \mathbb{C}^{m_1\times n_1}, \quad F_2 \in \mathbb{C}^{m_0\times n_2}, \\ &F_3 \in \mathbb{C}^{m_1\times n_3}, \quad F_4 \in \mathbb{C}^{m_2\times m_4}, \quad F_5 \in \mathbb{C}^{m_0\times n_5}, \end{aligned}$$

and

$$RA = AS_\sigma = \gamma_2P_2F_0\widehat{Q}_0+\gamma_1P_1F_1\widehat{Q}_1+\gamma_0P_0F_2\widehat{Q}_2+\gamma_1P_1F_3\widehat{Q}_3+\gamma_2P_2F_4\widehat{Q}_4+\gamma_0P_0F_5\widehat{Q}_5$$

for arbitrary $\gamma_0, \dots, \gamma_5$.

Example 3 All results obtained by assuming that R and S are involutions (and therefore have eigenvalues 1 and -1) can just as well be obtained by assuming only that R and S have the same two distinct eigenvalues, with possibly different multiplicities. The original idea in this area of research has its origins in the observation that A is centrosymmetric (skew-centrosymmetric) if and only if $AJ = JA$ ($AJ = -JA$). Since $J^2 = I$, these conditions can just as well be written as $JAJ = A$ ($JAJ = -A$); however, this and the invertibility of J are irrelevant. To illustrate this, suppose $n = 2r$, in which case

$$J = \begin{bmatrix} P_0 & P_1 \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & -I_r \end{bmatrix} \begin{bmatrix} P_0^T \\ P_1^T \end{bmatrix}$$

(i.e., $\widehat{P}_0 = P_0^T$ and $\widehat{P}_1 = P_1^T$), where

$$P_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} I_r \\ J_r \end{bmatrix} \quad \text{and} \quad P_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} I_r \\ -J_r \end{bmatrix}.$$

Starting from this, it can be shown $AJ = JA$ (or, equivalently, A is centrosymmetric) if and only if

$$A = \begin{bmatrix} P_0 & P_1 \end{bmatrix} \begin{bmatrix} B_0 & 0 \\ 0 & B_1 \end{bmatrix} \begin{bmatrix} P_0^T \\ P_1^T \end{bmatrix} = P_0 B_0 P_0^T + P_1 B_1 P_1^T \qquad (19)$$

with $B_0, B_1 \in \mathbb{C}^{r \times r}$. However, Theorem 2 implies that A has the form (19) if $RA = AR$ for some R of the form

$$R = \begin{bmatrix} P_0 & P_1 \end{bmatrix} \begin{bmatrix} \gamma_0 I_r & 0 \\ 0 & \gamma_1 I_r \end{bmatrix} \begin{bmatrix} P_0^T \\ P_1^T \end{bmatrix}$$

with $\gamma_0 \neq \gamma_1$, in which case

$$RA = AR = \gamma_0 P_0 B_0 P_0^T + \gamma_1 P_1 B_1 P_1^T.$$

for arbitrary γ_0 and γ_1.

According to the classical theorem, $AJ = -JA$ (or, equivalently, A is skew-centrosymmetric) if and only if

$$A = \begin{bmatrix} P_0 & P_1 \end{bmatrix} \begin{bmatrix} 0 & C_1 \\ C_0 & 0 \end{bmatrix} \begin{bmatrix} P_0^T \\ P_1^T \end{bmatrix} = P_1 C_0 \widehat{P}_0 + P_0 C_1 \widehat{P}_1 \qquad (20)$$

with $C_0, C_1 \in \mathbb{C}^{r \times r}$. Now let $\sigma(0) = 1$ and $\sigma(1) = 0$, so

$$R_\sigma = \begin{bmatrix} P_0 & P_1 \end{bmatrix} \begin{bmatrix} \gamma_1 I_r & 0 \\ 0 & \gamma_0 I_r \end{bmatrix} \begin{bmatrix} P_0^T \\ P_1^T \end{bmatrix}.$$

Theorem 2 implies that A has the form (20) if and only if $RA = AR_\sigma$ for some γ_0 and γ_1 with $\gamma_0 \neq \gamma_1$, in which case

$$RA = AR_\sigma = \gamma_1 P_1 C_0 \widehat{P}_0 + \gamma_0 P_0 C_1 \widehat{P}_1$$

for all γ_0 and γ_1.

Example 4 Let $R = [\delta_{r,s-1(\mathrm{mod}\ k)}]_{r,s=0}^{k-1}$, which is the 1-circulant with first row

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

By the Ablow-Brenner theorem [1], $C \in \mathbb{C}^{k\times k}$ is an α-circulant $C = [c_{s-\alpha r(\mathrm{mod}\ k)}]_{r,s=0}^{k-1}$ if and only if $RC = CR^\alpha$. Since

$$R = P\ \mathrm{diag}(1, \zeta, \zeta^2, \dots, \zeta^{k-1})P^*$$

where

$$P = \begin{bmatrix} p_0 & p_1 & \cdots & p_{k-1} \end{bmatrix} \quad \text{with} \quad p_\ell = \frac{1}{\sqrt{k}}\begin{bmatrix} 1 \\ \zeta^\ell \\ \zeta^{2\ell} \\ \vdots \\ \zeta^{(k-1)\ell} \end{bmatrix}, \quad 0 \le \ell \le k-1,$$

and

$$R^\alpha = P\ \mathrm{diag}(1, \zeta^\alpha, \zeta^{2\alpha}, \dots, \zeta^{(k-1)\alpha})P^*,$$

