An Elementary View of Weyl's Theory of Equal Distribution

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Abstract

Suppose that $-\infty < a < b < \infty$, $a \le u_{1n} \le u_{2n} \le \cdots \le u_{nn} \le b$, and $a \le v_{1n} \le v_{2n} \le \cdots \le v_{nn} \le b$ for $n \ge 1$. We simplify and strengthen Weyl's definition of equal distribution of $\{\{u_{in}\}_{i=1}^n\}_{n=1}^\infty$ and $\{\{v_{in}\}_{i=1}^n\}_{n=1}^\infty$ by showing that the following statements are equivalent:

(i) $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n (F(u_{in}) - F(v_{in})) = 0$ for all $F \in C[a,b]$,

(ii) $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n |u_{in} - v_{in}| = 0$,

(iii) $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n |F(u_{in}) - F(v_{in})| = 0$ for all $F \in C[a,b]$.

We relate this to Weyl's definition of uniform distribution and Szegö's distribution formula for the eigenvalues of a family of Toeplitz matrices $\{[t_{r-s}]_{r,s=1}^n\}_{n=1}^\infty$, where $t_r = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-irx} g(x)\,dx$ and g is real-valued and continuous on $[-\pi,\pi]$.

1 Introduction

We consider four definitions of "distribution" that can be traced back to H. Weyl. We assume throughout that the doubly-indexed sequences

$$\mathbf{U} = \{\{u_{in}\}_{i=1}^n\}_{n=1}^\infty \quad \text{and} \quad \mathbf{V} = \{\{v_{in}\}_{i=1}^n\}_{n=1}^\infty$$

are contained in a finite interval $[a,b]$. As usual, $C[a,b]$ is the family of real-valued continuous functions on $[a,b]$. To avoid annoying repetition, every occurence of "distributed" is to be interpreted as "distributed in $[a,b]$."

We have presented part of this discussion in [4] and [5]. However, [4] is interesting mainly to linear algebraists and operator theorists, and [5] is not widely circulated. Moreover, the arguments given here are simpler and we think that the conclusions will be interesting to a wider audience.

Our first definition is stated and attributed to H. Weyl in [1, p. 62].

Definition 1 $\mathbf{U}$ and $\mathbf{V}$ are *equally distributed* if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (F(u_{in}) - F(v_{in})) = 0 \ \text{ for all } \ F \in C[a,b]. \tag{1}$$

Definition 2 $\mathbf{V}$ is *uniformly distributed* if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(v_{in}) = \frac{1}{b-a} \int_a^b F(x)\, dx \ \text{ for all } \ F \in C[a,b]. \tag{2}$$

Definition 3 A sequence $\{x_i\}_{i=1}^{\infty} \subset [a,b]$ is *uniformly distributed* if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(x_i) = \frac{1}{b-a} \int_a^b F(x)\, dx \ \text{ for all } \ F \in C[a,b]. \tag{3}$$

Put another way, $\{x_i\}_{i=1}^{\infty}$ is uniformly distributed if $\{\{x_i\}_{i=1}^{n}\}_{n=1}^{\infty}$ is uniformly distributed as in Definition 2.

Definition 4 If a and b are respectively the minimum and maximum values of a continuous function g on a closed interval $[c,d]$, then $\mathbf{U}$ is *distributed like the values of* g if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} F(u_{in}) = \frac{1}{d-c} \int_c^d F(g(x))\, dx \ \text{ for all } \ F \in C[a,b]. \tag{4}$$

In the setting of linear algebra and operator theory, the members of $\mathbf{U}$ and $\mathbf{V}$ could be the eigenvalues of two families $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ of Hermitian matrices, and the problem is to find conditions on $\{A_n - B_n\}_{n=1}^{\infty}$ which imply that $\mathbf{U}$ and $\mathbf{V}$ are equally distributed.

It is well known that (2) is equivalent to

$$\lim_{n\to\infty} \frac{C_n(\mathcal{I})}{n} = \frac{\ell(\mathcal{I})}{b-a}$$

for every subinterval $\mathcal{I}$ of $[a,b]$, where $\ell(\mathcal{I})$ is the length of $\mathcal{I}$ and $C_n(\mathcal{I})$ is the cardinality of $\{u_{in}\}_{i=1}^{n} \cap \mathcal{I}$.

