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ASYMPTOTIC INTEGRATION OF LINEAR DIFFERENTIAL EQUATIONS SUBJECT TO INTEGRAL SMALLNESS CONDITIONS INVOLVING ORDINARY CONVERGENCE*

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Abstract. The problem of asymptotic behavior of solutions of an nth order linear differential equation is reconsidered, and a result obtained by Hartman under integral smallness conditions requiring absolute integrability is shown to hold with most of the conditions stated in terms of ordinary integrability. Results of Fubini and Halanay for linear perturbations of nonoscillatory second order equations are similarly extended.

1. Introduction. We study the behavior as $t \to \infty$ of solutions of the scalar equation

$$x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = 0, \qquad t > 0, \tag{1}$$

with $n \geqq 2$. Except when stated otherwise, all functions are permitted to be complex-valued; t is a real variable throughout.

Our main result is the following theorem.

THEOREM 1. *If* $p_1, \cdots, p_n \in C[0, \infty)$,

$$\int_0^\infty |p_1(t)|t^q \, dt < \infty, \tag{2}$$

and the integrals

$$\int_0^\infty p_k(t)t^{q+k-1} \, dt, \qquad 2 \leqq k \leqq n, \tag{3}$$

converge—perhaps conditionally—for some $q > 0$, *then* (1) *has solutions* $x_0, \cdots, x_{n-1}$ *which satisfy*

$$x_r^{(j)}(t) = \begin{cases} \dfrac{t^{r-j}}{(r-j)!}(1 + o(t^{-q})), & 0 \leqq j \leqq r, \\ o(t^{r-j-q}), & r+1 \leqq j \leqq n-1. \end{cases} \tag{4}$$

Hartman [4, Thm. 17.1, p. 315] has shown that the conclusion of Theorem 1 holds if

$$\int_0^\infty |p_k(t)|t^{q+k-1} \, dt < \infty, \qquad 1 \leqq k \leqq n,$$

for some $q \geqq 0$, and Hartman and Wintner [5] had earlier obtained the result for $q = 0$. (For a history of the problem with $q = 0$, see [4, p. 321].) The contribution here is that ordinary—rather than absolute—convergence is sufficient in (3) if $q > 0$.

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A special case of Hartman's result, due to Hille [6, part of Thm. 3], is that if $\int_0^\infty t^2|p(t)|\,dt < \infty$, then

$$y'' + p(t)y = 0$$

has solutions x_0 and x_1 such that $\lim_{t\to\infty} x_0(t) = 1$ and $\lim_{t\to\infty} (x_1(t) - t) = 0$. Theorem 1 shows that this conclusion holds even if $\int_0^t t^2 p(t)\,dt$ converges conditionally.

2. Proof of Theorem 1. To avoid unnecessary subscripts, we let r be a fixed integer ($0 \leqq r \leqq n - 1$) throughout. For convenience, let

$$Mx = \sum_{k=1}^{n} p_k x^{(n-k)} \tag{5}$$

(thus, (1) can be written as $x^{(n)} + Mx = 0$) and define the transformation $y = Tx$ by

$$y(t) = \frac{t^r}{r!} + \int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!}(Mx)(s)\,ds \tag{6}$$

if $0 \leqq r \leqq q$, or by

$$y(t) = \frac{t^r}{r!} + \int_{t_0}^t \frac{(t-\lambda)^{r-[q]-1}}{(r-[q]-1)!}\,d\lambda \int_\lambda^\infty \frac{(\lambda-s)^{n-r+[q]-1}}{(n-r+[q]-1)!}(Mx)(s)\,ds \tag{7}$$

if $q < r \leqq n - 1$. Here $[q]$ is the integer part of q and $t_0 \geqq 0$.

