A Data Combination Method for Postflight Trajectory Analysis

William F. Trench*
Drexel Institute of Technology, Philadelphia, Pa.

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During the powered flight of a space vehicle, such as the Saturn vehicle, the onboard accelerometers provide a source of trajectory information that can be used to obtain a record of position and velocity. However, the trajectory information so obtained usually is degraded by systematic errors that tend to become large, but whose form is essentially predictable, given a reasonably valid model for the errors in the inertial measurements. This paper describes a postflight technique for employing a relatively small amount of ground-based tracking data to eliminate these systematic errors. The technique has been applied to a simulated two-stage Saturn flight. Bias errors were introduced into the accelerometer data, and noisy ground-based data were simulated. The trajectory estimates were derived from inertial data alone, and from the present method. In all cases considered, a small number of ground-based position measurements yielded position and velocity estimates that were essentially free of bias errors. Even though the analytical basis for the method rests on the assumption of linear time-invariant error model, the method yielded good results in the presence of large, nonlinear, time-varying errors.

1. Introduction

TWO independent classes of trajectory data are usually available from the powered flight of a space vehicle: data from ground instrumentation along the tracking range, and data telemetered to the ground. (The Saturn vehicle has in its inertial system three mutually perpendicular accelerometers, mounted on a stable platform, from which thrust acceleration data are telemetered.) A postflight trajectory analysis technique, which depends entirely on one or the other of these data sources, is usually inadequate. No single ground instrument can track the vehicle during the entire flight, and abrupt changes in position estimates often occur when the track is handed over from one to another. These discontinuities are caused by differences in the error characteristics of the instruments. Another shortcoming of ground instrumentation is that its accuracy can be severely limited by the geometry of the tracking problem; thus, the term "geometrical dilution of precision" in common use among trajectory analysts.

Trajectory estimates based on data from the inertial system exhibit prohibitively large bias errors, introduced by platform drift and misalignment, scale factor errors, and nonorthogonality of the accelerometers.

This paper presents a method for combining these two kinds of data to obtain improved postflight estimates of the vehicle's position and velocity during powered flight. It is assumed that errors in the inertial system cause the measured thrust acceleration vector $A_M(t)$ to be related to the true thrust acceleration $A_T(t)$ by

$$A_M(t) = (I + K)A_T(t) \tag{1}$$

where K is a constant 3×3 matrix whose elements are unknown a priori. It is shown below, subject to certain linearizing assumptions, that

$$X(t) = Y(t) + \sum_{j=1}^{9} q_j P_j(t) \tag{2}$$

where $X(t)$ is the true trajectory, and $Y(t)$ is the trajectory that would be obtained from $A_M(t)$ alone. Each of the vectors $\{P_j(t)\}$ is the solution of a 3×3 system of linear time-varying differential equations, which can be computed from $A_M(t)$. The constants $q_1, \ldots, q_9$ are unknown a priori, and cannot be determined from the accelerometer data alone. Using standard methods of weighted least squares, they are chosen so that the right side of (2) gives the best fit, of the form (2), to the ground-based measurements. As shown below, $q_1, \ldots, q_9$ are used to calculate $A_T(t)$ from $A_M(t)$, and an improved estimate of $X(t)$ is obtained by integrating the equations of motion, using the corrected thrust acceleration data.

2. Analytical Basis for the Method

Let $X(t)$ be the vehicle's position at time t, relative to an inertial frame. The equation of motion is

$$\ddot{X} = A_T + G(X) \tag{3}$$

where $G(X)$ is the acceleration due to gravity. The vector $Y(t)$, which is the position estimate based on A_M, is a solution of

$$\ddot{Y} = A_M + G(Y)$$

The error $E = X - Y$ satisfies

$$\begin{aligned} \ddot{E} &= A_T - A_M + G(X) - G(Y) \\ E(0) &= \dot{E}(0) = 0 \end{aligned} \tag{4}$$

We assume that the error committed in writing

$$G(X) - G(Y) = HE$$

where

$$(H)_{ij} = \partial g_i/\partial x_j \qquad (i, j = 1, 2, 3)$$

is negligible. Substituting this in (4) yields

$$\ddot{E} - HE = JA_M \qquad E(0) = \dot{E}(0) = 0 \tag{5}$$

where

$$J = -I + (I + K)^{-1}$$

(It is reasonable to assume that $(I + K)^{-1}$ exists; if it did not, the null space of $I + K$ would be nontrivial, and there would be a direction in which the inertial system would be unable to detect thrust accelerations.)

