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STABILITY OF A CLASS OF DISCRETE MINIMUM VARIANCE SMOOTHING FORMULAS*

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Abstract. We study stability of midpoint smoothing formulas matched to discrete data consisting of equally spaced samples of an unknown polynomial of known maximal degree plus a random error with known spectral density. Stability is established for a class of minimum variance smoothing formulas which includes least squares and minimum R_m smoothing formulas, previously shown to be stable by T. N. E. Greville.

1. Introduction. We consider the problem of smoothing a sequence of observations,

$$v_r = f(r) + \varepsilon_r, \tag{1}$$

where f is an unknown polynomial of degree not exceeding $2k$ and $\{\varepsilon_r\}$ is a sample sequence from a real-valued stationary time series with zero mean and continuous spectral density

$$\Phi(\lambda) = \sum_{-\infty}^{\infty} \phi_r \cos r\lambda;$$

that is,

$$E(\varepsilon_j \varepsilon_{j+r}) = \phi_r.$$

We apply to (1) the smoothing formula

$$u_r = \sum_{s=-q}^{q} w_s v_{r-s}, \tag{2}$$

where the weighting coefficients $w_{-q}, \cdots, w_q$ are chosen to minimize

$$Q(w_{-q}, \cdots, w_q) = \sum_{r,s=-q}^{q} \phi_{r-s} w_r w_s$$

subject to the constraints

$$\sum_{-q}^{q} w_s s^r = \delta_{0r}, \qquad 0 \leqq r \leqq 2k. \tag{3}$$

If $\{w_{-q}, \cdots, w_q\}$ is any solution of (3) and

$$u_r^* = \sum_{s=-q}^{q} w_s v_{r-s},$$

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then

$$Eu_r^* = f(r) \tag{4}$$

whenever f is a polynomial of degree not exceeding $2k$, and

$$E(u_r^* - f(r))^2 = Q(w_{-q}, \cdots, w_q).$$

For these reasons we shall follow the convention introduced in [11], and refer to (2) as MV$(q, k; \Phi)$, which stands for "minimum variance smoothing formula, with respect to Φ, of span $2q + 1$ and degree $2k + 1$."

If $\Phi \not\equiv 0$, the constrained minimum problem has a unique solution for every q and k. Moreover, it happens that

$$w_s = w_{-s}, \tag{5}$$

so that MV$(q, k; \Phi)$ is symmetric, and (4) holds even if f is of degree $2k + 1$, rather than $2k$.

If $q \leqq k$, then (3) has only the uninteresting solution

$$w_0 = 1,$$
$$w_s = 0, \qquad s \neq 0;$$

therefore we shall assume that $q > k$.

The characteristic function of MV$(q, k; \Phi)$ is defined to be

$$C(\lambda) = \sum_{-q}^{q} w_r \cos r\lambda.$$

It follows [6] from (3) and (5) that

$$C(\lambda) = 1 + O(\lambda^{2k+2}), \qquad \lambda \to 0. \tag{6}$$

Schoenberg [5] has shown that a symmetric smoothing formula is stable under repeated application if and only if

$$|C(\lambda)| < 1, \qquad 0 < |\lambda| \leqq \pi. \tag{7}$$

(For a different interpretation of (7), see [11] and the footnote reference to Lanczos in [6].)

Results on stability of minimum variance smoothing formulas are quite limited. Greville [1] has shown that MV$(q, k; \Phi)$ is stable for all $q \geqq k + 1 \geqq 1$ if

$$\Phi(\lambda) = \sin^{2m}(\lambda/2),$$

where m is a nonnegative integer. If $m = 0$, this is equivalent to least-squares smoothing, the stability of which had been conjectured by Schoenberg; if $m \geqq 1$, it is equivalent to minimum R_m smoothing, as defined by Wolfenden [13]. Trench [11] has obtained the following result.

