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NONNEGATIVE AND ALTERNATING EXPANSIONS OF ONE SET OF ORTHOGONAL POLYNOMIALS IN TERMS OF ANOTHER*

WILLIAM F. TRENCH†

Abstract. Let $\{p_n(x)\}$ and $\{q_n(x)\}$ be monic polynomials orthogonal with respect to the distributions $du(x)$ and $dv(x) = w(x)\,du(x)$. Conditions are given on $w(x)$ which imply that, for all n, the coefficients in the expansion of $p_n(x)$ in terms of $q_0(x), \cdots, q_n(x)$ are nonnegative, and those in the expansion of $q_n(x)$ in terms of $p_0(x), \cdots, p_n(x)$ alternate in sign.

1. Introduction. Several recent papers have been concerned with finding conditions under which the constants $c_{0n}, c_{1n}, \cdots, c_{nn}$ in the expansion

(1) $$q_n(x) = \sum_{r=0}^{n} c_{rn}p_r(x), \qquad n = 0, 1, \cdots,$$

are all nonnegative, where $\{p_n(x)\}$ and $\{q_n(x)\}$ are suitably normalized polynomials orthogonal with respect to different distributions. Askey [1], [2], [3], Askey and Gasper [4], and Wilson [7] have obtained results on this question. Askey [3] gives references to areas in which this problem arises.

We shall say that the expansion (1) is *nonnegative* if $c_{rn} \geqq 0$ for $0 \leqq r \leqq n$, or *alternating* if $(-1)^{n-r}c_{rn} \geqq 0$ for $0 \leqq r \leqq n$. An alternating expansion can be transformed into a nonnegative expansion (and vice versa) by the renormalization

(2) $$P_n(x) = (-1)^n p_n(x), \qquad Q_n(x) = (-1)^n q_n(x), \qquad n = 0, 1, 2, \cdots.$$

2. Formulation of the problem. Throughout this paper we assume that $u(x)$ is nondecreasing and $w(x)$ nonnegative on an interval (a, b), that the distributions $du(x)$ and $dv(x) = w(x)\,du(x)$ have finite moments

$$\int_a^b x^r\,du(x) \quad \text{and} \quad \int_a^b x^r\,dv(x)$$

for all nonnegative integers r, and that $\{p_n(x)\}$ and $\{q_n(x)\}$ are the monic polynomials orthogonal over (a, b) with respect to $du(x)$ and $dv(x)$, respectively; i.e.,

(3) $$p_n(x) = x^n + \cdots, \qquad q_n(x) = x^n + \cdots,$$

and

$$\int_a^b p_n(x)p_m(x)\,du(x) = \int_a^b q_n(x)q_m(x)\,dv(x) = 0, \qquad n > m \geqq 0.$$

We shall give conditions under which the expansions

(4) $$q_n(x) = p_n(x) + \sum_{r=0}^{n-1} a_{rn}p_r(x)$$

and

(5) $$p_n(x) = q_n(x) + \sum_{r=0}^{n-1} b_{rn}q_r(x)$$

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† Department of Mathematics, Drexel University, Philadelphia, Pennsylvania 19104.

are, respectively, alternating and nonnegative for all n. (If $u(x)$ has only finitely many, say N, points of increase, the phrase "for all n" should be interpreted as "for $n = 0, 1, \cdots, N - 1$.")

3. Results. The following is a known result [6, Thm. 3.1.4, § 3.1].

LEMMA 1. *Suppose x_0 is not in (a, b) and $w(x) = |x - x_0|$. Then* (4) *and* (5) *reduce to*

$$q_n(x) = p_n(x) + \sum_{r=0}^{n-1} \frac{p_r(x_0)}{p_n(x_0)} p_r(x) \tag{6}$$

and

$$p_n(x) = q_n(x) - \frac{p_{n-1}(x_0)}{p_n(x_0)} q_{n-1}(x). \tag{7}$$

LEMMA 2. *If $-\infty < x_0 \leqq a$, then* (6) *is alternating and* (7) *is nonnegative for all n. If $b \leqq x_0 < \infty$, then* (6) *is nonnegative and* (7) *is alternating for all n.*

Proof. The roots of $p_j(x)$ are all in (a, b). Because of the normalization (3), $(-1)^j p_j(x_0) > 0$ if $x_0 \leqq a$, and $p_j(x_0) > 0$ if $x_0 \geqq b$. This yields the conclusion.

Suppose $\{p_n(x)\}$, $\{q_n(x)\}$ and $\{r_n(x)\}$ are sequences of polynomials such that, for all n, the expansion of $p_n(x)$ in terms of $q_0(x), q_1(x), \cdots, q_n(x)$ and the expansion of $q_n(x)$ in terms of $r_0(x), r_1(x), \cdots, r_n(x)$ are both alternating (nonnegative); then the expansion of $p_n(x)$ in terms of $r_0(x), r_1(x), \cdots, r_n(x)$ is also alternating (nonnegative) for all n. This and repeated application of Lemma 2 yield the following theorem.