the Ablow-Brenner theorem can be interpreted to mean that C is (R, R_σ)-commutative with $\sigma(\ell) = \alpha\ell \pmod k$, $0 \le \ell \le k-1$. Therefore Theorem 2 implies that

$$C = \sum_{\ell=0}^{k-1} p_{\alpha\ell(\mathrm{mod}\ k)} f_\ell p_\ell^*,$$

where $f_0, f_1, \dots, f_{k-1}$ are scalars. As a matter of fact, if

$$R = P\ \mathrm{diag}(\gamma_0, \gamma_1, \dots, \gamma_{k-1})P^*$$

with arbitrary $\gamma_0, \gamma_1, \dots, \gamma_{k-1}$, then

$$RC = CR_\sigma = \sum_{\ell=0}^{k-1} \gamma_{\alpha\ell(\mathrm{mod}\ k)} p_{\alpha\ell(\mathrm{mod}\ k)} f_\ell p_\ell^*.$$

Example 5 Let R and S be as in (1) and (2) and let $\sigma(\ell) = \alpha\ell + \mu \pmod k$, so

$$S_\sigma = Q\ \mathrm{diag}\left(\zeta^\mu I_{m_0}, \zeta^{\alpha+\mu} I_{m_1}, \dots, \zeta^{(k-1)\alpha+\mu} I_{m_{k-1}}\right) Q^{-1}.$$

Then the (R, S, α, μ)-symmetric matrix A in (4) is (R, S_σ)-commutative. More generally, if R and S are as in (5) and (6) and $\sigma(\ell) = \alpha\ell + \mu \pmod k$, then

$$RA = AS_\sigma = \sum_{\ell=0}^{k-1} \gamma_{\alpha\ell+\mu(\mathrm{mod}\ k)} P_{\alpha\ell+\mu(\mathrm{mod}\ k)} F_\ell \widehat{Q}_\ell$$

for arbitrary $\gamma_0, \gamma_1, \dots, \gamma_{k-1}$.

Renaming the variables in Theorem 2 yields the following theorem.

Theorem 3 *If $\rho : \mathbb{Z}_k \to \mathbb{Z}_k$, then $B \in \mathbb{C}^{n\times m}$ is (S, R_ρ)-commutative if and only if*

$$B = \sum_{\ell=0}^{k-1} Q_{\rho(\ell)} G_\ell \widehat{P}_\ell \quad \textit{with} \quad G_\ell \in \mathbb{C}^{n_{\rho(\ell)}\times m_\ell}, \quad 0 \le \ell \le k-1, \tag{21}$$

in which case

$$G_\ell = \widehat{Q}_{\rho(\ell)} B P_\ell, \quad 0 \le \ell \le k-1,$$

and

$$SB = BR_\rho = \sum_{\ell=0}^{k-1} \gamma_{\rho(\ell)} Q_{\rho(\ell)} G_\ell \widehat{P}_\ell$$

for arbitrary $\gamma_0, \gamma_1, \ldots, \gamma_{k-1}$.

4 General Results

Remark 1 If σ or ρ is a permutation of $\mathbb{Z}_k$, we can replace ℓ by $\sigma(\ell)$ or ℓ by $\rho(\ell)$ in a summation $\sum_{\ell=0}^{k-1}$, as in the proof of the following theorem, where "$\circ$" denotes composition; i.e., $\sigma \circ \rho(\ell) = \sigma(\rho(\ell))$ and $\rho \circ \sigma(\ell) = \rho(\sigma(\ell))$. Also,

$$\widehat{P}_{\sigma(r)} P_{\sigma(s)} = \delta_{rs} I_{m_{\sigma(r)}} \quad \text{and} \quad \widehat{Q}_{\rho(r)} Q_{\rho(s)} = \delta_{rs} I_{n_{\sigma(r)}}, \quad 0 \le r, s \le k-1, \tag{22}$$

if and only if σ and ρ are permutations. We will use this frequently without specifically invoking it.

Theorem 4 *Suppose $A \in \mathbb{C}^{m\times n}$ is (R, S_σ)-commutative and $B \in \mathbb{C}^{n\times m}$ is (S, R_ρ)-commutative. Then*: (a) *AB is $(R, R_{\sigma\circ\rho})$-commutative if ρ is a permutation and* (b) *BA is $(S, S_{\rho\circ\sigma})$-commutative if σ is a permutation.*

PROOF. From Theorems 2 and 3, our assumptions imply that A is as in (15) and B is as in (21). If ρ is a permutation then replacing ℓ by $\rho(\ell)$ in (15) yields

$$A = \sum_{\ell=0}^{k-1} P_{\sigma(\rho(\ell))} F_{\rho(\ell)} \widehat{Q}_{\rho(\ell)}.$$

From this, (21), and (22),

$$AB = \sum_{\ell=0}^{k-1} P_{\sigma(\rho(\ell))} F_{\rho(\ell)} G_\ell \widehat{P}_\ell,$$

so (9) and (11) imply that

$$R(AB) = (AB)R_{\sigma\circ\rho} = \sum_{\ell=0}^{k-1} \gamma_{\sigma(\rho(\ell))} P_{\sigma(\rho(\ell))} F_{\rho(\ell)} G_\ell \widehat{P}_\ell,$$

which proves (a).