Under the hypotheses of Theorem 1, we will show that T maps the space $V[t_0, \infty)$, consisting of functions in $C^{n-1}[t_0, \infty)$ and satisfying

$$x^{(j)}(t) = O(t^{r-j}), \qquad 0 \leqq j \leqq n-1, \tag{8}$$

and

$$(t^{j-r}x^{(j)}(t))' = O(t^{-q-1}), \qquad 0 \leqq j \leqq n-2, \tag{9}$$

into itself, and is a contraction mapping with respect to the norm

$$\|x\| = \sup_{t \geqq t_0}\left\{\sum_{j=0}^{n-1} t^{j-r}|x^{(j)}(t)| + t^{q+1}\sum_{j=0}^{n-2}|(t^{j-r}x^{(j)}(t))'|\right\} \tag{10}$$

if t_0 is sufficiently large. (The condition (8) is partially redundant, since (9) and the condition that $x(t) = O(t^r)$ imply (8); however, it is convenient in the following proof to define $\|x\|$ as in (10).) Since $V[t_0, \infty)$ is a Banach space under this norm, it will then follow from the contraction mapping principle [1, p. 11] that T has a fixed point (function) which, we will show, is essentially the solution of (1) which satisfies (4).

Throughout the rest of the paper, it is to be understood that all estimates hold for $t \geqq t_0$.

The following lemma is the key to the proof of Theorem 1.

LEMMA 1. *Suppose the hypotheses of Theorem 1 hold and $x \in V[t_0, \infty)$. Then*

$$\left|\int_t^\infty \frac{(t-s)^i}{i!}(Mx)(s)\,ds\right| \leqq \|x\|m(t)t^{i-n-q+r+1}, \qquad 0 \leqq i \leqq n+q-r-1, \tag{11}$$

where m is continuous on $(0, \infty)$, decreases monotonically to zero as $t \to \infty$, and does not depend on x or t_0.

Proof. First observe that

$$(12) \qquad \left| \int_t^\infty s^j p_1(s) x^{(n-1)}(s)\, ds \right| \leqq \|x\| E_1(t) t^{j-n-q+r+1}, \qquad 0 \leqq j \leqq n+q-r-1,$$

where

$$E_1(t) = \int_t^\infty s^q |p_1(s)|\, ds,$$

which exists, because of (2). For $k = 2, \cdots, n-1$, define

$$e_k(t) = \int_t^\infty s^{k+q-1} p_k(s)\, ds,$$

which exists because of the assumed convergence of (3). Then

$$\int_t^{t_1} s^{n+q-r-1} p_k(s) x^{(n-k)}(s)\, ds = -\int_t^{t_1} e_k'(s) s^{n-k-r} x^{(n-k)}(s)\, ds,$$

which, by integration by parts, equals

$$-e_k(s) s^{n-k-r} x^{(n-k)}(s) \Big|_t^{t_1} + \int_t^{t_1} e_k(s)(s^{n-k-r} x^{(n-k)}(s))'\, ds.$$

This converges to a finite limit as t_1 approaches ∞, since

$$|e_k(t_1) t_1^{n-k-r} x^{(n-k)}(t_1)| \leqq \|x\|\, |e_k(t_1)|,$$

$$|e_k(s)(s^{n-k-r} x^{(n-k)}(s))'| \leqq \|x\|\, |e_k(s)| s^{-q-1},$$

and $\lim_{t\to\infty} e_k(t) = 0$. Therefore, the integral

$$I_k(t) = \int_t^\infty s^{n+q-r-1} p_k(s) x^{(n-k)}(s)\, ds$$

$$= e_k(t) t^{n-k-r} x^{(n-k)}(t) + \int_t^\infty e_k(s)(s^{n-k-r} x^{(n-k)}(s))'\, ds$$

converges and satisfies

$$(13) \qquad |I_k(t)| \leqq \|x\| E_k(t),$$

where

$$E_k(t) = (1 + t^{-q}/q) \sup_{s \geqq t} |e_k(s)|,$$

because, from (10),

$$|t^{n-k-r} x^{(n-k)}(t)| \leqq \|x\|$$

and

$$\left| \int_t^\infty e_k(s)(s^{n-k-r} x^{(n-k)}(s))'\, ds \right| \leqq \|x\| \left(\sup_{s \geqq t} |e_k(s)|\right) \int_t^\infty s^{-q-1}\, ds.$$

(Here we need the assumption that $q > 0$.)