THEOREM 1. *Suppose* MV$(q, k; \Phi)$ *is stable for all* $q \geqq k + 1 \geqq 1$, *and let*

$$Q(x) = \frac{x^t \prod_{i=1}^{r} (1 + \theta_i x)}{\prod_{j=1}^{s} (1 - \gamma_j x)},$$

where t is a nonnegative integer, $\theta_i \geqq 0$, and $0 \leqq \gamma_j < 1$. Define

$$\eta(\lambda) = Q(\sin^2(\lambda/2))\Phi(\lambda).$$

Then $\mathrm{MV}(q, k; \eta)$ *is stable for all* $q \geqq k + 1 \geqq 1$.

Wilf [12], Lorch and Szegö [2], [3], Lorch, Muldoon and Szegö [4], and Trench [8], [9], [10] have considered related questions for continuous smoothing formulas.

In this paper we obtain sufficient conditions (Theorem 3) for stability of $\mathrm{MV}(q, k; \Phi_{\mu\nu})$, where

$$\Phi_{\mu\nu}(\lambda) = (\sin^2(\lambda/2))^{\mu}(\cos^2(\lambda/2))^{\nu}, \qquad \mu, \nu > -1/2.$$

These results are extended to more general spectral densities in Theorem 4.

2. Characteristic function of $\mathrm{MV}(q, k; \Phi_{\mu\nu})$. Throughout this paper

$$(u)_s = u(u + 1) \cdots (u + s - 1)$$

and

$$(u)^{(s)} = u(u - 1) \cdots (u - s + 1).$$

The following result reduces to Sheppard's formula for the characteristic function of minimum R_m smoothing [1], [7] when $\mu = m$ and $\nu = 0$.

THEOREM 2. *The characteristic function of* $\mathrm{MV}(q, k; \Phi_{\mu\nu})$ *is*

$$C(\lambda) = 1 - \frac{(-1)^{k+1}}{k!} \sum_{s=k+1}^{q} \frac{(-q)_s(q + \mu + \nu + 1)_s}{s(s - k - 1)!(k + \mu + 3/2)_s} \sin^{2s}(\lambda/2). \tag{8}$$

Proof. The variance of the output of $\mathrm{MV}(q, k; \Phi_{\mu\nu})$ is

$$\sigma^2 = \sum_{r,s=-q}^{q} \phi_{r-s} w_r w_s,$$

where $\{\phi_r\}$ are the Fourier coefficients of $\Phi_{\mu\nu}$. This can be written as

$$\sigma^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |C(\lambda)|^2 \Phi_{\mu\nu}(\lambda)\, d\lambda. \tag{9}$$

Since $\cos r\lambda$ is a polynomial of degree $|r|$ in

$$x = \sin^2(\lambda/2),$$

$C(\lambda)$ is a polynomial of degree q in x, which, from (6), is of the form

$$C(\lambda) = P(x) = 1 - \sum_{s=k+1}^{q} b_s x^s. \tag{10}$$

Substituting this into (9) and taking x as the new variable of integration yields

$$\sigma^2 = \frac{1}{\pi} \int_0^1 \left(1 - \sum_{s=k+1}^{q} b_s x^s\right)^2 x^{\mu - 1/2}(1 - x)^{\nu - 1/2}\, dx. \tag{11}$$

Thus, $P(x)$ (and therefore $C(\lambda)$) can be obtained by minimizing (11) with respect to $b_{k+1}, \cdots, b_q$.

We complete the proof of Theorem 2 with the following lemma.