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* Associate Professor of Mathematics.

Denote the elements of J by

$$J = \begin{pmatrix} q_1 & q_4 & q_7 \\ q_2 & q_5 & q_8 \\ q_3 & q_6 & q_9 \end{pmatrix}$$

From (5), E can be written in the form

$$E(t) = \sum_{j=1}^{9} q_j P_j(t)$$

where

$$\dot{P}_j = HP_j + F_j \qquad P_j(0) = 0$$

If we denote the components of A_M by $(\ddot{\xi}_M, \ddot{\eta}_M, \ddot{\zeta}_M)$, the forcing functions are given by

$$F_1 = \begin{pmatrix} \ddot{\xi}_M \\ 0 \\ 0 \end{pmatrix} \qquad F_2 = \begin{pmatrix} 0 \\ \ddot{\xi}_M \\ 0 \end{pmatrix} \qquad F_3 = \begin{pmatrix} 0 \\ 0 \\ \ddot{\xi}_M \end{pmatrix}$$

F_4, F_5, F_6, and F_7, F_8, F_9 are similar, with $\ddot{\xi}_M$ replaced by $\ddot{\eta}_M$ and $\ddot{\zeta}_M$, respectively. We will refer to $P_1, \ldots, P_9$ as error functions, and to $q_1, \ldots, q_9$ as error coefficients.

Consistent with the linearizing assumption made in the derivation of (5), the derivatives appearing in H can be evaluated at points on the uncorrected trajectory (that is, the trajectory obtained from A_M). Hence, the error functions can be calculated from A_M, and if the error coefficients were known, the true trajectory could be obtained from

$$X(t) = Y(t) + \sum_{j=1}^{9} q_j P_j(t) \tag{6}$$

As an alternative, one can compute $A_T = (I + J)A_M$ and obtain the corrected trajectory from (3). This was the method used in the calculations described subsequently.

3. Estimation of the Error Coefficients: Theoretical Considerations

If estimates of $X(t)$ are available from other data sources, such as ground instrumentation, the error coefficients can be obtained by weighted least squares estimation, where the data are fitted to a linear combination of $P_1(t), \ldots, P_9(t)$. Let $\hat{X}(t_1), \ldots, \hat{X}(t_K)$ be estimates of $X(t)$ at times $t_1, \ldots, t_K$, derived from ground instruments. Assume that the errors in $\hat{X}(t_k)$ have zero mean and known covariance matrix C_k. The i, jth element of C_k is given by

$$c_{ijk} = \overline{[\hat{x}_i(t_k) - x_i(t_k)][\hat{x}_j(t_k) - x_j(t_k)]} \qquad i, j = 1, 2, 3$$

(The bar denotes expected value.) The joint covariance matrix of the errors in $\hat{X}(t_1), \ldots, \hat{X}(t_K)$ is

$$C = \begin{pmatrix} C_1 & 0 & \ldots & 0 \\ 0 & C_2 & \ldots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 \ldots & & C_K \end{pmatrix}$$

where the zeros are 3×3 matrices with zero elements, so that C is of order $3K$.

Define the $3K$-dimensional vectors

$$U = \begin{pmatrix} X(t_1) \\ \cdot \\ \cdot \\ X(t_K) \end{pmatrix} \qquad \hat{U} = \begin{pmatrix} \hat{X}(t_1) \\ \cdot \\ \cdot \\ \hat{X}(t_K) \end{pmatrix} \qquad V = \begin{pmatrix} Y(t_1) \\ \cdot \\ \cdot \\ Y(t_K) \end{pmatrix}$$

the nine-dimensional vector

$$Q = \begin{pmatrix} q_1 \\ \cdot \\ \cdot \\ \cdot \\ q_9 \end{pmatrix}$$

and the $3K$ by 9 matrix

$$P = \begin{pmatrix} P_1(t_1) & P_2(t_1) & \ldots & P_9(t_1) \\ P_1(t_2) & P_2(t_2) & \ldots & P_9(t_2) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ P_1(t_K) & P_2(t_K) & \ldots & P_9(t_K) \end{pmatrix}$$