Definition 3 is a special case of Definition 2; nevertheless, a special case of Definition 3 is probably the most famous of all the definitions that we are considering. If x is an arbitrary real, let $[x]$ denote the greatest integer not greater than x, and let $\hat{x} = x - [x]$, so $0 \le \hat{x} < 1$. According to another definition of Weyl, $\{x_i\}_{i=1}^{n}$ is *equidistributed modulo* 1 or *uniformly distributed modulo* 1 if $\{\hat{x}_i\}_{i=1}^{\infty}$ is uniformly distributed in $[0,1]$ as in Definition 3, with $a = 0$ and $b = 1$.

The most famous example of Definition 4 is related to a special case of Szegö's distribution theorem [1, p. 64]. Suppose g is real-valued and continuous on $[-\pi,\pi]$. Let

$$t_r = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-irx} g(x)\, dx, \quad r = 0, \pm 1, \pm 2, \ldots,$$

and

$$T_n = [t_{r-s}]_{r,s=1}^n, \quad n = 1, 2, 3 \dots.$$

These are *Toeplitz* matrices. Since g is real-valued, $t_{-\ell} = \bar{t}_\ell$, so T_n is Hermitian and therefore has real eigenvalues λ_{1n}, λ_{2n}, $\dots$, λ_{nn}; in fact, they are all in $[a, b]$, where a and b are respectively the minimum and maximum values of g on $[-\pi, \pi]$.

Szegö showed that $\{\{\lambda_{in}\}_{i=1}^n\}_{n=1}^\infty$ is distributed like the values of g; i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n F(\lambda_{in}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(g(x))\, dx \quad \text{for all} \quad F \in C[a, b]$$

if g is essentially bounded and Lebesgue integrable on $[-\pi, \pi]$. Moreover, there are many results on this question under still weaker assumptions on g. We consider only the case where g is continous.

Although we have stated four definitions to provide a historical perspective, Definitions 2–4 are special cases of Definition 1. In connection with Definitions 2 and 3, let

$$w_{in} = a + \frac{i}{n}(b - a) \quad \text{for } 1 \le j \le n \quad \text{and } n = 2, 3, \dots. \tag{5}$$

From first year calculus, we know that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n F(w_{in}) = \frac{1}{b - a} \int_a^b F(x)\, dx \quad \text{for all} \quad F \in C[a, b].$$

From this and (1), $\mathbf{U}$ is uniformly distributed if and only if $\mathbf{U}$ and $\{\{w_{in}\}_{i=1}^n\}_{n=1}^\infty$ are equally distributed. Similarly, from (3), $\{x_i\}_{i=1}^\infty$ is uniformly distributed if and only if $\{\{x_i\}_{i=1}^n\}_{n=1}^\infty$ and $\{\{w_{in}\}_{i=1}^n\}_{n=1}^\infty$ are equally distributed.

As for Definition 4, let

$$y_{in} = c + \frac{i}{n}(d - c) \quad \text{for } 1 \le j \le n \quad \text{and } n = 2, 3 \dots. \tag{6}$$

Since g is continuous on $[c, d]$, it follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n F(g(y_{in})) = \frac{1}{d - c} \int_c^d F(g(x))\, dx \quad \text{for all} \quad F \in C[a, b].$$

From this and (4), $\mathbf{U}$ is distributed like the values of F if and only $\mathbf{U}$ and $\{\{g(y_{in})\}_{i=1}^n\}_{n=1}^\infty$ are equally distributed.

2 The Main Theorem and Corollaries

Henceforth we assume – without loss of generality – that

$$a \le u_{1n} \le u_{2n} \le \dots \le u_{nn} \le b \quad \text{and } a \le v_{1n} \le v_{2n} \le \dots \le v_{nn} \le b.$$

Here is our main result. We will prove it in Section 4.

Theorem 1 *The following assertions are equivalent*:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(F(u_{in})-F(v_{in}))=0 \ \ \text{for all } F\in C[a,b]; \tag{7}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|u_{in}-v_{in}|=0; \tag{8}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|F(u_{in})-F(v_{in})|=0 \ \ \text{for all } F\in C[a,b]. \tag{9}$$

This theorem and the discussion of Definitions 2–4 in Section 1 yield the following corollaries.