THEOREM 1. *Let $R(a, b)$ be the set of rational functions with only real zeros and poles, which are positive on (a, b), with finite zeros, if any, confined to $(-\infty, a]$, and finite poles, if any, confined to $[b, \infty)$. If $w(x)$ is in $R(a, b)$, then* (4) *is alternating and* (5) *is nonnegative for all n.*

Example 1. The Jacobi polynomials, defined by

$$P_n^{(\alpha,\beta)}(x) = (1 - x)^{-\alpha}(1 + x)^{-\beta} \frac{(-1)^n}{2^n n!} \left(\frac{d}{dx}\right)^n [(1 - x)^{n+\alpha}(1 + x)^{n+\beta}], \quad \alpha, \beta > -1,$$

are orthogonal with respect to the distribution

$$du(x) = (1 - x)^\alpha (1 + x)^\beta \, dx, \qquad -1 < x < 1,$$

and have positive leading coefficients. From Theorem 1, the expansion

$$P_n^{(\gamma,\delta)}(x) = \sum_{r=0}^{n} A_{rn}(\alpha, \beta; \gamma, \delta) P_r^{(\alpha,\beta)}(x) \tag{8}$$

is alternating for all n if $\gamma = \alpha - r > -1$ and $\delta = \beta + s$, with r and s nonnegative integers, and nonnegative for all n if $\gamma = \alpha + r$ and $\delta = \beta - s > -1$, with r and s nonnegative integers.

For other cases in which (8) is known to be nonnegative for all n, and for a conjecture on this point, see Askey and Gasper [4].

Example 2. Askey [1] has shown that (4) is alternating for all n if $a = 0$ and $w(x) = x^\alpha$, where α is a positive integer, and has conjectured that the result remains valid if α is an arbitrary positive number. (Actually, Askey speaks of nonnegative expansions, but his normalization differs from ours as in (2).) Theorem 1 contains

Askey's result for positive integral α, and also implies that in this case (5) is nonnegative for all n. For this reason it is tempting to extend Askey's conjecture: namely, to conjecture that (4) is alternating and (5) is nonnegative for all n if $a = 0$ and $w(x) = x^\alpha$, with α an arbitrary positive number. However, this extended conjecture is false, as can be seen by taking

$$u(x) = 1, \quad w(x) = x^\alpha, \quad a = 0, \quad b = 1;$$

then straightforward computations yield

$$q_0(x) = 1,$$

$$q_1(x) = x - \frac{\alpha + 1}{\alpha + 2},$$

$$q_2(x) = x^2 - \frac{2(\alpha + 2)}{\alpha + 4}x + \frac{(\alpha + 1)(\alpha + 2)}{(\alpha + 3)(\alpha + 4)},$$

$$p_0(x) = 1,$$

$$p_1(x) = x - \tfrac{1}{2},$$

$$p_2(x) = x^2 - x + \tfrac{1}{6}.$$

Therefore,

$$p_2(x) = q_2(x) + \frac{\alpha}{\alpha + 4}q_1(x) + \frac{\alpha(\alpha - 1)}{6(\alpha + 2)(\alpha + 3)}q_0(x),$$

which is not nonnegative if $0 < \alpha < 1$.

The coefficients of $p_n(x)$ and $q_n(x)$, as well as the coefficients a_{rn} and b_{rn} in (4) and (5), are continuous functions of the moments of $du(x)$ and $dv(x)$. The next lemma follows easily from this.

LEMMA 3. *Suppose $du_m(x)$ and $dv_m(x)$ are sequences of distributions on (a, b) such that*

(9) $$\lim_{m \to \infty} \int_a^b x^r \, du_m(x) = \int_a^b x^r \, du(x), \qquad r = 0, 1, \cdots,$$

(10) $$\lim_{m \to \infty} \int_a^b x^r \, dv_m(x) = \int_a^b x^r \, dv(x), \qquad r = 0, 1, \cdots.$$

Let $\{p_{nm}(x)\}_{n=0}^\infty$ and $\{q_{nm}(x)\}_{n=0}^\infty$ be the sequences of monic polynomials orthogonal over (a, b) with respect to $du_m(x)$ and $dv_m(x)$, respectively. For each m, let the expansions

$$q_{nm}(x) = p_{nm}(x) + \sum_{r=0}^{n-1} a_{rnm} p_{rm}(x)$$

and

$$p_{nm}(x) = q_{nm}(x) + \sum_{r=0}^{n-1} b_{rnm} q_{rm}(x)$$

be, respectively, alternating and nonnegative for all n. Then (4) is alternating and (5) is nonnegative for all n.

