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# ORTHOGONAL POLYNOMIAL EXPANSIONS WITH NONNEGATIVE COEFFICIENTS*

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**Abstract.** This paper presents general theorems implying that the coefficients in the expansions of one set of orthogonal polynomials in terms of another are positive or nonnegative. These theorems imply several results, previously obtained by special arguments, for the classical orthogonal polynomials. A special case of one of the theorems settles affirmatively a conjecture of Askey.

**1. Introduction.** The question of when the coefficients in the expansions

$$q_n(x) = \sum_{r=0}^{n} a_{rn} p_r(x) \tag{1}$$

of one set of orthogonal polynomials in terms of another are nonnegative has been studied in several recent papers (for examples and applications, see the references); in addition, there are several older results on this question for the classical orthogonal polynomials. Most of these results have been obtained by special arguments, often involving explicit computation of the coefficients. Askey [3] and Askey and Gasper [5] observed as recently as 1971 that there were only two general theorems [4], [11] implying nonnegativity of the coefficients in (1), and that many of the classical results had not been shown to follow from them. Since then, the author [8] has considered the case where $\{p_n(x)\}$ and $\{q_n(x)\}$ are orthogonal with respect to distributions $du(x)$ and $dv(x) = w(x)\,du(x)$, and has given conditions on $w(x)$ which imply that the coefficients in (1) are nonnegative for all $n$, while those in the "inverse" expansions

$$p_n(x) = \sum_{r=0}^{n} b_{rn} q_r(x)$$

alternate in sign; i.e., $(-1)^{n-r} b_{rn} \geqq 0$.

**2. Main results.** Here we present general theorems which imply several known results on the classical polynomials. Our starting point is the following lemma.

LEMMA 1. *For $s = 0, 1, \cdots, n$, let $p_s(x)$ be a polynomial of degree $s$ with $s$ roots in an interval $(a, b)$. Suppose $x_0 \notin (a, b)$, $p_s(x_0) > 0$ $(0 \leqq s \leqq n)$, and $Q(x)$ is a polynomial of degree $n$. Then*

$$Q(x) = \sum_{s=0}^{n} c_s p_s(x), \qquad c_s \geqq 0, \quad 0 \leqq s \leqq n, \tag{2}$$

*if there is a distribution function $F(x)$ with at least $n + 1$ points of increase in $(a, b)$ such that $\int_a^b x^k\, dF(x)$ exists for $0 \leqq k \leqq 2n$,*

$$(-1)^j \int_a^b |x - x_0|^j p_s(x)\, dF(x) \leqq 0, \qquad 0 \leqq j < s \leqq n, \tag{3}$$

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*and*

$$(-1)^j \int_a^b |x - x_0|^j Q(x)\, dF(x) \geqq 0, \qquad 0 \leqq j \leqq n. \tag{4}$$

*Moreover,* $c_s > 0$ $(0 \leqq s \leqq n)$ *if at least one of the inequalities in* (3) *is strict for each* $s$.

*Proof.* Descartes' rule of signs implies that

$$p_r(x) = \sum_{j=0}^{r} p_{jr}|x - x_0|^j, \qquad a < x < b,$$

with

$$(-1)^j p_{jr} > 0; \tag{5}$$

therefore (3) and symmetry imply that

$$\int_a^b p_r(x)p_s(x)\, dF(x) \leqq 0, \qquad 0 \leqq r, s \leqq n, \quad r \neq s, \tag{6}$$

and (4) implies that

$$\int_a^b p_r(x)Q(x)\, dF(x) \geqq 0, \qquad 0 \leqq r \leqq n. \tag{7}$$

From (2), $c_0, \cdots, c_n$ satisfy the system

$$\int_a^b p_r(x)Q(x)\, dF(x) = \sum_{s=0}^{n} c_s \int_a^b p_r(x)p_s(x)\, dF(x), \qquad 0 \leqq r \leqq n,$$