Corollary 1 **U** *and* **V** *are equally distributed if and only if* (8) *is true.*

Corollary 2 **V** *is uniformly distributed if and only if*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|v_{in}-w_{in}|=0, \tag{10}$$

with $\{\{w_{in}\}_{i=1}^{n}\}_{n=1}^{\infty}$ *as in* (5). *Moreover, each of the following statements is equivalent to* (10):

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(F(v_{in})-F(w_{in}))=0 \quad \text{for all } F\in C[a,b],$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|F(v_{in})-F(w_{in})|=0 \quad \text{for all } F\in C[a,b].$$

Corollary 3 *For each* $n\geq 2$, *let* σ_n *be a permutation of* $\{1,2,\dots n\}$ *such that*

$$x_{\sigma_n(1)}\leq x_{\sigma_n(2)}\leq\cdots\leq x_{\sigma_n(n)}.$$

Then $\{x_i\}_{i=1}^{\infty}$ *is uniformly distributed if and only if*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|x_{\sigma_n(i)}-w_{in}|=0. \tag{11}$$

Moreover, each of the following statements is equivalent to (11):

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(F(x_i)-F(w_{in}))=0 \quad \text{for all } F\in C[a,b],$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}|F(x_{\sigma_n(i)})-F(w_{in})|=0 \quad \text{for all } F\in C[a,b].$$

Corollary 4 *Let $\{\{y_{in}\}_{i=1}^{n}\}_{n=1}^{\infty}$ be as in (6) and, for each $n \ge 2$, let ρ_n be a permutation of $\{1, 2, \dots n\}$ such that*

$$g(y_{\rho_n(1),n}) \le g(y_{\rho_n(2),n}) \le \cdots \le g(y_{\rho_n(n),n}).$$

Then **U** *is distributed like the values of g if and only*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |u_{in} - g(y_{\rho_n(i),n})| = 0. \qquad (12)$$

Moreover, each of the following statements is equivalent to (12):

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (F(u_{in}) - F(g(y_{in})) = 0 \quad \textit{for all } F \in C[a, b],$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} |F(u_{in}) - F(g(y_{\rho_n(i),n})| = 0 \quad \textit{for all } F \in C[a, b].$$

We hope that the following suggestion will be taken as constructive rather than offensive: if (7), (8), and (9) are equivalent, then regarding (7) as the definition of equal distribution is putting the cart before the horse. Therefore – with some trepidation – we suggest that (8) should be the definition. Analogous suggestions apply to (10), (11), and (12) in connection with Definitions 2–4.

For examples that support this suggestion, suppose

$$\mathbf{U}_\ell = \{\{u_{in}^{(\ell)}\}_{i=1}^{n}\}_{n=1}^{\infty} \subset [a, b], \ \mathbf{V}_\ell = \{\{v_{in}^{(\ell)}\}_{i=1}^{n}\}_{n=1}^{\infty} \subset [a, b] \ \text{ for } 1 \le \ell \le k,$$

$$\lambda_{in} \ge 0, \quad 1 \le i \le n, \quad \text{and} \quad \lambda_{1n} + \lambda_{2n} + \cdots + \lambda_{nn} = 1.$$

Further, let $\mathbf{U} = \{\{u_{in}\}_{i=1}^{n}\}_{n=1}^{\infty}$ and $\mathbf{V} = \{\{v_{in}\}_{i=1}^{n}\}_{n=1}^{\infty}$, where

$$u_{in} = \lambda_{1n} u_{in}^{(1)} + \lambda_{2n} u_{in}^{(2)} + \cdots + \lambda_{kn} u_{in}^{(k)} \ \text{ and } v_{in} = \lambda_{1n} v_{in}^{(1)} + \lambda_{2n} v_{in}^{(2)} + \cdots + \lambda_{kn} v_{in}^{(k)}.$$

Then Corollary 1 obviously implies that **U** and **V** are equally distributed if $\mathbf{U}_\ell$ and $\mathbf{V}_\ell$ are equally distributed for $i = 1, 2, \dots, k$, and Corollary 2 obviously implies that **V** is uniformly distributed if $\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_k$ are uniformly distributed. These conclusions are not obvious from Definitions 1 and 2.