LEMMA 1. *Suppose* $\alpha, \beta > -1$, p *is a positive integer, and* n *is a nonnegative integer. Then the minimum value of*

$$\int_0^1 (F(x))^2 x^\alpha (1-x)^\beta \, dx$$

for $F(x)$ *of the form*

$$F(x) = 1 - x^p \sum_{s=0}^{n} a_s x^s,$$

is attained with

$$F(x) = 1 - \frac{(-1)^p}{(p-1)!} \sum_{s=p}^{n+p} \frac{(-n-p)_s (n+p+\alpha+\beta+2)_s}{s(s-p)!(p+\alpha+1)_s} x^s. \tag{12}$$

Proof. Differentiating

$$\int_0^1 \left(1 - x^p \sum_{s=0}^{n} a_s x^s\right)^2 x^\alpha (1-x)^\beta \, dx \tag{13}$$

with respect to $a_0, \cdots, a_n$ and equating the results to zero yields

$$\int_0^1 x^{p+r+\alpha}(1-x)^\beta \, dx = \sum_{s=0}^{n} a_s \int_0^1 x^{2p+r+s+\alpha}(1-x)^\beta \, dx, \qquad 0 \leqq r \leqq n. \tag{14}$$

From the properties of the beta function,

$$\int_0^1 x^\xi (1-x)^\eta \, dx = \frac{\Gamma(\xi+1)\Gamma(\eta+1)}{\Gamma(\xi+\eta+2)}, \qquad \xi, \eta > -1.$$

Applying this to (14) and cancelling common factors yields

$$\frac{(p+r+\alpha+\beta+2)_p}{(p+r+\alpha+1)_p} = \sum_{s=0}^{n} \frac{(2p+r+\alpha+1)_s}{(2p+r+\alpha+\beta+2)_s} a_s, \qquad 0 \leqq r \leqq n. \tag{15}$$

Subtracting the rth equation from the $(r+1)$st and using the relationship

$$\frac{(x+1)_j}{(y+1)_j} - \frac{(x)_j}{(y)_j} = j\frac{(x+1)_{j-1}}{(y)_{j+1}}(y-x)$$

yields

$$\frac{(p+r+\alpha+\beta+3)_{p+1}}{(p+r+\alpha+1)_{p+1}} = \sum_{s=0}^{n-1} \frac{(2p+r+\alpha+2)_s}{(2p+r+\alpha+\beta+4)_s}\left[-\frac{(s+1)a_{s+1}}{p}\right], \qquad 0 \leqq r \leqq n-1. \tag{16}$$

Denote the solution of (15) more precisely by $a_{sn}(\alpha, \beta, p)$; writing (15) for $\alpha - 1$, $\beta + 1, p + 1$ and $n - 1$, and comparing the result with (16) yields

$$a_{sn}(\alpha, \beta, p) = -\frac{p}{s} a_{s-1,n-1}(\alpha - 1, \beta + 1, p + 1), \qquad 1 \leqq s \leqq n. \tag{17}$$

Given $a_{0,n-1}, \cdots, a_{n-1,n-1}$ for all α, β and p, this yields $a_{1n}, \cdots, a_{nn}$ but not a_{0n}; hence we need another recursion formula. Multiplying (15) by $(2p + r + \alpha + \beta + 2)_n/(2p + r + \alpha + 1)_n$ yields, after some manipulation,

$$\frac{(p + r + \alpha + \beta + 2)_{n+p}}{(p + r + \alpha + 1)_{n+p}} = \sum_{s=0}^{n} \frac{(2p + r + s + \alpha + \beta + 2)_{n-s}}{(2p + r + s + \alpha + 1)_{n-s}} a_{sn}(\alpha, \beta, p), \qquad 0 \leqq r \leqq n. \tag{18}$$

Subtracting the rth equation from the $(r + 1)$st yields

$$\frac{(p + r + \alpha + \beta + 3)_{n+p-1}}{(p + r + \alpha + 1)_{n+p-1}} = \sum_{s=0}^{n-1} \frac{(2p + r + s + \alpha + \beta + 3)_{n-s-1}}{(2p + r + s + \alpha + 1)_{n-s-1}} \frac{n - s}{n + p} a_{sn}(\alpha, \beta, p), \qquad 0 \leqq r \leqq n - 1.$$

