HOUSTON JOURNAL OF MATHEMATICS, Volume 7, No. 2, 1981.

LIMITS OF CERTAIN SEQUENCES ASSOCIATED WITH CYLINDER FUNCTIONS

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ABSTRACT. This paper presents a theorem on the asymptotic behavior of the sequence of values of the r-th derivative of the cylinder function $\mathcal{C}_\mu = AJ_\mu + BY_\mu$ at the zeros of $\mathcal{C}_\upsilon^{(s)} = AJ_\upsilon^{(s)} + BY_\upsilon^{(s)}$, where $\mu - \upsilon$ is an integer.

The following theorem is our main result.

THEOREM 1. *Suppose*

$$\mathcal{C}_\upsilon = AJ_\upsilon + BY_\upsilon \text{ and } \mathcal{C}_{\upsilon+m} = AJ_{\upsilon+m} + BY_{\upsilon+m},$$

where J_υ *and* Y_υ *are the Bessel functions of the first and second kinds and* A *and* B *are real numbers such that*

(1) $A^2 + B^2 = 1.$

Let m,r, *and* s *be nonnegative integers, and denote the successive positive zeros of* $\mathcal{C}_\upsilon^{(s)}$ *and* $\mathcal{C}_{\upsilon+m}^{(r)}$ *by* $\{x_n\}$ *and* $\{y_n\}$, *respectively. Then*

$$\text{(a)}\quad \lim_{n\to\infty} x_n^{1/2}|\mathcal{C}_{\upsilon+m}^{(r)}(x_n)| = \lim_{n\to\infty} y_n^{1/2}|\mathcal{C}_\upsilon^{(s)}(y_n)| = \sqrt{2/\pi}$$

if $m + r + s$ *is odd, and*

$$\text{(b)}\quad \lim_{n\to\infty} x_n^{3/2}|\mathcal{C}_{\upsilon+m}^{(r)}(x_n)| = \lim_{n\to\infty} y_n^{3/2}|\mathcal{C}_\upsilon^{(s)}(y_n)| = \frac{|s-r-m(2\upsilon+m)|}{\sqrt{2\pi}}$$

if $m + r + s$ *is even. If*

(2) $s - r - m(2\upsilon + m) = 0$

in (b), then either

(3) $\mathcal{C}_{\upsilon+m}^{(r)} = f_{rsm\upsilon}\mathcal{C}_\upsilon^{(s)}$

where $f_{rsm\upsilon}$ *is a rational function independent of* A *and* B, *such that*

(4) $\lim_{x\to\infty} f_{rsm\upsilon}(x) = 1,$

or

(5) $\lim_{n\to\infty} x_n^{2k+3/2}|C_{\upsilon+m}^{(r)}(x_n)|$ *and* $\lim_{n\to\infty} y_n^{2k+3/2}|C_{\upsilon}^{(s)}(y_n)|$

are finite, nonzero, and equal for some positive integer k. *Their common value is also independent of* A *and* B.

Before proceeding with the proof, we observe that parts (a) and (b) can be stated more simply as follows, although the statement given above is convenient for the proof.

COROLLARY 1. *Let* A *and* B *be real numbers satisfying (1) and*

$$C_\upsilon = AJ_\upsilon + BY_\upsilon \text{ and } C_\mu = AJ_\mu + BY_\mu,$$

where $\mu - \upsilon$ *is an arbitrary integer, and let* $\{x_n\}$ *be the sequence of successive positive zeros of* $C_\upsilon^{(s)}$. *Then*

$$\lim_{n\to\infty} x_n^{1/2}|C_\mu^{(r)}(x_n)| = \sqrt{2/\pi}$$

if $\mu - \upsilon + r + s$ *is odd, and*

$$\lim_{n\to\infty} x_n^{3/2}|C_\mu^{(r)}(x_n)| = \frac{|s-r-\mu^2+\upsilon^2|}{\sqrt{2\pi}}$$

if $\mu - \upsilon + r + s$ *is even.*

PROOF OF THEOREM 1. Solving the system

$$C_{\upsilon+m}^{(r)} = AJ_{\upsilon+m}^{(r)} + BY_{\upsilon+m}^{(r)}$$

$$C_\upsilon^{(s)} = AJ_\upsilon^{(s)} + BY_\upsilon^{(s)}$$

for A and B and invoking (1) yields

$$(6)\quad 1 = \frac{[C_{\upsilon+m}^{(r)}Y_\upsilon^{(s)} - C_\upsilon^{(s)}Y_{\upsilon+m}^{(r)}]^2 + [J_{\upsilon+m}^{(r)}C_\upsilon^{(s)} - J_\upsilon^{(s)}C_{\upsilon+m}^{(r)}]^2}{[J_{\upsilon+m}^{(r)}Y_\upsilon^{(s)} - Y_{\upsilon+m}^{(r)}J_\upsilon^{(s)}]^2}$$

for all values of x for which the denominator is nonzero. In fact, it is easily verified from (1) that the equation obtained by multiplying both sides of (6) by the denominator on the right is an identity in the complex plane.