THEOREM 2. *If $\gamma > 0$ and the distribution $dv(x) = e^{\gamma x}\, du(x)$ has moments of all orders on (a, b), then (4) is alternating and (5) is nonnegative for all n.*

Proof. If $a > -\infty$, let $du_m(x) = du(x)$ and $dv_m(x) = w_m(x)\, du(x)$, where

$$w_m(x) = e^{\gamma a}\left(1 + \frac{\gamma(x-a)}{m}\right)^m, \quad x \geqq a.$$

Then (9) is obvious and, since $w_m(x) \leqq e^{\gamma x}$ and $\lim_{m\to\infty} w_m(x) = e^{\gamma x}$, Lebesgue's bounded convergence theorem implies (10). Moreover, $w_m(x)$ is in $R(a, b)$ for every m. Thus, if a is finite, the conclusion follows from Theorem 1 and Lemma 3.

If $a = -\infty$, we again apply Lemma 3, this time with

$$u_m(x) = \begin{cases} u(x), & x \geqq -m, \\ u(-m), & x < -m, \end{cases}$$

and $dv_m(x) = e^{\gamma x}\, du_m(x)$. From the result just proved for finite a, the hypotheses of Lemma 3 are satisfied, and therefore the conclusion follows.

Example 3. Suppose $\alpha > -1$ and

$$du(x) = x^{\alpha} e^{-x}\, dx, \quad x > 0;$$

then

$$p_n(x) = (-1)^n c_n L_n^{(\alpha)}(x), \tag{11}$$

where $L_n^{(\alpha)}(x)$ is the Laguerre polynomial and $c_n > 0$ [6, § 5.1]. If $\rho > 0$, the change of variable $x = \rho y$ transforms the orthogonality condition

$$\int_0^{\infty} e^{-x} x^{\alpha} p_n(x) p_m(x)\, dx = 0, \qquad n \neq m,$$

into

$$\int_0^{\infty} e^{-\rho x} y^{\alpha} p_n(\rho y) p_m(\rho y)\, dy = 0, \qquad n \neq m;$$

hence, the monic polynomials $q_n(x) = \rho^{-n} p_n(\rho x)$, $n = 0, 1, \cdots$, are orthogonal over $(0, \infty)$ with respect to the distribution

$$dv(x) = e^{-(\rho-1)x}\, du(x).$$

Bearing in mind the difference in normalization indicated in (11), we conclude from Theorem 2 that the expansion

$$L_n^{(\alpha)}(\rho x) = \sum_{r=0}^{n} A_{rn}^{(\alpha)}(\rho) L_r^{(\alpha)}(x)$$

is nonnegative for all n if $0 < \rho < 1$, and alternating for all n if $\rho > 1$. This is a known result; see [5, § 119].

Example 4. If

$$du(x) = e^{-x^2}\, dx, \qquad -\infty < x < \infty,$$

then

$$p_n(x) = d_n H_n(x),$$

where $H_n(x)$ is the nth Hermite polynomial and $d_n > 0$ [6, § 5.5]. The change of variable $x = y - x_0$ transforms the orthogonality condition

$$\int_{-\infty}^{\infty} e^{-x^2} p_n(x) p_m(x)\, dx = 0, \qquad m \neq n,$$

into

$$\int_{-\infty}^{\infty} e^{-(y-x_0)^2} p_n(y - x_0) p_m(y - x_0)\, dy, \qquad m \neq n;$$

hence, the monic polynomials $q_n(x) = p_n(x - x_0)$, $n = 0, 1, \cdots$, are orthogonal over $(-\infty, \infty)$ with respect to the distribution

$$dv(x) = e^{2x_0 x}\, du(x).$$

It follows from Theorem 2 that the expansion

$$H_n(x - x_0) = \sum_{r=0}^{n} K_{rn}(x_0) H_r(x)$$

is alternating for all n if $x_0 > 0$, and nonnegative for all n if $x_0 < 0$. This is also a known result; see [6, Prob. 68, p. 385].

We conclude with the following theorem, which can be obtained from Theorem 1, Lemma 3 and Theorem 2.

THEOREM 3. *Suppose* $-\infty < a < b < \infty$, *and let*

$$w(x) = e^{\gamma x} \frac{(x - a)^m \prod_{r=1}^{\infty}[1 + c_r(x - a)]}{(b - x)^n \prod_{s=1}^{\infty}[1 - d_s(x - b)]}, \tag{12}$$

where m *and* n *are nonnegative integers,* $\gamma \geqq 0$, $c_r \geqq 0$, $d_s \geqq 0$, $\sum_1^{\infty} c_r < \infty$, *and* $\sum_1^{\infty} d_s < \infty$. *If the distribution* $dv(x) = w(x)\, du(x)$ *has moments of all orders on* (a, b), *then* (4) *is alternating and* (5) *is nonnegative for all* n.

Remark. If $-\infty = a < b < \infty$, a similar result holds with (12) replaced by

$$w(x) = e^{\gamma x}(b - x)^{-n}\left(\sum_{s=1}^{\infty} [1 - d_s(x - b)]\right)^{-1}.$$

If $-\infty < a < b = \infty$, the appropriate form for $w(x)$ is

$$w(x) = e^{\gamma x}(x - a)^m \sum_{r=1}^{\infty} [1 + c_r(x - a)].$$

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