which, since $F(x)$ has at least $n + 1$ points of increase, has a positive definite Gram matrix $G$ with nonpositive off-diagonal elements (cf. (6)). Stieltjes [7] (see also [10, § 3.5]) showed that the inverse of such a matrix is nonnegative. This and (7) imply that $c_s \geqq 0$, $0 \leqq s \leqq n$. If at least one of the inequalities in (3) is strict for each $s$, then all of those in (6) are strict because of (5), and so the off-diagonal elements of $G$ are negative; in this case, Stieltjes' result implies that $G^{-1} > 0$. Since at least one of the inequalities in (7) must be strict, it then follows that $c_s > 0$, $0 \leqq s \leqq n$.

The idea of applying Stieltjes' theorem here came from a paper by M. W. Wilson [11].

Except where stated otherwise, we assume throughout the rest of the paper that $\{p_r(x)\}$ and $\{q_r(x)\}$ are orthogonal over a finite or semi-infinite interval $(a, b)$ with respect to distributions $du(x)$ and $dv(x)$, respectively, and normalized so as to be positive at some point $x_0 \notin (a, b)$. It is to be understood that the distributions have enough moments and points of increase so that the polynomials are defined and unique up to normalization.

For convenience below, we state the following obvious "principle of composition": *If* $p_k(x)$, $q_k(x)$ *and* $r_k(x)$ *are polynomials of exact degree* $k = 0, 1, 2, \cdots$ *such that*

$$q_k(x) = \sum_{i=0}^{k} a_{ik}p_i(x)$$

*and*

$$r_k(x) = \sum_{i=0}^{k} b_{ik} q_i(x),$$

*with*

(8) (a) $a_{ik} \geqq 0$, (b) $b_{ik} \geqq 0$, $0 \leqq i \leqq k \leqq n$,

*then*

$$r_k(x) = \sum_{i=0}^{k} c_{ik} p_i(x),$$

*with*

(9) $c_{ik} \geqq 0, \quad 0 \leqq i \leqq k \leqq n;$

*moreover, if the inequalities in either* (8a) *or* (8b) *are all strict, then so are those in* (9).

THEOREM 1. *If*

(10) $$(-1)^j \int_a^b |x - x_0|^j p_s(x)\, dv(x) \leqq 0, \qquad 0 \leqq j < s \leqq n,$$

*then* $a_{rn} \geqq 0$ *in* (1); *moreover,* $a_{rn} > 0$ *if at least one of the inequalities in* (10) *is strict for each s.*

*Proof.* The polynomials $\{p_s(x)\}$ satisfy the conditions of Lemma 1, with $F(x) = v(x)$. Since

$$\int_a^b (x - x_0)^j q_n(x)\, dv(x) = \frac{\delta_{jn} n!}{q_n^{(n)}(x_0)} \int_a^b (q_n(x))^2\, dv(x), \qquad 0 \leqq j \leqq n,$$

inequality (4) also holds with $F(x) = v(x)$ and $Q(x) = q_n(x)$; to see this, observe that since $q_n(x)$ has $n$ roots in $(a, b)$, Descartes' rule of signs implies that $q_n^{(n)}(x_0) > 0$ if $x_0 \geqq b$ and $(-1)^n q_n^{(n)}(x_0) > 0$ if $x_0 \leqq a$. Now the conclusion follows from Lemma 1.

Because of the difficulty of verifying (10), Theorem 1 may be too general to yield specific results; however, the following special case is applicable, as we will see below from examples.

THEOREM 2. *Suppose*

(11) $$dv(x) = \sigma(x)\, du(x),$$

*where* $\sigma(x)$ *is nonnegative* ($\not\equiv 0$) *and n times differentiable on* $(a, b)$. *If* $x_0 \leqq a$, *then* $a_{rn} \geqq 0$ *in* (1) *if*

(12) $$(-1)^{s-j}[(x - x_0)^j \sigma(x)]^{(s)} \leqq 0, \qquad a < x < b, \quad 0 \leqq j < s \leqq n;$$