Comparing this with (18) for α, $\beta + 1$, p and $n - 1$ yields

$$a_{sn}(\alpha, \beta, p) = \frac{n + p}{n - s} a_{s,n-1}(\alpha, \beta + 1, p), \qquad 0 \leqq s \leqq n - 1. \tag{19}$$

Starting from (15) with $n = 0$, induction on n using (17) and (19) implies that

$$a_{sn}(\alpha, \beta, p) = \frac{(-1)^p(-n - p)_{p+s}(n + p + \alpha + \beta + 2)_{p+s}}{(p - 1)!(s + p)s!(p + \alpha + 1)_{p+s}}, \qquad 0 \leqq s \leqq n,$$

which yields (12).

Comparing (11) and (13) shows that $P(x)$ can be obtained by setting

$$p = k + 1, \quad n = q - k - 1, \quad \alpha = \mu - 1/2, \quad \beta = \nu - 1/2 \tag{20}$$

in (12). This and (10) yield (8), which completes the proof of Theorem 2.

3. Main results. From (10), $\mathrm{MV}(q, k; \Phi_{\mu\nu})$ is stable if and only if

$$|P(x)| < 1, \qquad 0 < x \leqq 1;$$

however, it is convenient to consider the polynomial $F(x)$ defined by (12).

LEMMA 2. *If*

$$|F(1)| < 1, \tag{21}$$

then

$$|F(x)| < 1, \qquad 0 < x \leqq 1. \tag{22}$$

Therefore, $\mathrm{MV}(q, k; \Phi_{\mu\nu})$ *is stable if and only if* (21) *holds, with parameters* n, p, α *and* β *given by* (20).

Proof. From a result of Greville (see the proof of Lemma 2 of [1]), $(F(x))^2$ is interpolated at $x = 0$, at the relative extrema of $F(x)$ in $(0, 1)$, and at $x = 1$ by the polynomial

$$f(x) = 1 + \int_0^x t^{-p+1} q(t)(F'(t))^2 \, dt,$$

where

$$q(x) = \sum_{s=0}^{p-1} d_s r(s, x) x^s,$$

$$d_s = \frac{-(p-1)!(p+\alpha+1)_{p-s-1}}{s!(n+s+1)_{p-s}(n+p+\alpha+\beta+2)_{p-s}}$$

and

(23) $$r(s, x) = (3p + 2\alpha - 3s)(1-x) - (2\beta + 1)x, \qquad 0 \leqq s \leqq p - 1.$$

Clearly

$$r(s, 0) > 0, \quad \alpha > -1, \quad 0 \leqq s \leqq p - 1;$$

consequently, since $d_s < 0$, $q(x)$ is negative near $x = 0$. Moreover,

$$r'(s, x) < 0, \quad \alpha, \beta > -1, \quad 0 \leqq s \leqq p - 1,$$

so that $q(x)$ is monotone increasing. Hence $q(x)$ either remains negative for all x on $(0, 1)$ (from (23), this is true if and only if $-1 < \beta \leqq -1/2$) or changes sign exactly once, from negative to positive. In either case,

$$f(1) < 1$$

implies

$$f(x) < 1, \qquad 0 < x \leqq 1.$$

From the manner in which $f(x)$ interpolates $(F(x))^2$, it now follows that (21) implies (22), which completes the proof of Lemma 2.

The next theorem is our main result on stability of $\mathrm{MV}(q, k; \Phi_{\mu\nu})$.

THEOREM 3. (a) $\mathrm{MV}(q, 0; \Phi_{\mu\nu})$ *is stable if and only if* $-1/2 < \nu < \mu + 1$.

(b) *If* $-1/2 < \nu \leqq 1/2$, *then* $\mathrm{MV}(q, k; \Phi_{\mu\nu})$ *is stable for all* $q \geqq k + 1 \geqq 1$ *and* $\mu > -1/2$.

(c) *For each* k, μ *and* τ, $\mathrm{MV}(q, k; \Phi_{\mu\nu})$ *is stable for all* q *sufficiently large if* $-1/2 < \nu < \mu + 1$, *or unstable for all* q *sufficiently large if* $\tau > \mu + 1$.