Now, if $0 \leqq j < n + q - r - 1$,

$$\int_t^\infty s^j p_k(s) x^{(n-k)}(s)\,ds = -\int_t^\infty s^{j-n-q+r+1} I_k'(s)\,ds$$

$$= I_k(t)t^{j-n-q+r+1} + (j - n - q + r + 1)\int_t^\infty I_k(s)s^{j-n-q+r}\,ds$$

and, because of (13) and the obvious monotonicity of E_k,

$$\left|\int_t^\infty s^j p_k(s) x^{(n-k)}(s)\,ds\right| \leqq 2\|x\| E_k(t) t^{j-n-q+r+1}, \quad 0 \leqq j < n + q - r - 1. \tag{14}$$

This inequality also holds for $j = n + q - r - 1$, because then the integral on the left is just $I_k(t)$ (cf. (13)). Now, from (5), (12) and (14),

$$\left|\int_t^\infty s^j (Mx)(s)\,ds\right| \leqq \|x\|\left(E_1(t) + 2\sum_{k=2}^n E_k(t)\right) t^{j-n-q+r+1},$$

$$0 \leqq j \leqq n + q - r - 1,$$

and so (11) holds, with

$$m(t) = 2^{n+q-r-1}\left(E_1(t) + 2\sum_{k=2}^n E_k(t)\right).$$

Since $E_1, \cdots, E_n$ all decrease monotonically to zero as $t \to \infty$, this completes the proof of Lemma 1.

Returning to the proof of Theorem 1, we consider two cases.

Case 1. Suppose $0 \leqq r \leqq q$. Then Lemma 1 implies that the integral

$$H(t) = \int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!}(Mx)(s)\,ds \tag{15}$$

is defined whenever $x \in V[t_0, \infty)$, and that

$$|H^{(j)}(t)| \leqq \|x\| m(t) t^{r-j-q}, \qquad 0 \leqq j \leqq n - 1. \tag{16}$$

Since (6) can be rewritten as

$$y(t) = \frac{t^r}{r!} + H(t),$$

(16) implies that

$$y^{(j)}(t) = O(t^{r-j}), \qquad 0 \leqq j \leqq n - 1. \tag{17}$$

Moreover,

$$(t^{j-r}y^{(j)}(t))' = (t^{j-r}H^{(j)}(t))' = t^{j-r-1}[(j-r)H^{(j)}(t) + tH^{(j+1)}(t)],$$

$$0 \leqq j \leqq n - 2,$$

so (16) also implies that

(18) $$|(t^{j-r}y^{(j)}(t))'| = |(t^{j-r}H^{(j)}(t))'| \leqq \|x\|(|j-r|+1)m(t)t^{-q-1}, \qquad 0 \leqq j \leqq n-2,$$

which, with (17), implies that $y \in V[t_0, \infty)$; thus, T maps $V[t_0, \infty)$ into itself. If $\tilde{x}, \tilde{\tilde{x}} \in V[t_0, \infty)$, then

$$T\tilde{x}(t) - T\tilde{\tilde{x}}(t) = \int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!}[M(\tilde{x} - \tilde{\tilde{x}})](s)\, ds$$

and, by setting $x = \tilde{x} - \tilde{\tilde{x}}$ in (15) and using (16), (18) and the monotonicity of m, we find that

$$\|T\tilde{x} - T\tilde{\tilde{x}}\| \leqq \|\tilde{x} - \tilde{\tilde{x}}\| m(t_0)\left(nt_0^{-q} + \sum_{j=0}^{n-2}(|j-r|+1)\right).$$

Since $m(t) = o(1)$, this implies that

$$\|T\tilde{x} - T\tilde{\tilde{x}}\| < \tfrac{1}{2}\|\tilde{x} - \tilde{\tilde{x}}\|$$

if t_0 is sufficiently large. Hence, T is a contraction mapping of $V[t_0, \infty)$ into itself, and therefore has a unique fixed point (function) x_r such that $Tx_r = x_r$; i.e.,

$$x_r(t) = \frac{t^r}{r!} + \int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!}(Mx_r)(s)\, ds.$$

Clearly, x_r satisfies (1) on (t_0, ∞), and it can therefore be extended as a solution of (1) over $(0, \infty)$. That x_r satisfies (4) can be seen from (16), with $x = x_r$ in (15).