If σ is a permutation, replacing ℓ by $\sigma(\ell)$ in (21) yields

$$B = \sum_{\ell=0}^{k-1} Q_{\rho(\sigma(\ell))} G_{\sigma(\ell)} \widehat{P}_{\sigma(\ell)}.$$

From this, (15), and (22),

$$BA = \sum_{\ell=0}^{k-1} Q_{\rho(\sigma(\ell))} G_{\sigma(\ell)} F_\ell \widehat{Q}_\ell,$$

so (10) and (12) imply that

$$S(BA) = (AB)S_{\rho\circ\sigma} = \sum_{\ell=0}^{k-1} \gamma_{\rho(\sigma(\ell))} Q_{\rho(\sigma(\ell))} G_{\sigma(\ell)} F_\ell \widehat{Q}_\ell,$$

which proves (b). □

Corollary 1 *If σ is a permutation, $A \in \mathbb{C}^{m\times n}$ is (R, S_σ)-commutative, and $B \in \mathbb{C}^{n\times m}$ is $(S, R_{\sigma^{-1}})$-commutative, then AB commutes with R and BA commutes with S.*

Theorem 5 *Suppose $j > 1$ and $A_j \in \mathbb{C}^{m\times m}$ is (R, R_{σ_j})-commutative, where σ_j is a permutation if $j > 1$. Then $A_1 A_2 \cdots A_j$ is $(R, R_{\sigma_1\circ\sigma_2\circ\cdots\circ\sigma_j})$-commutative; specifically, if*

$$A_j = \sum_{\ell=0}^{k-1} P_{\sigma_j(\ell)} F_\ell^{(j)} \widehat{P}_\ell, \tag{23}$$

then

$$A_1 A_2 = \sum_{\ell=0}^{k-1} P_{\sigma_1\circ\sigma_2(\ell)} F^{(1)}_{\sigma_2(\ell)} F_\ell^{(2)} \widehat{P}_\ell,$$

$$A_1 A_2 A_3 = \sum_{\ell=0}^{k-1} P_{\sigma_1\circ\sigma_2\circ\sigma_3(\ell)} F^{(1)}_{\sigma_2\circ\sigma_3(\ell)} F^{(2)}_{\sigma_3(\ell)} F_\ell^{(3)} \widehat{P}_\ell,$$

and, in general,

$$A_1 A_2 \cdots A_j = \sum_{\ell=0}^{k-1} P_{\sigma_1\circ\sigma_2\circ\cdots\circ\sigma_j(\ell)} F^{(1)}_{\sigma_2\circ\cdots\circ\sigma_j(\ell)} F^{(2)}_{\sigma_3\circ\cdots\circ\sigma_j(\ell)} \cdots F^{(j-1)}_{\sigma_j(\ell)} F_\ell^{(j)} \widehat{P}_\ell.$$

PROOF. To minimize complicated notation, suppose

$$B_j = \sum_{\ell=0}^{k-1} P_{\sigma_1\circ\sigma_2\circ\cdots\circ\sigma_j(\ell)} G_\ell^{(j)} \widehat{P}_\ell$$

for some $j \geq 1$. Since σ_{j+1} is a permutation, we can replace ℓ by $\sigma_{j+1}(\ell)$ to obtain

$$B_j = \sum_{\ell=0}^{k-1} P_{\sigma_1\circ\sigma_2\circ\cdots\circ\sigma_j\circ\sigma_{j+1}(\ell)} G_{\sigma_{j+1}(\ell)} \widehat{P}_{\sigma_{j+1}(\ell)}.$$

Therefore, from (23) with j replaced by $j+1$,

$$\begin{aligned} B_j A_{j+1} &= \left(\sum_{\ell=0}^{k-1} P_{\sigma_1\circ\sigma_2\circ\cdots\circ\sigma_j\circ\sigma_{j+1}(\ell)} G_{\sigma_{j+1}(\ell)} \widehat{P}_{\sigma_{j+1}(\ell)}\right)\left(\sum_{\ell=0}^{k-1} P_{\sigma_{j+1}(\ell)} F_\ell^{(j+1)} \widehat{P}_\ell\right) \\ &= \sum_{\ell=0}^{k-1} P_{\sigma_1\circ\sigma_2\circ\cdots\circ\sigma_j\circ\sigma_{j+1}(\ell)} G_\ell^{(j+1)} \widehat{P}_\ell \quad \text{with} \quad G_\ell^{(j+1)} = G_{\sigma_{j+1}(\ell)} F_\ell^{(j+1)}. \end{aligned}$$

This provides the basis for a straightforward induction proof of the assertion. □

Corollary 2 *If σ is a permutation, $A \in \mathbb{C}^{m\times m}$ is (R, R_σ)-commutative, and j is a positive integer, then A^j is (R, R_{σ^j})-commutative; explicitly,*

$$A^j = \sum_{\ell=0}^{k-1} P_{\sigma^j(\ell)} F_{\sigma^{(j-1)}(\ell)} \cdots F_{\sigma(\ell)} F_\ell \widehat{P}_\ell \tag{24}$$

and

$$RA = AR_{\sigma^j} = \sum_{\ell=0}^{k-1} \gamma_{\sigma^j(\ell)} P_{\sigma^j(\ell)} F_{\sigma^{(j-1)}(\ell)} \cdots F_{\sigma(\ell)} F_\ell \widehat{P}_\ell$$

for arbitrary $\gamma_0, \gamma_1, \ldots, \gamma_{k-1}$.