Proof. From (12),

$$F(1) = 1 - \frac{(-1)^p}{(p-1)!} \sum_{s=p}^{n+p} \frac{(-n-p)_s (n+p+\alpha+\beta+2)_s}{s(s-p)!(p+\alpha+1)_s},$$

which can be rewritten (see the Appendix) as

(24) $$F(1) = \frac{(-1)^{n+1}(n+\beta+1)^{(n+1)}}{(n+p+\alpha+1)^{(n+1)}} \sum_{s=0}^{p-1} \frac{(n+1)_s (n+p+\alpha+\beta+2)_s}{s!(n+p+\alpha+2)_s}.$$

If $p = 1$, then

$$|F(1)| = \frac{(n+\beta+1)^{(n+1)}}{(n+\alpha+2)^{(n+1)}};$$

hence $|F(1)| < 1$ if and only if $\beta < \alpha + 1$, and (a) follows from (20) and Lemma 2.

If $\beta \leqq 0$, then (24) implies

$$
\begin{aligned}
|F(1)| &\leqq \frac{(n+1)!}{(n+p+\alpha+1)^{(n+1)}} \sum_{s=0}^{p-1} \frac{(n+1)_s}{s!} \\
&= \frac{(n+1)!(n+2)_{p-1}}{(n+p+\alpha+1)^{(n+1)}(p-1)!} \qquad (25)\\
&= \frac{(n+p)^{(n+1)}}{(n+p+\alpha+1)^{(n+1)}} < 1 \quad \text{if } \alpha > -1;
\end{aligned}
$$

hence (b) follows from (20) and Lemma 2.

(The first equality in (25) can be obtained from the identity $\sum_{r=0}^{q} (u)_r/r! = (u+1)_q/q!$.)

To prove (c), we rewrite (24) as

$$
|F(1)| = \frac{(n+\beta+1)^{(n-p+2)}}{(n+\alpha+2)^{(n-p+2)}} \cdot \left[\frac{(\beta+p-1)^{(p-1)}}{(n+p+\alpha+1)^{(p-1)}} \sum_{s=0}^{p-1} \frac{(n+1)_s(n+p+\alpha+\beta+2)_s}{s!(n+p+\alpha+2)_s} \right].
$$

The expression in brackets approaches $(\beta+p-1)^{(p-1)}/(p-1)!$ as n approaches infinity, and

$$
\lim_{n\to\infty} \frac{(n+\beta+1)^{(n-p+2)}}{(n+\alpha+2)^{(n-p+2)}} = \begin{cases} 0 & \text{if } \beta < \alpha+1, \\ \infty & \text{if } \beta > \alpha+1; \end{cases}
$$

hence (c) follows from (20) and Lemma 2.

Parts (a) and (b) of the next theorem follow from Theorems 1 and 3. Part (c) requires a minor modification of Theorem 1: namely, replacement of the phrase "for all $q \geqq k+1 \geqq 1$" by "for each fixed k and sufficiently large q." This modified version of Theorem 1 also follows from the proof given in [11].

THEOREM 4. *Let*

$$
Q(x) = \frac{x^\mu (1-x)^\nu \prod_{i=1}^{r} (1+\theta_i x)}{\prod_{j=1}^{s} (1-\gamma_j x)},
$$

where $\mu, \nu > -1/2$, $\theta_i \geqq 0$ and $0 \leqq \gamma_j < 1$. *Define*

$$
\Phi(\lambda) = Q(\sin^2(\lambda/2)).
$$

Then

(a) $\mathrm{MV}(q, 0; \Phi)$ *is stable if* $\nu < \mu + 1$.

(b) *If* $-1/2 < \nu \leqq 1/2$, *then* $\mathrm{MV}(q, k; \Phi)$ *is stable for all* $q \geqq k+1 \geqq 1$ *and* $\mu > -1/2$.