3 Required Lemmas

We need the following lemmas, in which $V_a^b(\phi)$ is the total variation of a function ϕ on $[a, b]$.

Lemma 1 (Helly's First Theorem) *Let $\{\phi_m\}_{m=1}^{\infty}$ be an infinite sequence of functions on $[a, b]$ and suppose that*

$$|\phi_m(x)| \le K < \infty \ \textit{ for } a \le x \le b \ \textit{ and } V_a^b(\phi_m) \le K, \quad m \ge 1.$$

Then there is a subsequence of $\{\phi_m\}_{m=1}^{\infty}$ that converges at every point of $[a, b]$ to a function of bounded variation on $[a, b]$.

Lemma 2 (Helly's Second Theorem) *Let $\{\phi_m\}_{m=1}^{\infty}$ be an infinite sequence of functions on $[a,b]$ such that $V_a^b(\phi_m) \le K < \infty$, $m \ge 1$, and*

$$\lim_{m\to\infty} \phi_m(x) = \phi(x) \quad \text{for } a \le x \le b.$$

Then $V_a^b(\phi) \le K$ and

$$\lim_{m\to\infty} \int_a^b F(x)\, d\phi_m(x) = \int_a^b F(x)\, d\phi(x) \quad \text{for all } F \in C[a,b].$$

Lemma 3 *Suppose $\phi(a) = \phi(b) = 0$, ϕ is of bounded variation on $[a,b]$, and*

$$\int_a^b F(x)\, d\phi(x) = 0, \quad \text{for all } F \in C[a,b].$$

Then $\phi(x) = 0$ at all points of continuity of ϕ. Thus, $\phi(x) \ne 0$ for at most countably many values of x.

For proofs of Lemmas 1–3, see [2, p. 222], [2, p. 233], and [3, p. 111].

Lemma 4 *Suppose $x_1 \le x_2 \le \cdots \le x_n$ and $y_1 \le y_2 \le \cdots \le y_n$. Let $\{\ell_1, \ell_2, \dots \ell_n\}$ be a permutation of $\{1, 2, \dots, n\}$ and define*

$$S(\ell_1, \ell_2, \dots, \ell_n) = \sum_{i=1}^{n} |x_i - y_{\ell_i}|. \tag{13}$$

Then

$$S(\ell_1, \ell_2, \dots, \ell_n) \ge S(1, 2, \dots, n) = \sum_{i=1}^{n} |x_i - y_i|. \tag{14}$$

Proof The proof is by induction. Let P_n be the stated proposition. P_1 is trivial. Suppose that $n > 1$ and P_{n-1} is true. If $\ell_n = n$ then P_{n-1} implies P_n. If $\ell_n = s < n$ then choose r so that $\ell_r = n$, and define

$$\ell_i' = \begin{cases} \ell_i & \text{if } i \ne r \text{ and } i \ne n, \\ s & \text{if } i = r, \\ n & \text{if } i = n. \end{cases}$$

Then

$$S(\ell_1, \ell_2, \dots, \ell_n) - S(\ell_1', \ell_2', \dots, \ell_n') = \sigma(x_n) - \sigma(x_r), \tag{15}$$

where

$$\sigma(x) = |x - y_s| - |x - y_n| = \begin{cases} y_s - y_n, & x < y_s, \\ 2x - y_s - y_n, & y_s \le x \le y_n, \\ y_n - y_s, & x > y_n. \end{cases}$$

Since σ nondecreasing, (15) implies that

$$S(\ell_1, \ell_2, \ldots, \ell_n) \geq S(\ell'_1, \ell'_2, \ldots, \ell'_n).$$

Since $\ell'_n = n$, P_{n-1} implies that

$$S(\ell'_1, \ell'_2, \ldots, \ell'_n) \geq S(1, 2, \ldots, n).$$

Therefore (15) implies (14), which completes the induction.