5 Generalized Inverses and Singular Value Decompositions

If A is an arbitrary complex matrix then A^- is a reflexive inverse of A if $AA^-A = A$ and $A^-AA^- = A^-$. The Moore-Penrose inverse $A^\dagger$ of A is the unique matrix that satisfies the Penrose conditions

$$(AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = AA^\dagger, \quad AA^\dagger A = A, \quad \text{and} \quad A^\dagger AA^\dagger = A^\dagger.$$

Theorem 6 *Suppose σ is a permutation and $A \in \mathbb{C}^{m\times n}$ is (R, S_σ)-commutative, so*

$$A = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F_\ell \widehat{Q}_\ell, \tag{25}$$

by Theorem 2. Let $F_0^-, F_1^-, \ldots, F_{k-1}^-$ be reflexive inverses of $F_0, F_1, \ldots, F_{k-1}$, and define

$$B = \sum_{\ell=0}^{k-1} Q_\ell F_\ell^- \widehat{P}_{\sigma(\ell)}. \tag{26}$$

Then B is a reflexive inverse of A. Moreover, if P and Q are unitary, then

$$A^\dagger = \sum_{\ell=0}^{k-1} Q_\ell F_\ell^\dagger P^*_{\sigma(\ell)}. \tag{27}$$

PROOF. From (9), (10), (22), (25), and (26),

$$AB = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F_\ell F_\ell^- \widehat{P}_{\sigma(\ell)}, \quad BA = \sum_{\ell=0}^{k-1} Q_\ell F_\ell^- F_\ell \widehat{Q}_\ell, \tag{28}$$

$$ABA = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F_\ell F_\ell^- F_\ell \widehat{Q}_\ell = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F_\ell \widehat{Q}_\ell = A, \tag{29}$$

and

$$BAB = \sum_{\ell=0}^{k-1} Q_\ell F_\ell^- F_\ell F_\ell^- \widehat{P}_{\sigma(\ell)} = \sum_{\ell=0}^{k-1} Q_\ell F_\ell^- \widehat{P}_{\sigma(\ell)} = B. \tag{30}$$

The last two equations show that B is a reflexive inverse of A. If P and Q are unitary and we redefine

$$B = \sum_{\ell=0}^{k-1} Q_\ell F_\ell^\dagger P^*_{\sigma(\ell)},$$

then (28)–(30) become

$$AB = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F_\ell F_\ell^\dagger P^*_{\sigma(\ell)}, \quad BA = \sum_{\ell=0}^{k-1} Q_\ell F_\ell^\dagger F_\ell Q_\ell^*, \tag{31}$$

$$ABA = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F_\ell F_\ell^\dagger F_\ell Q_\ell^* = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F_\ell Q_\ell^* = A,$$

and

$$BAB = \sum_{\ell=0}^{k-1} Q_\ell F_\ell^\dagger F_\ell F_\ell^- P^*_{\sigma(\ell)} = \sum_{\ell=0}^{k-1} Q_\ell F_\ell^\dagger P^*_{\sigma(\ell)} = B.$$

Moreover, from (31)

$$(AB)^* = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} (F_\ell F_\ell^\dagger)^* P^*_{\sigma(\ell)} = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F_\ell F_\ell^\dagger P^*_{\sigma(\ell)} = AB$$

and

$$(BA)^* = \sum_{\ell=0}^{k-1} Q_\ell (F_\ell^\dagger F_\ell)^* Q_\ell^* = \sum_{\ell=0}^{k-1} Q_\ell F_\ell^\dagger F_\ell Q_\ell^* = BA.$$

Therefore $B = A^\dagger$, which implies (27). □

Corollary 3 *If σ is a permutation, P and Q are unitary, and $A \in \mathbb{C}^{m\times n}$ is (R, S_σ)-commutative, then $A^\dagger$ is $(S, R_{\sigma^{-1}})$-commutative.*

PROOF. From (9)–(11), (22), and (27),

$$SA^\dagger = A^\dagger R_{\sigma^{-1}} = \sum_{\ell=0}^{k-1} \gamma_\ell Q_\ell F_\ell^\dagger P_{\sigma(\ell)}^*$$

for arbitrary $\gamma_0, \gamma_1, \dots, \gamma_{k-1}$. □

Remark 2 It is well known – and straightforward to verify – that if $G \in \mathbb{C}^{p\times q}$ and $\operatorname{rank} G = q$, then $G^\dagger = (G^*G)^{-1}G^*$. Hence, (27) implies the folllowing corollary.

Corollary 4 *In addition to the assumptions of Theorem 6, suppose that $\operatorname{rank}(F_\ell) = n_\ell$, $0 \le \ell \le k-1$ (or, equivalently, $\operatorname{rank}(A) = n$). Then*

$$A^\dagger = \sum_{\ell=0}^{k-1} Q_\ell (F_\ell^* F_\ell)^{-1} F_\ell^* P_{\sigma(\ell)}^*.$$

Theorem 7 *Suppose σ is a permutation, P and Q are unitary, and A is (R, S_σ)-commutative and therefore of the form*

$$A = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F_\ell Q_\ell^*,$$

by Theorem 2. Let

$$F_\ell = \Omega_\ell \Gamma_\ell \Phi_\ell^*, \quad 0 \le \ell \le k-1,$$

with

$$\Omega_\ell \in \mathbb{C}^{m_{\sigma(\ell)}\times m_{\sigma(\ell)}}, \quad \Gamma_\ell \in \mathbb{C}^{m_{\sigma(\ell)}\times n_\ell}, \quad \text{and} \quad \Phi_\ell \in \mathbb{C}^{n_\ell\times n_\ell}, \quad 0 \le \ell \le k-1,$$

be singular value decompositions of F_ℓ, $0 \le \ell \le k-1$. Let

$$\Omega = \begin{bmatrix} P_{\sigma(0)}\Omega_0 & P_{\sigma(1)}\Omega_1 & \cdots & P_{\sigma(k-1)}\Omega_{k-1} \end{bmatrix}$$

and

$$\Phi = \begin{bmatrix} Q_0\Phi_0 & Q_1\Phi_1 & \cdots & Q_{k-1}\Phi_{k-1} \end{bmatrix}$$

Then

$$A = \Omega \operatorname{diag}(\Gamma_0, \Gamma_1, \dots, \Gamma_{k-1})\Phi^*$$

is a singular value decomposition of A, except that the singular values are not necessarily arranged in decreasing order. Thus, for $0 \le \ell \le k-1$, each singular value of F_ℓ is a singular value of A with an associated left singular vector in the column space of $P_{\sigma(\ell)}$ and a right singular vector in the column space of Q_ℓ.