Now denote

$$(7)\quad J_{\upsilon+m}^{(r)}(x)Y_\upsilon^{(s)}(x) - Y_{\upsilon+m}^{(r)}(x)J_\upsilon^{(s)}(x) = \frac{2}{\pi}w_{rsm\upsilon}(1/x)$$

and

$$p_{\mu k}(x) = (J_\mu^{(k)}(x))^2 + (Y_\mu^{(k)}(x))^2.$$

Then (6) and the definition of $\{x_n\}$ and $\{y_n\}$ imply that the function

(8) $I_1(x) = \frac{2}{\pi} p_{\upsilon s}^{-1/2}(x) |w_{rsm\upsilon}(1/x)|$

interpolates $\{|C_{\upsilon+m}^{(r)}(x_n)|\}$ for those n such that $p_{\upsilon s}(x_n) \neq 0$, and that the function

(9) $I_2(x) = \frac{2}{\pi} p_{\upsilon+m,r}^{-1/2}(x) |w_{rsm\upsilon}(1/x)|$

interpolates $\{|C_{\upsilon}^{(s)}(y_n)|\}$ for those n such that $p_{\upsilon+m,r}(y_n) \neq 0$. Since $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty$ and, as we will see below, $p_{\mu k}$ has at most finitely many zeros, the key to the proof is to study the behavior of $p_{\mu k}$ for large values of its argument and that of $w_{rsm\upsilon}$ for small values of its argument.

We will first show that

(10) $p_{\mu k}(x) = 2/\pi x + O(1/x^3)$.

This is known for k = 0 and k = 1 [1, page 138], but perhaps not for $k \geqslant 2$. To prove it, we first observe that $J_\mu^{(k)}$ and $Y_\mu^{(k)}$ can be written as

$$J_\mu^{(k)}(x) = \sqrt{2/\pi x}\,[P_k(x,\mu)\cos(x - \mu\pi/2 - \pi/4) - Q_k(x,\mu)\sin(x - \mu\pi/2 - \pi/4)]$$

and

$$Y_\mu^{(k)}(x) = \sqrt{2/\pi x}\,[P_k(x,\mu)\sin(x - \mu\pi/2 - \pi/4) + Q_k(x,\mu)\cos(x - \mu\pi/2 - \pi/4)].$$

This is well known for k = 0 [2, page 206], and by using the identity

$$C_\mu^{(k)} = \frac{C_{\mu-1}^{(k-1)} - C_{\mu+1}^{(k-1)}}{2},$$

valid for any cyclinder function [2, page 82], it can be established by induction, with

$$(11)\quad \begin{aligned} P_k(x,\mu) &= \frac{-[Q_{k-1}(x,\mu-1)+Q_{k-1}(x,\mu+1)]}{2}, \\ Q_k(x,\mu) &= \frac{[P_{k-1}(x,\mu-1)+P_{k-1}(x,\mu+1)]}{2}, \end{aligned} \qquad k \geqslant 1.$$

Starting from the known asymptotic properties,

$$P_0(x,\mu) = 1 + O(1/x^2),\ Q_0(x,\mu) = O(1/x)$$

[2, page 199], (11) and induction imply that

$$P_{2j}(x,\mu) = (-1)^j + O(1/x^2),\ Q_{2j}(x,\mu) = O(1/x),$$

$$P_{2j+1}(x,\mu) = O(1/x),\ Q_{2j+1}(x,\mu) = (-1)^j + O(1/x^2).$$

Since

$$P_{\mu k}(x) = 2/\pi x\,[P_k^2(x,\mu) + Q_k^2(x,\mu)],$$

this implies (10).