*moreover,* $a_{rn} > 0$ *if at least one of the inequalities in* (12) *is strict for each s. The same conclusions hold if* $x_0 \geqq b$, *and* (12) *is replaced by*

(13) $$[(x - x_0)^j \sigma(x)]^{(s)} \leqq 0, \qquad a < x < b, \quad 0 \leqq j < s \leqq n.$$

*Proof.* Let $h(x)$ be the projection of $|x - x_0|^j\sigma(x)$ on the space of polynomials of degree $\leqq s$ ($\leqq n$) with respect to the inner product

$$(f, g) = \int_a^b f(x)g(x)\,du(x);$$

thus,

$$h(x) = b_0p_0(x) + \cdots + b_sp_s(x),$$

where

$$b_s = \frac{1}{\|p_s\|^2}\int_a^b |x - x_0|^j\sigma(x)p_s(x)\,du(x). \tag{14}$$

Since $x_0 \notin (a, b)$, $|x - x_0|^j\sigma(x)$ has $n$ derivatives on $(a, b)$. Moreover, the function

$$|x - x_0|^j\sigma(x) - h(x)$$

is orthogonal to every polynomial of degree $\leqq s$, and therefore has at least $s + 1$ zeros in $(a, b)$; hence, its $s$th derivative has at least one, and so

$$([|x - x_0|^j\sigma(x)]^{(s)} - b_sp_s^{(s)}(x_0))|_{x=x_1} = 0$$

for some $x_1$ in $(a, b)$, which implies that the sign of $b_s$ is the same as that of

$$p_s^{(s)}(x_0)([|x - x_0|^j\sigma(x)]^{(s)})|_{x=x_1}.$$

If $x_0 \leqq a$, then $(-1)^sp_s^{(s)}(x_0) > 0$, and (10) follows from (11), (12) and (14); if $x_0 \geqq b$, then $p_s^{(s)}(x_0) > 0$, and (10) follows from (11), (13) and (14). Theorem 1, therefore, implies that $a_{rn} \geqq 0$ in (1) in either case. It is straightforward to verify that the statements concerning strict positivity of $a_{rn}$ also follow from Theorem 1.

The following corollary settles affirmatively a conjecture of Askey [1], [3]; its proof has also been given separately elsewhere [9].

COROLLARY 1. *If* $c > 0$ *and*

$$dv(x) = |x - x_0|^c\,du(x),$$

*then* $a_{rn} > 0$ *in* (1) *for all* $n$.

*Proof.* With $\sigma(x) = |x - x_0|^c$ and $0 < c < 1$, the inequalities in (12) hold strictly for all $n$ if $x_0 \leqq a$, and those in (13) hold strictly for all $n$ if $x_0 \geqq b$. This gives the result for $0 < c < 1$, and it follows from this for all positive $c$, by the principle of composition.

*Example* 1. Corollary 1 implies known results for the Laguerre polynomials $\{L_r^\alpha(x)\}$ and the Jacobi polynomials $\{P_r^{(\alpha,\beta)}(x)\}$ (for definitions, see [6]); namely, that

$$L_n^{\alpha+\mu}(x) = \sum_{r=0}^{n} a_{rn}L_r^\alpha(x),$$

with $a_{rn} > 0$ if $\mu > 0$ and $\alpha > -1$, and that

$$P_n^{(\alpha+\mu,\beta)}(x) = \sum_{r=0}^{n} b_{rn} P_r^{(\alpha,\beta)}(x)$$

if $\mu > 0$ and $\alpha, \beta > -1$.

Askey [1] cited these results as evidence supporting his conjecture of Corollary 1.

