Asymptotic Integration of $y^{(n)} + P(t)y' = f(t)$ under Mild Integral Smallness Conditions

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This paper gives sufficient conditions for the equation

$$y^{(n)} + P(t)y' = f(t) \quad (n \geq 2)$$

to have solutions which behave like polynomials of degree \(<n\) as $t \to \infty$. This question has been investigated by many authors, but, to our knowledge, always under integral smallness conditions on $P$ and $f$ which require absolute convergence of all improper integrals in question. Also, most authors have assumed that $\tau > 0$. (For exceptions to this, see [1], [2], and [4].) Here we allow $\tau$ to be any real number. Moreover, our integral smallness conditions require only ordinary (i.e., perhaps conditional) convergence, except for a condition on $P$ in Theorem 2 which does require absolute convergence, but is still considerably weaker than the usual condition.

We assume throughout that $P$ and $f$ are real-valued and continuous on $(0, \infty)$ and that $\tau$ is real. When we say that an improper integral converges, we mean that it may converge conditionally, unless, of course, it is clear that the integrand is non-negative.

**Theorem 1.** Suppose $\nu$ and $m$ are integers, $0 \leq \nu \leq m \leq n - 1$, and let $\alpha$ be a non-negative number such that

$$\nu - \alpha \leq m;$$

moreover, suppose that $\alpha < 1$ if $\nu \neq 0$. Assume that the integrals

$$\int_{0}^{\infty} t^{n - \nu - 1 + \alpha + m} P(t) dt$$

and

$$\int_{0}^{\infty} t^{n - \nu - 1 + \alpha} f(t) dt$$

converge. Let

$$q(t) = \sum_{j=\nu}^{m} a_j \frac{t^j}{j!}.$$
where \( a_r, \ldots, a_m \) are given real constants, with \( a_m > 0 \). Then (1) has a solution \( y_0 \), defined for sufficiently large \( t \), such that

\[
y_0^{(r)}(t) = p^{(r)}(t) + o(t^{r-a}), \quad 0 \leq r \leq n - 1.
\]

The following two lemmas will be useful in proving Theorem 1 as well as Theorem 2, which is stated below.

**Lemma 1.** Suppose \( u \in C[t_0, \infty) \) for some \( t_0 \geq 0 \) and let \( a \) and \( b \) be constants, \( 0 \leq a \leq b \). Suppose also that \( \int_t^\infty t^b u(t) dt \) converges, and define

\[
\rho_0(t) = \sup_{t \geq t_0} \left| \int_t^\infty s^a u(s) ds \right|.
\]

Then

\[
\left| \int_t^\infty (s-t)^a u(s) ds \right| \leq 2 \rho_0(t) t^{a-b}, \quad t \geq t_0 \geq t_0.
\]

**Proof.** Let \( U(t) = \int_t^\infty s^b u(s) ds \), and note that

\[
|U(t)| \leq \rho_0(t).
\]

Now,

\[
(s-t)^a u(s) = -\left(1 - \frac{t_0}{s}\right)^a s^{a-b} U'(s),
\]

so integrating by parts yields

\[
\int_t^\infty (s-t)^a u(s) ds = -\left(1 - \frac{t_0}{s}\right)^a s^{a-b} U(s) \bigg|_t^{t_0} + \int_t^{t_0} U(s) \frac{d}{ds} \left[\left(1 - \frac{t_0}{s}\right)^a s^{a-b}\right] ds.
\]

But

\[
\frac{d}{ds} \left[\left(1 - \frac{t_0}{s}\right)^a s^{a-b}\right] = \left(1 - \frac{t_0}{s}\right)^a \frac{d}{ds} (s^{a-b}) + s^{a-b} \frac{d}{ds} \left[\left(1 - \frac{t_0}{s}\right)^a\right],
\]

where the first product is nonpositive and the second is positive if \( s > t_0 \). This enables us to let \( t \to \infty \) in (9) and infer that

\[
\left| \int_t^\infty (s-t)^a u(s) ds \right| \leq \left(1 - \frac{t_0}{t}\right)^a t^{a-b} |U(t)| - \int_t^\infty \left(1 - \frac{t_0}{s}\right)^a |U(s)| \frac{d}{ds} (s^{a-b}) ds
\]

\[
+ \int_t^\infty |U(s)| s^{a-b} \frac{d}{ds} \left(1 - \frac{t_0}{s}\right)^a ds
\]

\[
\leq 2 \rho_0(t) t^{a-b}.
\]
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(see (8)). This completes the proof of Lemma 1.