We invoke the first equality in (22) repeatedly in the proof of the following theorem.

Theorem 8 *Suppose σ is a permutation, P is unitary, and $A \in \mathbb{C}^{m\times m}$ is (R, R_σ)-commutative, so*

$$A = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F_\ell P_\ell^*,$$

by Theorem 2. Then:

(i) *A is Hermitian if and only if* $= F^*_{\sigma(\ell)} P^*_{\sigma^2(\ell)} = F_\ell P_\ell^*$, $0 \le \ell \le k-1$.

(ii) *A is normal if and only if* $F^*_{\sigma(\ell)} F_{\sigma(\ell)} = F_\ell F_\ell^*$, $0 \le \ell \le k-1$.

(iii) *A is EP (i.e., $AA^\dagger = A^\dagger A$) if and only if* $F^\dagger_{\sigma(\ell)} F_{\sigma(\ell)} = F_\ell F_\ell^\dagger$, $0 \le \ell \le k-1$.

PROOF. Since R is unitary, Theorems 2 and 6 imply that

$$A = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F_\ell P_\ell^*, \quad A^* = \sum_{\ell=0}^{k-1} P_\ell F_\ell^* P^*_{\sigma(\ell)}, \quad \text{and } A^\dagger = \sum_{\ell=0}^{k-1} P_\ell F_\ell^\dagger P^*_{\sigma(\ell)}. \tag{32}$$

Replacing ℓ by $\sigma(\ell)$ in the second sum in (32) yields

$$A^* = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F^*_{\sigma(\ell)} P^*_{\sigma^2(\ell)},$$

and comparing this with the first sum in (32) yields (i).

From (32),

$$AA^* = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F_\ell F_\ell^* P^*_{\sigma(\ell)} \tag{33}$$

and

$$A^*A = \sum_{\ell=0}^{k-1} P_\ell F_\ell^* F_\ell P_\ell^* = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F^*_{\sigma(\ell)} F_{\sigma(\ell)} P^*_{\sigma(\ell)}.$$

Comparing the second sum here with (33) yields (ii).

From (33),

$$AA^\dagger = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F_\ell F_\ell^\dagger P^*_{\sigma(\ell)} \tag{34}$$

and

$$A^\dagger A = \sum_{\ell=0}^{k-1} P_\ell F_\ell^\dagger F_\ell P_\ell^* = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F^\dagger_{\sigma(\ell)} F_{\sigma(\ell)} P^*_{\sigma(\ell)},$$

Comparing the second sum here with (34) yields (iii). □

6 Solving $Az = w$ and the least-squares problem

Throughout this section σ is a permutation and $A \in \mathbb{C}^{m\times n}$ is (R, S_σ)-commutative, and can therefore be written as in (15).

If $z \in \mathbb{C}^n$ and $w \in \mathbb{C}^m$, we write

$$z = Qu = \sum_{\ell=0}^{k-1} Q_\ell u_\ell \quad \text{and} \quad w = Pv = \sum_{\ell=0}^{k-1} P_\ell v_\ell, \tag{35}$$

with $u_\ell \in \mathbb{C}^{n_\ell}$ and $v_\ell \in \mathbb{C}^{m_\ell}$, $0 \le \ell \le k-1$.

Theorem 9 *If* (35) *holds then*

$$\text{(a)} \quad Az = w \quad \textit{if and only if} \quad \text{(b)} \quad F_\ell u_\ell = v_{\sigma(\ell)}, \quad 0 \le \ell \le k-1. \tag{36}$$

PROOF. From (10), (15), and (35),

$$\begin{aligned} Az - w &= \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F_\ell u_\ell - \sum_{\ell=0}^{k-1} P_\ell v_\ell = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F_\ell u_\ell - \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} v_{\sigma(\ell)} \\ &= \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} \left(F_\ell u_\ell - v_{\sigma(\ell)}\right), \end{aligned} \tag{37}$$

so (36)(b) implies (36)(a). From (22) and (37),

$$F_\ell u_\ell - v_{\sigma(\ell)} = \widehat{P}_{\sigma(\ell)}(Az - w), \quad 0 \le \ell \le k-1,$$

so (36)(a) implies (36)(b). □

Since $F_\ell \in \mathbb{C}^{m_{\sigma(\ell)} \times n_\ell}$, $0 \le \ell \le k-1$, (36) implies the following theorem.