(c) *For each* k, μ *and* ν, $\mathrm{MV}(q, k; \Phi)$ *is stable for all* q *sufficiently large if* $-1/2 < \nu < \mu + 1$.

Appendix. The purpose of this Appendix is to verify (24).

LEMMA A.1. *The following is an identity in u and v:*

$$\sum_{s=0}^{m} (-1)^s \binom{m}{s} \frac{(u+v)_s}{(v)_s} = \frac{(u)^{(m)}}{(-v)^{(m)}}. \tag{A.1}$$

Proof. If $v \neq -m+1, \cdots, -1, 0$, the left side of (A.1) is a polynomial of degree m in u. Call it $Q(u)$. If $r = 0, 1, \cdots, m-1$, then

$$\begin{aligned} Q(r) &= \sum_{s=0}^{m} (-1)^s \binom{m}{s} \frac{(v+r)_s}{(v)_s} \\ &= \frac{(v+r)_{m-r}}{(v)_m} \sum_{s=0}^{m} (-1)^s \binom{m}{s} (v+s)_r = 0, \end{aligned}$$

since the last sum is the mth difference of a polynomial of degree less than m. As a polynomial in u, the right side of (A.1) has the same zeros as the left; moreover, both sides equal 1 when $u = -v$. Hence (A.1) is an identity.

LEMMA A.2. *The following is an identity in x and y:*

$$\begin{aligned} 1 - \frac{(-1)^p}{(p-1)!} \sum_{s=p}^{n+p} \frac{(-n-p)_s (x+y)_s}{s(s-p)!(y)_s} \\ = \frac{(x)^{(n+1)}}{(-y)^{(n+1)}} \sum_{s=0}^{p-1} \frac{(n+1)_s (x+y)_s}{s!(y+n+1)_s}. \end{aligned} \tag{A.2}$$

Proof. For a fixed $y \neq -n-p+1, \cdots, -1, 0$, the left side of (A.2) is a polynomial of degree $n+p$ in x. Call it $P(x)$. Then

$$P(-y-r) = 1, \qquad 0 \leqq r \leqq p-1. \tag{A.3}$$

Also, $P(x)$ can be rewritten as

$$P(x) = \frac{(-1)^{p-1}}{(y)_{n+p}} \sum_{s=0}^{n+p} (-1)^s \binom{n+p}{s} \binom{s-1}{p-1} (x+y)_s (y+s)_{n+p-s}.$$

If $r = 0, 1, \cdots, n$, then

$$P(r) = \frac{(-1)^{p-1}(y+r)_{n+p-r}}{(y)_{n+p}} \sum_{s=0}^{n+p} (-1)^s \binom{n+p}{s} \binom{s-1}{p-1} (y+s)_r,$$

which is the $(n+p)$th difference of a polynomial of degree less than $n+p$; hence

$$P(r) = 0, \qquad r = 0, 1, \cdots, n.$$

The right side of (A.2) also vanishes at $x = 0, 1, \cdots, n$. Because of (A.3), the proof will be complete if we show that the right side of (A.2) equals 1 when $x = -y-r$, $r = 0, 1, \cdots, p-1$; that is, we must show that

$$\frac{(-y-r)^{(n+1)}}{(-y)^{(n+1)}} \sum_{s=0}^{r} \frac{(n+1)_s (-r)_s}{s!(y+n+1)_s} = 1. \tag{A.4}$$

This is accomplished by rewriting the left side of (A.4) as

$$\frac{(-y-r)^{(n+1)}}{(-y)^{(n+1)}}\sum_{s=0}^{r}(-1)^s\binom{r}{s}\frac{(n+1)_s}{(y+n+1)_s},$$

and invoking (A.1) with $m = r$, $u = -y$, and $v = y + n + 1$.

Now (24) can be obtained by setting $x = n + \beta + 1$ and $y = p + \alpha + 1$ in (A.2).

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