**Lemma 2.** Suppose \( u \in C[t_0, \infty) \) for some \( t_0 \geq 0 \) and

\[
\int_{t_0}^\infty t^{n-\nu-1+x} u(t) \, dt
\]

converges, where \( x \) and \( \alpha \) are as in Theorem 1. Define

\[
w(t) = \int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!} u(s) \, ds \quad \text{if} \quad \nu = 0,
\]

or

\[
w(t) = \int_{t_0}^t \frac{(t-s)^{\nu-1}}{(\nu-1)!} d\lambda \int_s^\infty \frac{(\lambda-s)^{n-\nu-1}}{(n-\nu-1)!} u(s) \, ds \quad \text{if} \quad 1 \leq \nu \leq n-1,
\]

and let

\[
\rho(t) = \sup_{s \geq t} \left| \int_s^\infty s^{n-\nu-1+x} u(s) \, ds \right|.
\]

Then \( w \in C^{(n)}[t_0, \infty) \),

\[
|w^{(r)}(t)| \leq \frac{2\rho(t)t^{r-x}}{(n-r-1)!}, \quad \nu \leq r \leq n-1,
\]

and, if \( \nu \geq 1 \),

\[
|w^{(r)}(t)| \leq \frac{2\rho(t_0)t^{r-x}}{(n-\nu-1)!} \frac{1}{\prod_{j=1}^{\nu-1} (j-r)}, \quad 0 \leq r \leq \nu-1.
\]

Moreover,

\[
w^{(r)}(t) = o(t^{r-x}), \quad 0 \leq r \leq n-1.
\]

**Proof.** From Lemma 1 with

\[
a = n-r-1, \quad b = n-\nu-1+\alpha, \quad t = t_0,
\]

\[
\left| \int_t^\infty (t-s)^{n-r} u(s) \, ds \right| \leq 2\rho(t)t^{r-x}, \quad \nu \leq r \leq n-1.
\]

This implies that \( w \) as defined by (10) or (11) is in \( C^{(n)}[t_0, \infty) \) and satisfies (12) and (14) for \( \nu \leq r \leq n-1 \). Therefore, the proof is complete if \( \nu = 0 \). If \( 0 \leq r \leq \nu-1 \), then

\[
|w^{(r)}(t)| \leq \frac{2}{(\nu-r-1)! (n-\nu-1)!} \int_{t_0}^t (t-\lambda)^{r-x-1}\lambda^{-\nu} \rho(\lambda) \, d\lambda
\]

from (11) and (15) (the latter with \( r = \nu \)). Since \( \rho \) is nonincreasing, we may replace
\( \rho(\lambda) \) by \( \rho(t_0) \) here, then replace \( t_0 \) by zero in the lower limit of integration (recall that \( \alpha<1 \)), and integrate repeatedly by parts to obtain (13).

From (16),
\[
|w^{(r)}(t)|t^{\nu+r+s} \leq \frac{2t^{s-1}}{(\nu-r-1)! (n-v-1)!} \int_{t_0}^{t} \rho(\lambda)\lambda^{-s}d\lambda,
\]
which implies (14) for \( 0 \leq r \leq \nu-1 \). (If \( \int_{0}^{\infty} \rho(\lambda)\lambda^{-s}d\lambda < \infty \), this is obvious; if \( \int_{0}^{\infty} \rho(\lambda)\lambda^{-s}d\lambda = \infty \), it follows from 1'Hospital's rule. Here again we have used the assumption that \( \alpha<1 \)). This completes the proof of Lemma 2.