From (6), $U = V + PQ$. For every nine-dimensional vector Q', we can define a vector, $U' = V + PQ'$, and a positive number,

$$\sigma^2 = (U' - \hat{U})^T C^{-1} (U' - \hat{U})$$

The quantity σ^2 is a function of $(q_1', \ldots, q_9')$, it can be written directly in terms of the vector Q' as follows:

$$\sigma^2(Q') = (V - \hat{U} + PQ')^T C^{-1} (V - U + PQ')$$

It can be shown that σ^2 has a unique minimum value, attained when $Q' = \tilde{Q}$, a solution of

$$\Lambda \tilde{Q} = P^T C^{-1} (\hat{U} - V) = L \tag{7}$$

where

$$\Lambda = P^T C^{-1} P$$

If Λ is nonsingular, $\tilde{Q}$ is unique; $\tilde{Q} = \Lambda^{-1} L$. If Λ is singular, there are infinitely many vectors $\tilde{Q}$ for which the same minimum value of σ^2 is attained. Since this case is of considerable interest, we will examine it further.

The singularity of Λ is equivalent to the existence of a nontrivial vector

$$Z = \begin{pmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_9 \end{pmatrix}$$

such that $\Lambda Z = 0$. From the positive definiteness of C, it follows that $\Lambda Z = 0$ if and only if $PZ = 0$ and from the definition of P, this is equivalent to

$$\sum_{j=1}^{9} z_j P_j(t_k) = 0 \qquad (k = 1, \ldots, K)$$

From this, we can conclude that either

$$\sum_{j=1}^{9} z_j P_j(t) = 0 \tag{8}$$

identically in t, or Λ can be rendered nonsingular by introducing additional ground-based data. We will assume that if Λ is singular, it is because of (8). It can be shown that (8) is satisfied if and only if A_M lies in the same inertial plane for all t; that is,

$$n_1 \ddot{\xi}_M + n_2 \ddot{\eta}_M + n_3 \ddot{\zeta}_M = 0$$

where n_1, n_2, and n_3 are the (constant) components of a normal to the plane of A_M. It follows from (1) that A_T lies in a fixed inertial plane if and only if A_M does. We will denote these planes by Γ_T and Γ_M.

It is reasonable to expect that Γ_T will be known in advance; in fact, Γ_T is often defined by

$$\zeta_T(t) = 0$$

because the ζ axis of the inertial system is erected normal to the plane of the nominal trajectory. From this, it can also be seen intuitively that $\tilde{Q}$ cannot be obtained uniquely in this situation, because, if it could, then the matrix K could be obtained uniquely. This would be absurd, since the coefficients k_{13}, k_{23}, and k_{33}, which cause errors arising from accelerations along the ζ axis, cannot be determined if there are no such accelerations. However, even though $\tilde{Q}$ is not unique, this does not preclude finding the best correction for the trajectory, since it can be shown that if $\tilde{\tilde{Q}}$ and $\tilde{Q}$ are both minimizing vectors, then

$$\sum_{j=1}^{9} (\tilde{q}_j - \tilde{\tilde{q}}_j)P_j(t) = 0$$

4. Estimation of the Error Coefficients: Practical Considerations

It is unlikely that the vector A_T will be strictly confined to a plane, since guidance errors will cause some motion of the vehicle out of the nominal plane of the trajectory. Thus, it is not likely that Λ will be exactly singular. Rather, the question of interest is how "nearly singular," or "ill conditioned" Λ is. An equivalent question is how "close" the vector A_M stays to a plane. We give a procedure that simultaneously leads to a precise definition of the latter situation, and also provides the appropriate reduction of the system (7) to a lower order, if necessary.

Let the times at which thrust accelerations are measured be of the form $t_n = 0, h, 2h, \ldots, Nh$, and define the 3×3 matrix $\Psi = (\Psi_{ij})$ by

$$\Psi_{ij} = \frac{1}{N+1}\sum_{n=0}^{N} A_i(nh)A_j(nh) \qquad (i, j = 1, 2, 3)$$

where we have adopted the notations $A_{1M} = \xi_M$, $A_{2M} = \eta_M$, $A_{3M} = \zeta_M$.

It can be shown that Ψ is symmetric and positive semidefinite, and it is singular if and only if Λ is singular. In this case,

$$n^T\Psi n = 0$$

where n is the unit normal to Γ_M.