COROLLARY 2. *Suppose $x_1, x_2, \cdots, x_m$ are in an interval $I$ which does not intersect $(a, b)$, $\{p_r(x)\}$ and $\{q_r(x)\}$ are normalized so as to be positive on $I$, and*

$$dv(x) = \sigma(x)\, du(x),$$

*where*

$$\sigma(x) = \sum_{k=1}^{m} b_k |x - x_k|^{c_k} \tag{15}$$

*with*

$$\text{(a)}\quad b_k > 0, \qquad \text{(b)}\quad 0 < c_k < 1, \qquad k = 1, \cdots, m. \tag{16}$$

*Then $a_{rn} > 0$ in (1) for all $n$.*

*Proof.* If $a > -\infty$ and $x_k \leqq a$, then

$$\begin{aligned} &(-1)^{s-j}[(x-a)^j(x-x_k)^{c_k}]^{(s)} \\ &= (-1)^{s-j} \sum_{i=0}^{j} \binom{j}{i} (x_k - a)^{j-i} [(x - x_k)^{c_k+i}]^{(s)} \\ &= (-1)^{s-j} s! \sum_{i=0}^{j} \binom{j}{i} (x_k - a)^{j-i} \binom{c_k + i}{s} (x - x_k)^{c_k+i-s}. \end{aligned} \tag{17}$$

Because of (16b),

$$(-1)^{s-i}\binom{c_k + i}{s} < 0, \qquad i = 0, \cdots, s-1,$$

and, therefore, the last member of (17) is negative if $0 \leqq j < s$ and $x \geqq a$. Now (16a) implies that $\sigma(x)$ satisfies (12) (with strict inequality) for all $n$ if $I = (-\infty, a]$, and so the conclusion follows from Theorem 2. The proof for the case where $I = [b, \infty)$ is similar.

Corollary 1 and the following lemma enable us to improve on a result obtained in [8].

LEMMA 2. *Suppose $y_0 \notin (a, b)$ and*

$$(-1)^r p_r(y_0) > 0, \qquad (-1)^r q_r(y_0) > 0, \qquad r = 0, 1, \cdots. \tag{18}$$

*Let $m$ be a positive integer, and suppose the distribution*

$$dv(x) = \frac{du(x)}{|x - y_0|^m}$$

*has moments of all orders on $(a, b)$. Then $a_{rn} \geqq 0$ in (1) for all $n$.*

*Proof.* For $m = 1$, it is known [6, Thm. 3.1.4, § 3.1] that

$$q_n(x) = A_n p_n(x) - B_n \frac{q_{n-1}(y_0)}{q_n(y_0)} p_{n-1}(x), \qquad (A_n, B_n > 0),$$

and the conclusion follows from (18); it follows for all positive integrals $m$ from this and the principle of composition.

Lemma 2 is not valid for arbitrary positive $m$. For a counter-example, see [8].

The following theorem improves on Theorem 1 of [8].

THEOREM 3. *Suppose $(a, b)$ is finite. Let $I$ be one of the intervals $(-\infty, a]$ or $[b, \infty)$, and let $J$ be the other. Suppose $p_r(x) > 0$ and $q_r(x) > 0$ $(r = 0, 1, \cdots)$ for $x$ in $I$. Let $x_1, \cdots, x_r$ be in $I$ and $z_1, \cdots, z_s$ be in $J$, and define*

$$dv(x) = \frac{\prod_{i=1}^{r} |x - x_i|^{c_i}}{\prod_{j=1}^{s} |x - z_j|^{m_j}} du(x), \qquad a < x < b,$$

*where $m_1, \cdots, m_s$ are nonnegative integers and $c_1, \cdots, c_s$ are arbitrary nonnegative numbers. Suppose $du(x)$ and $dv(x)$ have moments of all orders on $(a, b)$. Then $a_{rn} \geqq 0$ in (1) for all $n$; moreover, $a_{rn} > 0$ if at least one of $c_1, \cdots, c_s$ is positive.*

The proof consists of a straightforward application of Corollary 1, Lemma 2, and the principle of composition. (Notice that the assumptions imply that $(-1)^r \cdot p_r(x) > 0$ and $(-1)^r q_r(x) > 0$ on $J$, so that Lemma 2 is applicable.)