**Proof of Theorem 1.** For \( t_0 \geq 0 \), let \( H(t_0) \) be the Banach space of functions \( h \) in \( C^{(n-1)}[t_0, \infty) \) such that

\[
h^{(r)}(t) = 0(t^{-r-n}), \quad 0 \leq r \leq n-1,
\]

with norm

\[
\|h\| = \sup_{t \geq t_0} \left\{ t^{-r-s} \sum_{r=0}^{s-1} t^r |h^{(r)}(t)| \right\}.
\]

For \( M > 0 \), let

\[
H_M(t_0) = \{ h \in H(t_0) \mid \|h\| \leq M \}.
\]

Since \( \nu \leq m \) and \( a_m > 0 \) in (5), there are constants \( M, \lambda, \) and \( T_0 \) such that if

\[
t_0 \geq T_0 \quad \text{and} \quad h \in H_M(t_0),
\]

then

\[
q(t) + h(t) \geq \frac{1}{2} a_m t^m \frac{t^m}{m!}
\]

and

\[
|q^{(r)}(t) + h^{(r)}(t)| \leq \lambda t^{r-n}, \quad 0 \leq r \leq n-1,
\]

for all \( t \geq t_0 \). (From (20), \( (q+h)^r \) is defined and real-valued on \([t_0, \infty) \) if (19) holds.) We assume henceforth that \( h, h_1, \) and \( h_2 \) are in \( H_M(t_0) \) for some \( t_0 \geq T_0 \). The constants appearing in estimates that follow may depend upon \( T_0 \), but they do not depend upon \( t_0, h, h_1, h_2 \), etc. We assume that \( t \geq t_0 \) throughout.

We will show that the transformation

\[
\hat{h} = \mathcal{F} h
\]
defined by

\[ h(t) = \int_t^\infty \frac{(t-s)^{\nu-1}}{(\nu-1)!} \left[ -f(s) + P(s)(q(s) + h(s)) \right] ds \quad \text{if } \nu = 0 \]

or by

\[ \hat{h}(t) = \int_t^{t_0} \frac{(t-s)^{\nu-1}}{(\nu-1)!} ds \int_s^{t_0} \frac{(\lambda - s)^{\nu-1}}{(n-\nu-1)!} \left[ -f(s) + P(s)(q(s) + h(s)) \right] ds \]

\[ \text{if } 1 \leq \nu \leq n - 1, \]

is a contraction mapping of $H_\nu(t_0)$ into itself if $t_0$ is sufficiently large. To this end, we first study the integral

\[ F(t; h) = \int_t^\infty s^{\nu-1} P(s)(q(s) + h(s)) ds. \]

The convergence of this integral follows easily from Dirichlet's test and the convergence of (3); nevertheless, we will write out the details of the proof, because they will be useful in obtaining estimates that we need below. Let

\[ \Phi(t) = \int_t^\infty s^{\nu-1} P(s) ds, \]

which exists, because (3) converges. If $\tau \geq t$, then

\[ \int_t^{\tau} s^{\nu-1} P(s)(q(s) + h(s)) ds \]

\[ = -\int_t^{\tau} \Phi'(s)(s^{\nu}(q(s) + h(s))) ds \]

\[ = -\Phi(s)(s^{\nu}(q(s) + h(s))) |_{s=t} \]

\[ + \int_t^{\tau} \Phi(s)(s^{\nu}(q(s) + h(s))) |_{s=t}^{s^{\nu}(q(s) + h(s)))'} + (s^{\nu}h(s))' ds. \]

Now,

\[ (s^{\nu}q(s))' = 0(s^{\nu}) \]

and

\[ |(s^{\nu}h(s))'| \leq (m + 1)M s^{\nu-1} - \nu. \]

(See (17) and (18).) Since $\Phi(t) = o(1)$, the last two inequalities together with (2), (20) and (21) enable us to let $\tau \to \infty$ in (26) to obtain

\[ F(t; h) = \Phi(t)(t^{\nu}(q(t) + h(t)))' \]

\[ + \int_t^\infty \Phi(s)(s^{\nu}(q(s) + h(s)))' + (s^{\nu}h(s))' ds, \]
where the integral on the right converges absolutely.