The eigenvalues of Ψ are nonnegative. Denote them by $\nu_1^2 \geq \nu_2^2 \geq \nu_3^2 \geq 0$. Let l, m, and n be an orthonormal system of eigenvectors. Thus l, m, and n are the principal axes of Ψ and ν_1^2, ν_2^2, and ν_3^2 are the mean square values of the projection of A_M on l, m, and n, respectively.

Ψ is singular if and only if ν_3^2, its smallest eigenvalue, is equal to zero. In this case, the vector n is the normal to the plane Γ_M. However, even if Ψ is not exactly singular, a convenient measure of the "closeness" to singularity is afforded by

$$r = (\nu_1^2 + \nu_2^2)/\nu_3^2$$

which is the ratio of the mean square value of the component A_M in the plane of l and m, to the mean square value of the component perpendicular to it. If $r = \infty$, Ψ, and consequently Λ, are singular. Clearly, the bigger the value of r, the more nearly singular Ψ becomes. If $r = \infty$, or is very large, define the matrix

$$R = \begin{pmatrix} n_1 & 0 & 0 & n_2 & 0 & 0 & n_3 & 0 & 0 \\ 0 & n_1 & 0 & 0 & n_2 & 0 & 0 & n_3 & 0 \\ 0 & 0 & n_1 & 0 & 0 & n_2 & 0 & 0 & n_3 \\ l_1 & 0 & 0 & l_2 & 0 & 0 & l_3 & 0 & 0 \\ 0 & l_1 & 0 & 0 & l_2 & 0 & 0 & l_3 & 0 \\ 0 & 0 & l_1 & 0 & 0 & l_2 & 0 & 0 & l_3 \\ m_1 & 0 & 0 & m_2 & 0 & 0 & m_3 & 0 & 0 \\ 0 & m_1 & 0 & 0 & m_2 & 0 & 0 & m_3 & 0 \\ 0 & 0 & m_1 & 0 & 0 & m_2 & 0 & 0 & m_3 \end{pmatrix}$$

where the nonzero entries are the components of l, m, and n, normalized so that $||l|| = ||m|| = ||n|| = 1$. If we define $\Lambda' = R\,\Lambda\,R^T$, $\tilde{L}' = R\tilde{L}$, and $\tilde{Q}' = R\tilde{Q}$, then we can write (7) in the equivalent form

$$\Lambda'\tilde{Q}' = \tilde{L}' \tag{9}$$

If $\nu_3^2 = 0$, the first three rows of R are eigenvectors of Λ, associated with the eigenvalue zero. From this, it is not hard to verify that the elements in the first three rows and columns of Λ' are zero, as are the first three elements of L'. Hence, (9) is really a 6×6 system, since $\tilde{q}_1'$, $\tilde{q}_2'$, and $\tilde{q}_3'$ are arbitrary, whereas the other 6 components of $\tilde{Q}'$ are the solution of the system obtained by striking out the first three rows and columns of Λ', and the first three elements of $\tilde{L}'$. If we define $\tilde{Q}_0'$ to be the solution of (9) obtained in this way, with $\tilde{q}_1'$, $\tilde{q}_2'$, and $\tilde{q}_3'$ taken to be zero, then

$$\tilde{Q}_0 = R^T/\tilde{Q}_0'$$

is a vector for which the $\sigma^2(Q)$ is minimized.

5. Results with Simulated Data

To test the method presented in this paper, two computer programs were written. One was a simulation program, that accepts thrust acceleration data as input, and produces the following outputs:

a) A reference trajectory, obtained by numerical integration of the equations of motion with error-free input data. This is used as a standard in later computations.

b) A simulated time history of the output of the accelerometers. This is obtained by means of a program that introduces systematic errors into the thrust acceleration data in accordance with an error model that accounts for scale factor errors, cross coupling between accelerometer axes, nonorthogonality of the axes, misalignment, and drift. The last two types of errors do not match the assumed time-invariant error model of Eq. (1).

c) A simulated range data tape, containing estimates of the position of the vehicle as a function of time, as would be provided by ground based data. This tape is obtained by computing the local coordinates of the target, corresponding to the reference trajectory, and adding noise and bias errors. The resulting estimates of the local coordinates then are transformed into the central coordinate system, and stored on a tape that represents the input that would be provided by ground instrumentation. In addition, this tape contains a covariance matrix for each time of observation.