Now we must consider $w_{rsm\upsilon}$ for small values of its argument. If m = 0, then the relations in (7) reduce to Bassett's formulas, several of which are listed in [2, page 76]. If $\mathcal{C}_\upsilon$ is any cylinder function of order υ, then - following Bassett - we can express $\mathcal{C}_\upsilon^{(k)}$ in terms of $\mathcal{C}_\upsilon$ and $\mathcal{C}_\upsilon'$ by repeated differentiation of Bessel's equation,

$$\mathcal{C}_\upsilon''+(1/x)\mathcal{C}_\upsilon' + (1 - \upsilon^2/x^2)\mathcal{C}_\upsilon = 0.$$

Induction on k leads to

(12) $\mathcal{C}_\upsilon^{(k)}(x) = F_{\upsilon k}(1/x)\mathcal{C}_\upsilon'(x) + G_{\upsilon k}(1/x)\mathcal{C}_\upsilon(x),$

where

(13) $F_{\upsilon 0} = 0,\ G_{\upsilon 0} = 1,\ \mathbf{F}_{\upsilon 1} = 1,\ G_{\upsilon 1} = 0,$

(14) $F_{\upsilon,k+1}(y) = -y^2F_{\upsilon k}'(y) + G_{\upsilon k}(y) - yF_{\upsilon k}(y),$

(15) $G_{\upsilon,k+1}(y) = -y^2G_{\upsilon k}'(y) - (1 - \upsilon^2y^2)F_{\upsilon k}(y).$

Taking $\mathcal{C}_\upsilon = J_\upsilon$ and then $\mathcal{C}_\upsilon = Y_\upsilon$ in (12) and using (7) and the known relation

$$J_\upsilon(x)Y_\upsilon'(x) - J_\upsilon'(x)Y_\upsilon(x) = 2/\pi x$$

[2, page 576] yields

$$w_{rs0\upsilon}(y) = y[F_{\upsilon s}(y)G_{\upsilon r}(y) - G_{\upsilon s}(y)F_{\upsilon r}(y)],$$

where y = 1/x.

From (13), (14), (15) and induction,

$$F_{\upsilon n}(-y) = (-1)^{n-1}F_{\upsilon n}(y),\ G_{\upsilon n}(-y) = (-1)^nG_{\upsilon n}(y),$$

$$F_{\upsilon,2k}(y) = (-1)^k ky + O(y^3),\ F_{\upsilon,2k+1}(y) = (-1)^k + O(y^2),$$

$$G_{\upsilon,2k}(y) = (-1)^k + O(y^2),\ G_{\upsilon,2k+1}(y) = (-1)^{k+1} ky + O(y^3).$$

From the last four equations it is straightforward to verify that $w_{rs0\upsilon}$ is a polynomial which is even or odd with r + s, and that

(16) $$w_{rs0\upsilon}(y) = \begin{cases} (-1)^{(3r+s)/2}((s-r)/2)y^2 + O(y^4) \text{ if } r + s \text{ is even,} \\ (-1)^{(3r+s-1)/2}y + O(y^3) \text{ if } r + s \text{ is odd.} \end{cases}$$

Since any zero of $p_{\upsilon r}$ is a zero of $w_{rs0\upsilon}$ for every s, it is now clear that $p_{\upsilon r}$ has at most finitely many zeros in the complex plane.

Now we wish to find the lowest degree term in $w_{rsm\upsilon}$, where $m \geqslant 0$. Differentiating the identity

$$C_{\upsilon+m}(x) = -C'_{\upsilon+m-1}(x) + \frac{\upsilon+m-1}{x}C_{\upsilon+m-1}(x)$$

[2, page 83] yields

$$C^{(r)}_{\upsilon+m}(x) = -C^{(r+1)}_{\upsilon+m-1}(x) + (\upsilon+m-1)\Sigma^{r}_{k=0}(-1)^k\frac{r!}{(r-k)!}x^{-k-1}C^{(r-k)}_{\upsilon+m-1}(x).$$

Applying this to $J^{(r)}_{\upsilon+m}$ and $Y^{(r)}_{\upsilon+m}$ in (7) yields

(17) $w_{rsm\upsilon}(y) = -w_{r+1,s,m-1,\upsilon}(y) + (\upsilon+m-1)\Sigma^{r}_{k=0}(-1)^k\frac{r!}{(r-k)!}y^{k+1}w_{r-k,s,m-1,\upsilon}(y).$

From this and (16), it can be shown by induction on m that $w_{rsm\upsilon}$ is a polynomial, even or odd with $m + r + s$; that