Theorem 10 *A is invertible if and only if $m_{\sigma(\ell)} = n_\ell$ and F_ℓ is invertible, $0 \le \ell \le k-1$ (which, from (3), implies that $m = n$). In this case,*

$$A^{-1} = \sum_{\ell=0}^{k-1} Q_\ell F_\ell^{-1} \widehat{P}_{\sigma(\ell)} \tag{38}$$

and the solution of $Az = w$ is

$$z = \sum_{\ell=0}^{k-1} Q_\ell F_\ell^{-1} v_{\sigma(\ell)}.$$

Moreover, A^{-1} is $(S, R_{\sigma^{-1}})$-commutative; specifically,

$$SA^{-1} = A^{-1} R_{\sigma^{-1}} = \sum_{\ell=0}^{k-1} \gamma_\ell Q_\ell F_\ell^{-1} \widehat{P}_{\sigma(\ell)}$$

for arbitrary $\gamma_0, \gamma_1, \ldots, \gamma_{k-1}$.

If $m = n$ and $R = S$ (so A is (R, R_σ)-commutative), then (38) becomes

$$A^{-1} = \sum_{\ell=0}^{k-1} P_\ell F_\ell^{-1} \widehat{P}_{\sigma(\ell)}.$$

In this case,

$$A^{-j} = \sum_{\ell=0}^{k-1} P_\ell F_\ell^{-1} F_{\sigma(\ell)}^{-1} \cdots F_{\sigma^{j-1}(\ell)}^{-1} \widehat{P}_{\sigma^j(\ell)},$$

which can be verified by simply multiplying the right hand side by A^j as written in (24).

Before turning to the least squares problem for A, we review some elementary facts about the least squares problem for a matrix $G \in \mathbb{C}^{p\times q}$ and a given $u \in \mathbb{C}^p$; i.e., find $v \in \mathbb{C}^q$ such that

$$\|Gv - u\| = \min_{\xi\in\mathbb{C}^q} \|G\xi - u\|,$$

where $\|\cdot\|$ is the 2-norm. An arbitrary $v \in \mathbb{C}^{p\times q}$ can be written as

$$v = G^\dagger u + G(v - G^\dagger u),$$

so

$$\|Gv - u\|^2 = \|(GG^\dagger - I_p)u\|^2 + \|G(v - G^\dagger u)\|^2,$$

since

$$\begin{aligned}[G(v - G^\dagger u)]^*(GG^\dagger - I_p)u &= [GG^\dagger G(v - G^\dagger)u]^*(GG^\dagger - I_p)u \\ &= [G(v - G^\dagger u)]^* GG^\dagger(GG^\dagger - I_p)u\end{aligned}$$

and

$$G^\dagger(GG^\dagger - I_p) = G^\dagger GG^\dagger - G^\dagger = 0.$$

Hence,

$$\min_{\xi\in\mathbb{C}^q} \|G\xi - u\| = \|(GG^\dagger - I)u\|,$$

and this minimum is attained with a given v if and only if $v = G^\dagger u + h$ where $Gh = 0$. In this case, $\|v\|^2 = \|G^\dagger u\|^2 + \|h\|^2$ since

$$h^* G^\dagger u = h^* G^\dagger G G^\dagger u = (G^\dagger G h)^* G^\dagger u = 0,$$

so $v_0 = G^\dagger u$ is the unique solution of (37) with minimal norm, and is therefore called the optimal solution. From Remark 2, $v_0 = (G^*G)^{-1}G^*u$ if $\operatorname{rank}(G) = q$. If P is unitary then (37) implies that

$$\|Az - w\|^2 = \sum_{\ell=0}^{k-1} \|F_\ell u_\ell - v_{\sigma(\ell)}\|^2,$$

so the least squares problem for A and a given w reduces to k independent least squares problems for $F_\ell \in \mathbb{C}^{m_{\sigma(\ell)}\times n_\ell}$ and a given $v_{\sigma(\ell)} \in \mathbb{C}^{m_{\sigma(\ell)}}$, $0 \le \ell \le k-1$. Therefore,

$$\|Az - w\| = \min_{\zeta\in\mathbb{C}^n} \|A\zeta - w\|$$

if and only if

$$z = \sum_{\ell=0}^{k-1} Q_\ell (F_\ell^\dagger v_{\sigma(\ell)} + h_\ell),$$

where $F_\ell h_\ell = 0$, $0 \le \ell \le k-1$. If Q is also unitary, then

$$\|z\|^2 = \sum_{\ell=0}^{k-1} \|F_\ell^\dagger v_{\sigma(\ell)} + h_\ell\|^2 = \sum_{\ell=0}^{k-1} \|F_\ell^\dagger v_{\sigma(\ell)}\|^2 + \sum_{\ell=0}^{k-1} \|h_\ell\|^2,$$

so the unique optimal (least norm) solution of the least squares problem is

$$z = \sum_{\ell=0}^{k-1} Q_\ell F_\ell^\dagger v_{\sigma(\ell)},$$

which can be written as

$$z = \sum_{\ell=0}^{k-1} Q_\ell (F_\ell^* F_\ell)^{-1} F_\ell^* v_{\sigma(\ell)} \quad \text{if } \operatorname{rank}(F_\ell) = n_\ell, \quad 0 \le \ell \le k-1,$$

or, equivalently, if $\operatorname{rank}(A) = n$.

7 The eigenvalue problem

Throughout this section $A \in \mathbb{C}^{m\times m}$ is (R, R_σ)-commutative, and can therefore be written as

$$A = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F_\ell \widehat{P}_\ell \quad \text{where} \quad F_\ell \in \mathbb{C}^{m_{\sigma(\ell)}\times m_\ell} \quad 0 \le \ell \le k-1, \tag{39}$$

and σ is a permutation.