The output tapes described by b) and c) are used as input to a second program that combines the data from the two tapes, corrects for the systematic errors in the tape b) data, and computes a corrected trajectory.

When the errors in the inertial data were of the form (1), the corrected trajectory was found to lie extremely close to the reference trajectory. However, the more interesting result is that, if errors in the guidance system are simulated with a time-varying error model in the simulation program, the method still provides excellent corrections to the trajectory, despite the fact that it is based on a constant error model.

The fact that good corrections to the trajectory can be obtained with the constant error model (1), even in the case when the actual errors are considerably more complicated, shows that the operational program is a useful tool for obtaining improved estimates of the trajectory under practical conditions.

Several simulated applications of the method were computed, using a thrust acceleration profile for a two-stage, 600-sec powered flight of a Saturn vehicle. Tracking range data were introduced every 20 sec. We will summarize some typical runs, where we used three different error models for the inertial system, described as follows:

Table 1 Results for the three error models (perturbations) at t = 560 sec after launch

	True trajectory	Perturbed trajectories No. 1	No. 2	No. 3
x, m	1,383,877.8	1,383,593.5	1,400,570.4	1,393,474.3
y, m	403,440.1	403,436.4	421,290.2	426,477.7
z, m	27,283.8	39,161.8	30,834.3	126,126.2
Position error, m ≅		11,900	24,700	102,000
$\dot{x}$, m/sec	5,805	5,803	5,868	5,826
$\dot{y}$, m/sec	−1,076	−1,076	−1,036	−992
$\dot{z}$, m/sec	162	216	178	717
Velocity error, m/sec ≅		54	85	562
Corrected values (Tracking range 1)				
x, m		1,383,882.8	1,383,883.4	1,384,002.0
y, m		403,420.7	403,417.1	403,624.4
z, m		27,296.5	27,295.9	27,271.1
Position error, m ≅		25	27	223
$\dot{x}$, m/sec		5,805	5,805	5,808
$\dot{y}$, m/sec		−1,076	−1,076	−1,070
$\dot{z}$, m/sec		163	162	162
Velocity error, m/sec ≅		1	0	7
Corrected values (Tracking range 2)				
x				1,383,918.1
y				403,525.8
z				27,750.2
Position error =				488
$\dot{x}$				5,807
$\dot{y}$				−1,072
$\dot{z}$				166
Velocity error =				6

Case 1: Platform drift about one axis at rate of 0.00015° sec/m.

Case 2: Scale factor error on each axis of 1%, nonorthogonality of axes = 0.1°.

Case 3: Constant scale factor error on each axis of 1%, nonorthogonality of axes = 0.1°. Platform drift about all three axes at rate of 0.0015° sec/m.

Case 1 is a time-varying error model. Case 2 is a constant error model of the form (1). Case 3 is again time varying, with very large errors. This case was computed primarily to test the linearizing assumptions.

The tracking range that was simulated can be described as follows:

Ground sensors: One radar and one Azusa in the launch area, and one radar and one Azusa down range.

Errors:

Tracking range 1:

$$\sigma_R = 20\ m \qquad \sigma_A = \sigma_E = 0.3\ mr$$

$$\sigma_l = \sigma_m = 1 \times 10^{-5}$$

Tracking range 2:

$$\sigma_R = \overline{\Delta R} = 100\ m \qquad \sigma_A = \overline{\Delta A} = \sigma_E = \overline{\Delta E} = 3mr$$

$$\sigma_l = \overline{\Delta_l} = \overline{\Delta m} = \sigma_m = 1 \times 10^{-4}$$

In tracking range 2, $\overline{\Delta A}$, $\overline{\Delta E}$, $\overline{\Delta l}$, and $\overline{\Delta m}$ are constant bias errors.

The results are given in terms of a tangent plane coordinate system, with origin at the launch point. The x coordinate is positive down range, y is positive along the vertical at the launch point, and z completes the right-hand system. The results are summarized in Table 1.

Reference

[1] Drucker, A. N., "Performance analysis of rocket vehicle guidance systems," *Inertial Guidance*, edited by G. R. Pitman (John Wiley & Sons Inc., New York, 1962).