(18) $w_{rsm\upsilon}(y) = (-1)^{(m+3r+s-1)/2}y + O(y^3)$

if $m + r + s$ is odd; and that

(19) $w_{rsm\upsilon}(y) = (-1)^{(m+3r+s)/2}[s - r - m(2\upsilon + m)]\ y^2/2 + O(y^4)$

if $m + r + s$ is even. In verifying (18) and (19), note that the induction assumption applied with subscript $m - 1$ permits us to rewrite (17) as

$$w_{rsm\upsilon}(y) = -w_{r+1,s,m-1,\upsilon}(y) + (\upsilon + m-1)yw_{r,s,m-1,\upsilon}(y) + \cdots,$$

where $\cdots$ stands for $O(y^4)$ if $m + r + s$ is even or $O(y^3)$ if $m + r + s$ is odd. In either case these terms can be ignored.

Now (8), (9), (10), (18), and (19) imply that

$$I_j(x) = \begin{cases} \sqrt{2/\pi x}\,[1 + O(1/x^2)] \text{ if } m + r + s \text{ is odd,} \\ (x^{-3/2}/\sqrt{2\pi})\ |s - r - m(2\upsilon + m)|[1 + O(1/x^2)] \text{ if } m + r + s \text{ is even,} \end{cases}$$

for $j = 1,2$. This implies (a) and (b).

If (2) holds with $m + r + s$ even and $w_{rsm\upsilon} \neq 0$, then

$$w_{rsm\upsilon}(y) = A_{rsm\upsilon}y^{2k+2} + O(y^{2k+4})$$

with k a positive integer and $A_{rsm\upsilon} \neq 0$; in this case the limits (5) have the stated properties. If $w_{rsm\upsilon} = 0$, then

(20) $J^{(r)}_{\upsilon+m}/J^{(s)}_{\upsilon} = Y^{(r)}_{\upsilon+m}/Y^{(s)}_{\upsilon}$

identically in the complex plane. But $f^{(r)}_{rsm\upsilon} = J_{\upsilon+m}/J^{(s)}_{\upsilon}$ is in any case meromorphic. If (20) holds, then any zero of $f_{rsm\upsilon}$ is a zero of $p_{\upsilon+m,k}$ and any pole of $f_{rsm\upsilon}$ is a zero of $p_{\upsilon k}$. Since $p_{\upsilon+m,k}$ and $p_{\upsilon k}$ have at most finitely many zeros, (20) implies that

$f_{rsm\upsilon}$ is rational. Since (20) also implies that

$$f_{rsm\upsilon}^2 = p_{\upsilon+m,k}^2/p_{\upsilon k}^2,$$

(10) implies (4) under the stated assumptions. This completes the proof.

It is perhaps worth noting that if $\upsilon = q + 1/2$, where q is an integer, then I_1^2 and I_2^2 are rational functions.

If $m = 0$ and (2) holds, then obviously $s = r$ and (3) holds, with $f_{rr0\upsilon} = 1$. Less trivial examples of this case are given by $m = 1$, $r = 0$, $s = 1$, $\upsilon = 0$, when (3) takes the form

$$\mathcal{C}_1 = -\mathcal{C}_0'$$

and by $m = 1$, $r = 0$, $s = 3$, $\upsilon = 1$, when it can be shown that (3) takes the form

$$\mathcal{C}_1''' = (1 - 3/x^2)\mathcal{C}_2.$$

An example where the limits (5) have the stated properties for some integer $k > 0$ is given by $m = 1$, $r = 1$, $s = 4$, $\upsilon = 1$. Applying the method used in the proof of Theorem 4 to this case shows that

$$J_2'(x)Y_1^{(4)}(x) - Y_2'(x)J_1^{(4)}(x) = -12/\pi x^4,$$

and, from this and (10), the interpolating functions (8) and (9) satisfy

$$I_j(x) = 6\sqrt{2/\pi}\, x^{-7/2}[1 + O(1/x^2)], j = 1,2;$$

hence, if $\{x_n\}$ and $\{y_n\}$ are the successive zeros of $\mathcal{C}_1^{(4)}$ and $\mathcal{C}_2'$, respectively, then

$$\lim_{n\to\infty} x_n^{7/2}|\mathcal{C}_2'(x_n)| = \lim_{n\to\infty} y_n^{7/2}|\mathcal{C}_1^{(4)}(y_n)| = 6\sqrt{2/\pi}.$$

REFERENCES

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Received January 29, 1980