Case 2. Suppose $q < r$. Then Lemma 1 implies that the integral

(19) $$g(t) = \int_t^\infty \frac{(t-s)^{n-r+[q]-1}}{(n-r+[q]-1)!}(Mx)(s)\, ds$$

is defined whenever $x \in V[t_0, \infty)$, and that

(20) $$|g^{(i)}(t)| \leqq \|x\| m(t) t^{[q]-q-i}, \qquad 0 \leqq i \leqq n-r+[q]-1.$$

Now (7) can be rewritten as

$$y(t) = \frac{t^r}{r!} + G(t),$$

where

(21) $$G^{(j)}(t) = \frac{1}{(r-[q]-j-1)!}\int_{t_0}^t (t-\lambda)^{r-[q]-j-1} g(\lambda)\, d\lambda, \qquad 0 \leqq j \leqq r-[q]-1,$$

and

(22) $$G^{(j)}(t) = g^{(j-r+[q])}(t), \qquad r-[q] \leqq j \leqq n-1.$$

From (20) (with $i = 0$), (21) and the monotonicity of m,

$$|G^{(j)}(t)| \leqq \frac{t^{r-[q]-j-1}}{(r-[q]-j-1)!}\int_{t_0}^{t}|g(\lambda)|\,d\lambda$$

$$\leqq \frac{\|x\|m(t_0)t^{r-[q]-j-1}}{(r-[q]-j-1)!}\int_{0}^{t}\lambda^{[q]-q}\,d\lambda \tag{23}$$

$$= \frac{\|x\|m(t_0)t^{r-j-q}}{(1+[q]-q)(r-[q]-j-1)!}, \qquad 0 \leqq j \leqq r-[q]-1.$$

From (20) (with $i = j - r + [q]$) and (22),

$$|G^{(j)}(t)| \leqq \|x\|m(t)t^{r-j-q}, \qquad r-[q] \leqq j \leqq n-1. \tag{24}$$

Using (23) and (24) and a computation similar to that of Case 1, it is straightforward to verify that y, as defined by (7), is in $V[t_0, \infty)$ and that T is a contraction mapping if t_0 is sufficiently large. The function left fixed by T satisfies

$$x_r(t) = \frac{t^r}{r!} + \int_{t_0}^{t}\frac{(t-\lambda)^{r-[q]-1}}{(r-[q]-1)!}\,d\lambda\int_{\lambda}^{\infty}\frac{(\lambda-s)^{n-r+[q]-1}}{(n-r+[q]-1)!}(Mx_r)(s)\,ds, \tag{25}$$

and so is a solution of (1) on (t_0, ∞), and can be extended as such over $(0, \infty)$. Since the integral on the right of (25) is $G(t)$ (cf. (21)), with $x = x_r$ in (19), it is clear from (24) that x_r satisfies (4) for $r - [q] \leqq j \leqq n - 1$. The same conclusion cannot be obtained from (23) for $0 \leqq j \leqq r - [q] - 1$, since the last member of (23) is $O(t^{r-j-q})$ rather than $o(t^{r-j-q})$; hence, a different analysis is needed for this case, as follows. Again let $x = x_r$ in (19). From (20) (with $i = 0$) and (21),

$$|G^{(j)}(t)| \leqq \frac{\|x_r\|t^{r-[q]-j-1}}{(r-[q]-j-1)!}\int_{t_0}^{t}m(\lambda)\lambda^{[q]-q}\,d\lambda,$$

$$0 \leqq j \leqq r-[q]-1;$$

hence

$$|t^{j-r+q}G^{(j)}(t)| \leqq \frac{\|x_r\|t^{q-[q]-1}}{(r-[q]-j-1)!}\int_{t_0}^{t}m(\lambda)\lambda^{[q]-q}\,d\lambda, \tag{26}$$

which shows that (4) also holds if $0 \leqq j \leqq r - [q] - 1$, since the right side of (26) approaches zero as $t \to \infty$. (This is obvious if the integral converges, and it follows from l'Hôpital's rule if it diverges, since $m(t) = o(1)$.)