*Example* 2. With

$$du(x) = (1 - x)^\alpha (1 + x)^\beta \, dx$$

and

$$dv(x) = (1 - x)^\mu (1 + x)^{-k} \, du(x),$$

Theorem 3 implies a known result for the Jacobi polynomials; namely, that

$$P_n^{(\alpha+\mu, \beta-k)}(x) = \sum_{r=0}^{n} a_{rn} P_r^{(\alpha,\beta)}(x),$$

with $a_{rn} > 0$ if $\mu > 0$, $k$ is a nonnegative integer, $\alpha > -1$ and $\beta > k - 1$.

*Example* 3. We introduce a class of orthogonal polynomials which includes Jacobi's and Heine's polynomials [6]. Suppose $k \geqq 2$ and $a_1 < a_2 < \cdots < a_k$. Let $v$ be a fixed integer in $\{1, \cdots, k - 1\}$ and let $A = (\alpha_1, \cdots, \alpha_k)$ be a $k$-tuple of real numbers restricted only by the requirement that $\alpha_v > -1$ and $\alpha_{v+1} > -1$. Let $\{P_r^{(A)}(x)\}$ be a sequence of polynomials orthogonal over $[a_v, a_{v+1}]$ with respect to

$$du(x) = \prod_{j=1}^{k} |x - a_j|^{\alpha_j} \, dx,$$

and normalized so that $P_r^{(A)}(\infty) = \infty$. Then Theorem 1 implies that

$$P_n^{(B)}(x) = \sum_{r=0}^{n} a_{rn} P_r^{(A)}(x),$$

with $a_{rn} \geqq 0$ for all $n$ if

$$B = (\alpha_1 - k_1, \cdots, \alpha_\nu - k_\nu, \alpha_{\nu+1} + \mu_{\nu+1}, \cdots, \alpha_k + \mu_k),$$

provided $k_1, \cdots, k_\nu$ are nonnegative integers, $k_\nu < 1 + \alpha_\nu$, and $\mu_{\nu+1}, \cdots, \mu_k$ are arbitrary nonnegative numbers; moreover, $a_{rn} > 0$ if at least one of the latter is positive.

**3. Special results concerning even distributions.** The case where $du(x)$ and $dv(x)$ are even distributions deserves special attention. If

$$(a, b) = (-R, R), \quad u(-x) = -u(x), \quad v(-x) = -v(x), \tag{19}$$

then

$$p_n(-x) = (-1)^n p_n(x), \qquad q_n(-x) = (-1)^n q_n(x),$$

and it is appropriate to consider separately the expansions

$$q_{2n}(x) = \sum_{r=0}^{n} b_{rn} p_{2r}(x) \tag{20}$$

and

$$q_{2n+1}(x) = \sum_{r=0}^{n} c_{rn} p_{2r+1}(x). \tag{21}$$

In this case, the sequences $\{P_n(y)\}$ and $\{Q_n(y)\}$, defined by

$$P_n(y) = p_{2n}(y^{1/2}), \qquad Q_n(y) = q_{2n}(y^{1/2}),$$

are orthogonal over $(0, R^2)$ with respect to $du(y^{1/2})$ and $dv(y^{1/2})$, and the sequences $\{\tilde{P}_n(y)\}$ and $\{\tilde{Q}_n(y)\}$, defined by

$$\tilde{P}_n(y) = y^{-1/2} p_{2n+1}(y^{1/2}), \qquad \tilde{Q}_n(y) = y^{-1/2} q_{2n+1}(y^{1/2}),$$

are orthogonal over $(0, R^2)$ with respect to $y\,du(y^{1/2})$ and $y\,dv(y^{1/2})$. Our earlier results, applied separately to these two pairs of sequences, yield conclusions not directly obtainable by considering $\{p_n(x)\}$ and $\{q_n(x)\}$.

The next two theorems follow from Theorem 1.