4 Proof of Theorem 1

Obviously, (9) implies (7). To see that (8) implies (9), suppose that $F \in C[a, b]$ and $\epsilon > 0$. By the Weierstrass approximation theorem, there is a polynomial P such that

$$|F(x) - P(x)| < \epsilon/2 \text{ for } a \leq x \leq b.$$

By the triangle inequality,

$$\begin{aligned} |F(u_{in}) - F(v_{in})| &\leq |F(u_{in}) - P(u_{in})| + |P(u_{in}) - P(v_{in})| \\ &\quad + |P(v_{in}) - F(v_{in})| \\ &< |P(u_{in}) - P(v_{in})| + \epsilon. \end{aligned} \tag{16}$$

Let $M = \max_{a \leq x \leq b} |P'(x)|$. By the mean value theorem,

$$|P(u_{in}) - P(v_{in})| \leq M|u_{in} - v_{in}|.$$

This and (16) imply that

$$\frac{1}{n} \sum_{i=1}^{n} |F(u_{in}) - F(v_{in})| < \epsilon + \frac{M}{n} \sum_{i=1}^{n} |u_{in} - v_{in}|.$$

From this and (8),

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} |F(u_{in}) - F(v_{in})| \leq \epsilon.$$

Since ϵ is arbitrary, this implies (9).

To complete the proof, we must show that (7) implies (8). The proof is by contradiction. If (8) is false, there is an $\epsilon_0 > 0$ and an increasing sequence $\{\ell_k\}_{k=1}^{\infty}$ of positive integers such that

$$\frac{1}{\ell_k} \sum_{i=1}^{\ell_k} |u_{i\ell_k} - v_{i\ell_k}| \geq \epsilon_0, \quad k \geq 1. \tag{17}$$

However, we will show that if (7) holds, then any increasing infinite sequence $\{\ell_k\}_{k=1}^{\infty}$ of positive integers has a a subsequence $\{n_k\}_{k=1}^{\infty}$ such that

$$\lim_{k \to \infty} \frac{1}{n_k} \sum_{i=1}^{n_k} |u_{in_k} - v_{in_k}| = 0, \tag{18}$$

which contradicts (17).

If S is a set, let $\operatorname{card}(S)$ be the cardinality of S. For $a \le x \le b$, let

$$\nu_n(x;\mathbf{U}) = \operatorname{card}(\{i \,|\, u_{in} < x\}) \text{ and } \nu_n(x;\mathbf{V}) = \operatorname{card}(\{i \,|\, v_{in} < x\}).$$

Define

$$\rho_n(x;\mathbf{U}) = \begin{cases} \nu_n(x;\mathbf{U})/n, & a \le x < b, \\ 1, & x = b, \end{cases} \tag{19}$$

and

$$\rho_n(x;\mathbf{V}) = \begin{cases} \nu_n(x;\mathbf{V})/n, & a \le x < b, \\ 1, & x = b. \end{cases} \tag{20}$$

Then

$$\frac{1}{n}\sum_{i=1}^{n} F(u_{in}) = \int_a^b F(x)\, d\rho_n(x;\mathbf{U}) \text{ for all } F \in C[a,b] \tag{21}$$

and

$$\frac{1}{n}\sum_{i=1}^{n} F(v_{in}) = \int_a^b F(x)\, d\rho_n(x;\mathbf{V}) \text{ for all } F \in C[a,b] \tag{22}$$

[2, p. 231]. If

$$\phi_n = \rho_n(\cdot;\mathbf{U}) - \rho_n(\cdot;\mathbf{V}),$$

then (7), (21), and (22) imply that

$$\lim_{n\to\infty} \int_a^b F(x)\, d\phi_n(x) = 0 \text{ for all } F \in C[a,b]. \tag{23}$$

Since

$$|\phi_n(x)| \le 1, \quad a \le x \le b, \text{ and } V_a^b(\phi_n) \le 2, \quad n \ge 1,$$

Lemma 1 implies that every sequence $\{\ell_k\}_{k=1}^{\infty}$ of positive integers has a subsequence $\{n_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} \phi_{n_k}(x) = \phi(x) \text{ for } a \le x \le b,$$

where ϕ is of bounded variation on $[a,b]$. From (23) and Lemma 2,

$$\int_a^b F(x)\, d\phi(x) = 0 \quad \text{for all } F \in C[a,b].$$

This and Lemma 3 imply that $\phi(x) = 0$ for all but countably many values of x.