An arbitrary $z \in \mathbb{C}^m$ can be written as

$$z = \sum_{\ell=0}^{k-1} P_\ell u_\ell \quad \text{with} \quad u_\ell \in \mathbb{C}^{m_\ell}, \quad 0 \le \ell \le k-1.$$

Therefore (9) and (39) imply that

$$Az - \lambda z = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} F_\ell u_\ell - \lambda \sum_{\ell=0}^{k-1} P_\ell u_\ell = \sum_{\ell=0}^{k-1} P_{\sigma(\ell)} (F_\ell u_\ell - \lambda u_{\sigma(\ell)}); \tag{40}$$

hence, $Az = \lambda z$ if and only if

$$F_\ell u_\ell = \lambda u_{\sigma(\ell)}, \quad 0 \le \ell \le k-1.$$

We first consider the case where σ is the identity. The next three theorems are essentially restatements of results from [21], recast so as to be consistent with viewpoint that we have taken in this paper.

Let $\mathcal{C}_\ell$ denote the column space of P_ℓ and let $\mathcal{C} = \cup_{\ell=0}^{k-1}\mathcal{C}_\ell$.

Theorem 11 *If A commutes with R then λ is an eigenvalue of A if and only if λ is an eigenvalue of one or more of the matrices $F_0, F_1, \dots, F_{k-1}$. Assuming this to be true, let*

$$S_A(\lambda) = \left\{\ell \in \{0, 1, \dots, k-1\} \,\middle|\, \lambda \text{ is an eigenvalue of } F_\ell\right\}.$$

If $\ell \in S_A(\lambda)$ and $\{u_\ell^{(1)}, u_\ell^{(2)}, \cdots, u_\ell^{(d_\ell)}\}$ is a basis for the set $\left\{u_\ell \in \mathbb{C}^{m_\ell \times m_\ell} \,\middle|\, F_\ell u_\ell = \lambda u_\ell\right\}$, then $P_\ell u_\ell^{(1)}, P_\ell u_\ell^{(2)}, \dots, P_\ell u_\ell^{(d_\ell)}$ are linearly independent λ-eigenvectors of A. Moreover,

$$\bigcup_{\ell \in S_A(\lambda)} \{P_\ell u_\ell^{(1)}, P_\ell u_\ell^{(2)}, \cdots, P_\ell u_\ell^{(d_\ell)}\}$$

is a basis for the λ-eigenspace of A. Finally, A is diagonalizable if and only if $F_0, F_1, \dots, F_{k-1}$ are all diagonalizable. In this case, A has m_ℓ linearly independent eigenvectors in $\mathcal{C}_\ell$, $0 \le \ell \le k-1$.

It seems useful to consider the case where A is diagonalizable more explicitly.

Theorem 12 *Suppose a diagonalizable matrix A commutes with R and and $F_\ell = \Omega_\ell D_\ell \Omega_\ell^{-1}$ is a spectral decomposition of F_ℓ, $0 \le \ell \le k-1$. Let*

$$\Omega = \begin{bmatrix} P_0\Omega_0 & P_1\Omega_1 & \cdots & P_{k-1}\Omega_{k-1} \end{bmatrix}$$

Then

$$A = \Omega \left(\bigoplus_{s=0}^{k-1} D_\ell \right) \Omega^{-1}$$

with

$$\Omega^{-1} = \begin{bmatrix} \Omega_0^{-1}\widehat{P}_0 \\ \Omega_1^{-1}\widehat{P}_1 \\ \vdots \\ \Omega_{k-1}^{-1}\widehat{P}_{k-1} \end{bmatrix}$$

is a spectral decomposition of A.

Remark 3 It is well known that commuting diagonalizable matrices are simultaneouly diagonalizable. Theorem 12 makes this explicit, since since $\Omega R \Omega^{-1}$ and $\Omega A \Omega^{-1}$ are both diagonal.

The original version of the following theorem, which dealt with centrosymmetric matrices, is due to Andrew [2, Theorem 6]. The proof is practically identical to Andrew's original proof.

Theorem 13

(i) *If A commutes with R and λ is an eigenvalue of A, then the λ-eigenspace of S has a basis in $\mathcal{C}$.*

(ii) *If A has n linearly independent eigenvectors in $\mathcal{C}$, then A commutes with R.*

PROOF. (i) See Theorem 11. (ii) If $z \in \mathcal{C}$ then $Rz = \gamma_\ell z$ for some $\ell \in \mathbb{Z}_k$. If $Az = \lambda z$ and $Rz = \gamma_\ell z$, then

$$RAz = \lambda Rz = \lambda\gamma_\ell z \quad \text{and} \quad ARz = \gamma_\ell Az = \gamma_\ell \lambda z;$$

hence, $RAz = ARz$. Now suppose that A has n linearly independent eigenvectors $\{z_1, z_2, \dots, z_n\}$ in $\mathcal{C}$. Then we can write an arbitrary $z \in \mathbb{C}^n$ as $z = \sum_{i=1}^n a_i z_i$. Since $RAz_i = ARz_i$, $1 \le i \le n$, it follows that $RAz = ARz$. Therefore $AR = RA$. □

For the remainder of this section we assume that A is (R, R_σ)-commutative and σ is a permutation other than the identity.

The following theorem shows that finding the null space of A reduces to finding the null spaces of $F_0, F_1, \dots, F_{k-1}$.

Theorem 14 *If A is (R, R_σ)-commutative and σ is a permutation then $Az = 0$ if and only if $z = \sum_{\ell=0}^{k-1} P_\ell u_\ell$, where*

$$F_\ell u_\ell = 0, \quad 0 \le \ell \le k-1; \tag{41}$$

hence, the null space if A is independent of σ (so long as σ is a permutation).