THEOREM 4. *Suppose* (19) *holds and*

$$p_r(R) > 0, \qquad q_r(R) > 0, \qquad r = 0, 1, \cdots. \tag{22}$$

*Then*: (i) $b_{rn} \geqq 0$ *in* (20) *if*

$$\int_0^R (x^2 - x_1^2)^j p_{2r}(x)\, dv(x) \leqq 0, \qquad 0 \leqq j < r \leqq n,$$

*for some* $x_1 \geqq R$, *and* $b_{rn} > 0$ *if at least one of these inequalities is strict for each* $r$; (ii) $c_{rn} \geqq 0$ *in* (21) *if*

$$\int_0^R (x^2 - x_2^2)^j x p_{2j+1}(x)\, dv(x) \leqq 0, \qquad 0 \leqq j < r \leqq n,$$

*for some* $x_2 \geqq R$, *and* $c_{rn} > 0$ *if at least one of these inequalities is strict for each* $r$.

THEOREM 5. *Suppose* (19) *holds and*

$$p_{2r}(0) > 0, \quad q_{2r}(0) > 0, \qquad p'_{2r+1}(0) > 0, \quad q'_{2r+1}(0) > 0, \qquad r = 0, 1, \cdots. \tag{23}$$

*Then*: (i) $b_{rn} \geqq 0$ *in* (20) *if*

$$(-1)^j \int_0^R (x^2 + x_1^2)^j p_{2r}(x)\, dv(x) \leqq 0, \qquad 0 \leqq j < r \leqq n,$$

*for some number* $x_1$, *and* $b_{rn} > 0$ *if at least one of these inequalities is strict for each* $r$; (ii) $c_{rn} \geqq 0$ *in* (21) *if*

$$(-1)^j \int_0^R (x^2 + x_2^2)^j x p_{2r+1}(x)\, dv(x) \leqq 0, \qquad 0 \leqq j < r \leqq n,$$

*for some* $x_2$, *and* $c_{rn} > 0$ *if at least one of these inequalities is strict for each* $r$.

The next theorem follows from Theorem 2.

THEOREM 6. *Suppose* (19) *holds and*

$$dv(x) = \rho(x^2)\, du(x),$$

*where* $\rho(y)$ *has* $n$ *derivatives on* $(0, R^2)$. *Then*: (i) $b_{rn} \geqq 0$ *and* $c_{rn} \geqq 0$ *in* (20) *and* (21) *if* (23) *holds and*

$$(-1)^{s-j}[(y + \gamma^2)^j \rho(y)]^{(s)} \leqq 0, \qquad 0 < y < R^2, \quad 0 \leqq j < s \leqq n, \tag{24}$$

*for some number* $\gamma$; *moreover,* $b_{rn} > 0$ *and* $c_{rn} > 0$ *if at least one of the inequalities in* (24) *is strict for each* $s$. (ii) *The same conclusions hold if* (22) *holds and* (24) *is replaced by*

$$[(y - \gamma^2)^j \rho(y)]^{(s)} \leqq 0, \qquad 0 < y < R^2, \quad 0 \leqq j < s \leqq n,$$

*for some* $\gamma \geqq R$.

Corollaries 1 and 2 and Theorem 3 can also be adapted to the special case (19). We present only the following adaptation of Corollary 1.

COROLLARY 3. *Suppose* (19) *holds. Then* $a_{rn} > 0$ *and* $b_{rn} > 0$ *for all* $n$ *in* (20) *and* (21) *if*: (i) (22) *holds and*

$$dv(x) = (x_0^2 - x^2)^c\, du(x)$$

*with* $c > 0$ *and* $x_0 \geqq R$; *or,* (ii) (23) *holds and*

$$dv(x) = (x_0^2 + x^2)^c\, du(x)$$

*where* $c > 0$ *and* $x_0$ *is any real number.*

*Example* 4. By taking

$$du(x) = (1 - x^2)^\alpha\, dx$$

and

$$dv(x) = (1 - x^2)^\mu\, du(x),$$

we obtain from (i) of Corollary 3 the following known result for Jacobi polynomials, due to Gegenbauer:

$$P_{2n}^{(\alpha+\mu,\alpha+\mu)}(x) = \sum_{r=0}^{n} b_{rn} P_{2r}^{(\alpha,\alpha)}(x)$$

and

$$P_{2n+1}^{(\alpha+\mu,\alpha+\mu)}(x) = \sum_{r=0}^{n} c_{rn} P_{2r+1}^{(\alpha,\alpha)}(x),$$

where $b_{rn} > 0$ and $c_{rn} > 0$ for all $n$ if $\alpha > -1$ and $\mu > 0$.