This completes the proof of Theorem 1.

3. A related result.

THEOREM 2. *Suppose* $p_1, \cdots, p_n \in C[0, \infty)$ *and* r *is a fixed integer,* $0 \leqq r \leqq n - 2$. *Then* (1) *has a solution* x_r *satisfying* (4) *if*

$$\int_0^{\infty}|p_1(t)|\,dt < \infty, \tag{27}$$

$$\int_0^{\infty}p_k(t)t^{k-1}\,dt \quad \textit{exists for } 2 \leqq k \leqq n-r-1, \tag{28}$$

and, for some $q > 0$,

$$\int_0^\infty p_k(t)t^{q+k-1}\,dt \quad \textit{exists for } n - r \leqq k \leqq n. \tag{29}$$

We omit the proof of this theorem, which is very similar to that of Theorem 1. The essential difference is the need to restrict further the domain $V[t_0, \infty)$ of the transformation T, by defining $V[t_0, \infty)$ to be the subset of $C^{n-1}[t_0, \infty)$ consisting of functions x which satisfy (9) and

$$x^{(j)}(t) = \begin{cases} O(t^{r-j}), & 0 \leqq j \leqq r, \\ O(t^{r-j-q}), & r+1 \leqq j \leqq n-1, \end{cases}$$

instead of (8), and defining $\|x\|$ by

$$\|x\| = \sup_{t \geqq t_0} \left\{ \sum_{j=0}^{r} t^{j-r}|x^{(j)}(t)| + \sum_{j=r+1}^{n-1} t^{j-r+q}|x^{(j)}(t)| + t^{q+1} \sum_{j=0}^{n-2} |(t^{j-r}x^{(j)}(t))'| \right\},$$

instead of (10). The other changes required to adapt the proof of Theorem 1 to that of Theorem 2 stem naturally from these and the differences between the hypotheses of the two theorems.

Hartman [4, p. 315] has shown that the conclusions of Theorem 2 hold if the integrals in (27), (28) and (29) all converge absolutely.

The essential difference between the conclusions of Theorems 1 and 2 is this: the former states that (1) has a fundamental system $\{x_0, \cdots, x_{n-1}\}$ consisting of functions which satisfy (4), while the latter implies that (1) has a "partial" system of $r + 1$ $(<n)$ solutions $\{x_0, \cdots, x_r\}$ such that

$$x_i^{(j)}(t) = \begin{cases} \dfrac{t^{i-j}}{(i-j)!}(1 + o(t^{-q})), & 0 \leqq j \leqq i, \\[2ex] o(t^{i-j-q}), & i+1 \leqq j \leqq n-1, \end{cases}$$

for $0 \leqq i \leqq r$.

4. Linear perturbations of a nonoscillatory second order equation. We now apply Theorem 1 to obtain a result on the asymptotic behavior of solutions of

$$(r(t)x')' + g(t)x = 0, \qquad t > 0, \tag{30}$$

considered as a perturbation of

$$(r(t)y')' + f(t)y = 0, \qquad t > 0, \tag{31}$$

which is assumed to be nonoscillatory. In this case it is known [4, p. 355] that (31) has solutions y_0 and y_1 such that

$$y_0(t) > 0 \quad \text{and} \quad y_1(t) > 0, \qquad t \geqq \bar{t} \text{ (for some } \bar{t}), \tag{32}$$

$$r(y_0y_1' - y_0'y_1) = 1 \tag{33}$$

and

$$\lim_{t\to\infty} \frac{y_1(t)}{y_0(t)} = \infty. \tag{34}$$

THEOREM 3. *Suppose r, f, and g are continuous, $r > 0$, and f is real-valued on $[0, \infty)$. Let (31) be nonoscillatory on $(0, \infty)$, suppose y_0 and y_1 are solutions of (31) which satisfy (32), (33) and (34), and suppose*

$$\int_0^\infty (g(t) - f(t))(y_1(t))^{q+1}(y_0(t))^{-q+1}\,dt$$

converges—perhaps conditionally—for some $q > 0$. Then (30) has solutions x_0 and x_1 such that