PROOF. Clearly, (41) implies that $Az = 0$ without any assumption on σ. For the converse, note from (22) and (40) that if σ is a permutation then $\widehat{P}_{\sigma(\ell)} Az = F_\ell u_\ell$, $0 \le \ell \le k-1$, so $Az = 0$ implies (41). □

Henceforth we assume that $\lambda \ne 0$. In this case, suppose that σ has p orbits $\mathcal{O}_0$, $\dots$, $\mathcal{O}_{p-1}$. If $p = 1$, then σ is a k-cycle and $\mathbb{Z}_k = \left\{\sigma^j(0) \,\middle|\, 0 \le j \le k-1\right\}$. In any case, if $\ell_r \in \mathcal{O}_r$, $0 \le r \le p-1$, then $\mathbb{Z}_k = \mathcal{O}_0 \cup \dots \cup \mathcal{O}_{p-1}$, where

$$\mathcal{O}_r = \left\{\sigma^j(\ell_r) \,\middle|\, 0 \le j \le k_r - 1\right\}, \quad 0 \le r \le p-1,$$

and $k_0 + \dots + k_{p-1} = k$. It is important to note that

$$\sigma^{k_r}(\ell_r) = \ell_r, \quad 0 \le r \le p-1, \tag{42}$$

and $k_0, k_1, \dots, k_{p-1}$ are respectively the smallest positive integers for which these equalities hold. In Example 1, $p = 3$, $\mathcal{O}_0 = \{0, 1, 3\}$, $\mathcal{O}_1 = \{2, 4\}$, $\mathcal{O}_2 = \{5\}$, so $k_0 = 3$, $k_1 = 2$, $k_3 = 1$, $\mathbb{Z}_6 = \mathcal{O}_0 \bigcup \mathcal{O}_1 \bigcup \mathcal{O}_2$, and we may choose $\ell_0 = 0$, $\ell_1 = 2$, and $\ell_2 = 5$.

To solve the eigenvalue problem, we rearrange the terms in $z = \sum_{\ell=0}^{k-1} P_\ell u_\ell$ as

$$z = \sum_{r=0}^{p-1} z_r \quad \text{with} \quad z_r = \sum_{j=0}^{k_r-1} P_{\sigma^j(\ell_r)} u_{\sigma^j(\ell_r)}, \quad 0 \le r \le p-1, \tag{43}$$

and rearrange the terms in (39) as

$$A = \sum_{r=0}^{p-1} A_r \quad \text{with} \quad A_r = \sum_{j=0}^{k_r-1} P_{\sigma^{j+1}(\ell_r)} F_{\sigma^j(\ell_r)} \widehat{P}_{\sigma^j(\ell_r)}, \quad 0 \le r \le p-1. \tag{44}$$

Since (9) implies that $A_r A_s = 0$ if $r \ne s$, we can replace (44) by

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_{p-1};$$

hence, $Az = \lambda z$ if and only if

$$A_r z_r = \lambda z_r, \quad 0 \le r \le p-1.$$

Therefore, the eigenvalue problem for A reduces to p independent eigenvalue problems for $A_0, A_1, \dots, A_{p-1}$.

From (43) and (44), $A_r z_r = \lambda z_r$ if and only if

$$\sum_{j=0}^{k_r-1} P_{\sigma^{j+1}(\ell_r)} F_{\sigma^j(\ell_r)} u_{\sigma^j(\ell_r)} = \lambda \sum_{j=0}^{k_r-1} P_{\sigma^j(\ell_r)} u_{\sigma^j(\ell_r)} = \lambda \sum_{j=0}^{k_r-1} P_{\sigma^{j+1}(\ell_r)} u_{\sigma^{j+1}(\ell_r)},$$

which is equivalent to

$$F_{\sigma^j(\ell_r)} u_{\sigma^j(\ell_r)} = \lambda u_{\sigma^{j+1}(\ell_r)}, \quad 0 \le j \le k_r - 1. \tag{45}$$

If $k_r = 1$ then $\sigma(\ell_r) = \ell_r$ and (44) becomes $F_{\ell_r} u_{\ell_r} = \lambda u_{\ell_r}$; hence, if (λ, u_{ℓ_r}) is an eigenpair of F_{ℓ_r} then $z_r = P_{\ell_r} u_{\ell_r}$ is λ-eigenvector of A.

If $k_r > 1$ then (42) and (44) imply that

$$G_r u_{\ell_r} = \lambda^k u_{\ell_r}, \quad \text{where} \quad G_r = F_{\sigma^{k_r-1}(\ell_r)} \cdots F_{\sigma(\ell_r)} F_{\ell_r} \in \mathbb{C}^{m_{\ell_r} \times m_{\ell_r}}.$$

Therefore, if ν is a nonzero eigenvalue of G_r and $\zeta = e^{2\pi i/k_r}$, then $\nu^{1/k}$, $\nu^{1/k}\zeta, \dots,$ $\nu^{1/k}\zeta^{k_r-1}$ are distinct eigenvalues of A_r (and therefore of A). If λ is any one of these eigenvalues, then the corresponding eigenvector z_r of A_r (and therefore of A) is given by (43), where $u_{\sigma^j(\ell_r)}$, $1 \le j \le k_{r-1}$, can be computed recursively from (44) as

$$u_{\sigma^j(\ell_r)} = \frac{1}{\lambda} F_{\sigma^{j-1}(\ell_r)} u_{\sigma^{j-1}(\ell_r)}, \quad 1 \le j \le k_r - 1.$$

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