Since $\lim_{k\to\infty} \phi_{n_k}(x) = 0$ for all but countably many values of x, (19) and (20) imply that

$$\lim_{k\to\infty} \frac{\nu_{n_k}(x,\mathbf{U}) - \nu_{n_k}(x,\mathbf{V})}{n_k} = 0$$

for all but countably many values of x. Therefore, given $\epsilon > 0$, we can choose x_0, x_1, …, x_m so that

$$a = x_0 < x_1 < \cdots < x_m = b,$$

$$x_j - x_{j-1} < \epsilon \quad \text{for} \quad 1 \le j \le m, \tag{24}$$

and

$$\lim_{k\to\infty} \frac{\nu_{n_k}(x_j, \mathbf{U}) - \nu_{n_k}(x_j, \mathbf{V})}{n_k} = 0. \tag{25}$$

Let

$$I_j = [x_{j-1}, x_j) \quad \text{for} \quad 1 \le j \le m-1, \quad \text{and} \quad I_m = [x_{m-1}, x_m],$$

and denote

$$U_{jk} = \text{card}\{i \,|\, u_{in_k} \in I_j\}, \quad V_{jk} = \text{card}\{i \,|\, v_{in_k} \in I_j\}.$$

Since

$$U_{jk} = \begin{cases} \nu_{n_k}(x_1; \mathbf{U}), & j = 1, \\ \nu_{n_k}(x_j; \mathbf{U}) - \nu_{n_k}(x_{j-1}; \mathbf{U}), & 2 \le j \le m-1, \\ n_k - \nu_{n_k}(x_{m-1}; \mathbf{U}), & j = m, \end{cases}$$

and

$$V_{jk} = \begin{cases} \nu_{n_k}(x_1; \mathbf{V}), & j = 1, \\ \nu_{n_k}(x_j; \mathbf{V}) - \nu_{n_k}(x_{j-1}; \mathbf{V}), & 2 \le j \le m-1, \\ n_k - \nu_{n_k}(x_{m-1}; \mathbf{V}), & j = m, \end{cases}$$

(25) implies that

$$\lim_{k\to\infty} \frac{U_{jk} - V_{jk}}{n_k} = 0 \quad \text{for} \quad 1 \le j \le m. \tag{26}$$

Since

$$\min(U_{jk}, V_{jk}) = \frac{U_{jk} + V_{jk} - |U_{jk} - V_{jk}|}{2},$$

and

$$\sum_{j=1}^{m} U_{jk} = \sum_{j=1}^{m} V_{jk} = n_k,$$

it follows that

$$\sum_{j=1}^{m} \min(U_{jk}, V_{jk}) = n_k - r_k, \tag{27}$$

where

$$r_k = \frac{1}{2} \sum_{j=1}^{m} |U_{jk} - V_{jk}|.$$

From (26),

$$\lim_{k\to\infty} \frac{r_k}{n_k} = 0. \tag{28}$$

From (24) and (27), there is a permutation τ_k of $\{1, \dots, n_k\}$ such that

$$|u_{in_k} - v_{\tau_k(i),n_k}| < \epsilon$$

for at least $n_k - r_k$ values of i; hence

$$\sum_{i=1}^{n_k} |u_{in_k} - v_{\tau_k(i),n_k}| < n_k\epsilon + r_k(b-a).$$

Now Lemma 4 implies that

$$\sum_{i=1}^{n_k} |u_{in_k} - v_{in_k}| < n_k\epsilon + r_k|b-a|.$$

Hence, from (28),

$$\limsup_{k\to\infty} \frac{1}{n_k}\sum_{i=1}^{n_k} |u_{in_k} - v_{in_k}| \le \epsilon.$$

Since ϵ is arbitrary, this implies (18), which completes the proof.

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Lemma 4 and its proof are similar to a well known result [6, p. 108] applicable in the case where (13) is replaced by

$$S(\ell_1, \ell_2, \dots, \ell_n) = \sum_{i=1}^{n} (x_i - y_{\ell_i})^2.$$

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