$$\begin{aligned} x_0(t) &= y_0(t)(1 + o(s^{-q})), \\ x_0'(t) &= y_0'(t)(1 + o(s^{-q})) + y_1'(t)o(s^{-q-1}), \end{aligned} \tag{35}$$

and

$$\begin{aligned} x_1(t) &= y_1(t)(1 + o(s^{-q})), \\ x_1'(t) &= y_1'(t)(1 + o(s^{-q})) + y_0'(t)o(s^{-q+1}), \end{aligned} \tag{36}$$

where

$$s = s(t) = \frac{y_1(t)}{y_0(t)}, \qquad t > \bar{t}. \tag{37}$$

Proof. From (33),

$$s'(t) = \frac{1}{r(t)(y_0(t))^2} > 0, \qquad t \geqq \bar{t}, \tag{38}$$

so (34) implies that $s = s(t)$ maps $[\bar{t}, \infty)$ one-to-one onto $[s(\bar{t}), \infty)$. By rewriting (30) as

$$(r(t)x')' + f(t)x + (g(t) - f(t))x = 0$$

and making the change of variables $s = s(t)$ and $u(s) = x(t)/y_0(t)$, it is straightforward to verify that (30) is equivalent to

$$\frac{d^2u}{ds^2} + p(s)u = 0, \tag{39}$$

with

$$p(s) = r(t)(y_0(t))^4(g(t) - f(t)), \qquad (s = s(t)).$$

From (37) and (38),

$$\int_{s(\bar{t})}^\infty s^{q+1}p(s)\,ds = \int_{\bar{t}}^\infty (g(t) - f(t))(y_1(t))^{q+1}(y_0(t))^{-q+1}\,dt,$$

which exists for some $q > 0$, by assumption; hence Theorem 1 implies that (39) has solutions u_0 and u_1 such that

$$u_0(s) = 1 + o(s^{-q}), \qquad \frac{du_0(s)}{ds} = o(s^{-q-1}),$$

and

$$u_1(s) = s(1 + o(s^{-q})), \qquad \frac{du_1(s)}{ds} = 1 + o(s^{-q}).$$

Now let $x_i(t) = y_0(t)u_i(s(t))$ $(i = 0, 1)$; then x_0 and x_1 are solutions of (30), and elementary manipulations (which make use of (33), (37) and (38)) show that they satisfy (35) and (36).

Halanay [3] obtained the conclusion of Theorem 3 for $r \equiv 1$ and $q = 1$ under the stronger assumption that

$$\int_0^\infty |g(t) - f(t)|(y_1(t))^2 \, dt < \infty.$$

He also obtained the conclusions of Theorem 3 for $r \equiv 1$ and $q = 0$ by assuming that

$$\int_0^\infty |g(t) - f(t)| y_0(t) y_1(t) \, dt < \infty; \tag{40}$$

of course, Theorem 3 does not improve on this, because it applies only if $q > 0$. (Hartman and Wintner obtained a similar result for $q = 0$, under an assumption weaker than (40); cf. [4, Thm. 9.1, p. 379].)

By considering

$$x'' - x + P(t)x = 0 \tag{41}$$

as a perturbation of $y'' - y = 0$ and taking $a = 2q$, we obtain the following corollary to Theorem 3.

COROLLARY 1. *If $P \in C[t_0, \infty)$ and $\int_0^\infty P(t) e^{at} \, dt$ converges—perhaps conditionally—for some $a > 0$, then* (41) *has solutions x_0 and x_1 such that*

$$\begin{aligned} x_0^{(j)}(t) &= (-1)^j e^{-t}(1 + o(e^{-at})), \\ x_1^{(j)}(t) &= e^t(1 + o(e^{-at})), \end{aligned} \qquad j = 0, 1. \tag{42}$$

This corollary contains a result obtained by Fubini [2] for $a = 2$, under the stronger assumption that $\int_0^\infty |P(t)| e^{2t} \, dt < \infty$; however, Fubini did not specify the order of convergence in (42).

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