*Example* 5. As applications of Corollary 3 (ii), we show that

$$P_n^{(\alpha,\beta)}(2x^2 - 1) = \sum_{r=0}^{n} (-1)^{n-r} \phi_{rn} P_{2r}^{(\alpha,\alpha)}(x) \tag{25}$$

with $\phi_{rn} > 0$ for all $n$ if $\alpha > -1$ and $\beta > -1/2$, and

$$xP_n^{(\alpha,\beta)}(2x^2 - 1) = \sum_{r=0}^{n} (-1)^{n-r} \psi_{rn} P_{2r+1}^{(\alpha,\alpha)}(x) \tag{26}$$

with $\psi_{rn} > 0$ for all $n$ if $\alpha < -1$ and $\beta > 1/2$.

By substituting $y = 2x^2 - 1$ in the orthogonality relation

$$\int_{-1}^{1} P_r^{(\alpha,\beta)}(y) P_s^{(\alpha,\beta)}(y)(1 - y)^\alpha (1 + y)^\beta \, dy = 0, \qquad r \neq s,$$

and using the evenness of the resulting integrand, we find that

$$\int_{-1}^{1} P_r^{(\alpha,\beta)}(2x^2 - 1) P_s^{(\alpha,\beta)}(2x^2 - 1)(1 - x^2)^\alpha (x^2)^{\beta + 1/2} \, dx = 0, \qquad r \neq s. \tag{27}$$

This can be interpreted to mean that $\{P_r^{(\alpha,\beta)}(2x^2 - 1)\}$ is the "even-degree" subsequence of a sequence of polynomials orthogonal over $(-1, 1)$ with respect to

$$dv(x) = (1 - x^2)^\alpha (x^2)^{\beta + 1/2} \, dx.$$

Since $\{P_r^{(\alpha,\alpha)}(x)\}$ is orthogonal over $(-1, 1)$ with respect to

$$du(x) = (1 - x^2)^\alpha \, dx,$$

we infer the stated conclusion concerning (25) by applying (ii) of Corollary 3, with $x_0 = 0$ and $c = \beta + 1/2$, to the sequences $\{p_{2n}(x)\}$ and $\{q_{2n}(x)\}$ defined by

$$p_{2n}(x) = (-1)^n P_{2n}^{(\alpha,\alpha)}(x)$$

and

$$q_{2n}(x) = (-1)^n P_n^{(\alpha,\beta)}(2x^2 - 1).$$

(The factor $(-1)^n$ adjusts the normalizations of $\{p_{2n}(x)\}$ and $\{q_{2n}(x)\}$ so that they satisfy (23), as required in (ii).)

To prove the assertion concerning (26), we interpret (27) to mean that $\{xP_n^{(\alpha,\beta)}(2x^2 - 1)\}$ is the "odd-degree" subsequence of a sequence of polynomials orthogonal over $(-1, 1)$ with respect to

$$dv(x) = (1 - x^2)^\alpha (x^2)^{\beta - 1/2}\, dx,$$

and apply (ii) of Corollary 3, with $x_0 = 0$ and $c = \beta - 1/2$, to the sequences $\{\tilde{P}_{2n+1}(x)\}$ and $\{\tilde{q}_{2n+1}(x)\}$ defined by

$$\tilde{p}_{2n+1}(x) = (-1)^n P_{2n+1}^{(\alpha,\alpha)}(x)$$

and

$$\tilde{q}_{2n+1}(x) = (-1)^n x P_n^{(\alpha,\beta)}(2x^2 - 1).$$

The results in this example can also be deduced from earlier known properties of the Jacobi polynomials.

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