We will now show that $F(t; h)$ satisfies a Lipschitz condition with respect to $h$. Applying the mean value theorem to $G(u)=u^r$ and invoking (20) if $r<0$ or (21) with $r=0$ if $r>0$ yields the inequality

$$\begin{align*}
|\left[ t^{-m}(q(t)+h_1(t)) \right]^r - \left[ t^{-m}(q(t)+h_2(t)) \right]^r | \\
\leq A_1 t^{-m} |h_1(t) - h_2(t)| \\
\leq A_1 t^{-m-a} \|h_1 - h_2\|
\end{align*}$$

(30)

(see (17)) for some constant $A_1$. With $Q_j(s)(j=1, 2)$ defined by

$$Q_j(s) = [s^{-m}(q(s)+h_j(s))]^{-\frac{1}{r}}[(s^{-m}q(s))^r+(s^{-m}h_j(s))^r],$$

applying the mean value theorem to $G(u, v)=u^{-1}v$ and invoking (20) and (21) yields

$$|Q_1(s) - Q_2(s)| \leq A_2 s^{-m-1} |h_1(s) - h_2(s)| + A_3(s^{-m}h_1(s))^r - (s^{-m}h_2(s))^r$$

(31)

for suitable constants $A_2$ and $A_3$. (Here we have also used (2), (27), and (28) to obtain the first term on the right.) From (17) and (31),

$$|Q_1(s) - Q_2(s)| \leq A_4 \|h_1 - h_2\| s^{-m-1-a}$$

(32)

for some constant $A_4$.

From (2), (29), (30) and (32),

$$|F(t; h_1) - F(t; h_2)| \leq A_5 \|h_1 - h_2\| t^{-m-a} \phi(t),$$

(33)

for some constant $A_5$, with

$$\phi(t) = \sup_{T \geq t} \phi(T) = o(1).$$

(34)

Here we have used (2) again.

The convergence of (4) and (25) imply that the function

$$G(t; h) = \int_t^\infty s^{u-v-1+a}[-f(s) + P(s)(q(s)+h(s))^r]ds$$

is defined on $[t_0, \infty)$. Moreover,

$$|G(t; h)| \leq |G(t; 0)| + |G(t; h) - G(t; 0)| = |G(t; 0)| + |F(t; h) - F(t; 0)|,$$

so that invoking (33) with $h_1 = h$ and $h_2 = 0$ (and recalling that $\|h\| \leq M$) yields

$$|G(t; h)| \leq \sigma(t) = A_5 M t^{s-m-a} + \sup_{T \geq t} |G(T; 0)|.$$
Now Lemma 2 with $u = -f + P(q + h)^r$ implies that $\hat{h}$ as defined by (10) or (11) is in $H(t_0)$, and that

\[ ||\hat{h}|| \leq K\sigma(t_0) \]

for a suitable constant $K$. Moreover, if $\hat{h}_i = \mathcal{F}h_i$ ($i = 1, 2$), we can apply Lemma 2 with

\[ u = P[(q + h)^r - (q + h_0)^r], \]

and conclude from (33) that

\[ ||\hat{h}_1 - \hat{h}_2|| \leq KA_4 t_0^{n-\alpha} \phi(t_0) ||h_1 - h_2||. \]

Since $\sigma$ and $\phi$ both decrease to zero as $t \to \infty$, we can choose $t_0$ so that

\[ K\sigma(t_0) \leq M \]

and

\[ KA_4 t_0^{n-\alpha} \phi(t_0) < 1. \]

Now (35) and (37) imply that $\mathcal{F}$ maps $H_p(t_0)$ into itself, and (36) and (38) imply that $\mathcal{F}$ is a contraction mapping. Therefore there is a function $h_0$ in $H_p(t_0)$ such that $h_0 = \mathcal{F}h_0$. Since (23) or (24) holds with $\hat{h} = h = h_0$, the function $y_0 = q + h_0$ satisfies (1). Moreover, Lemma 2 (specifically, (14)) with $u = w = h_0$ implies that

\[ h_0^{(r)}(t) = o(t^{\nu-r}), \quad 0 \leq r \leq n-1, \]

and this implies (6). This completes the proof of Theorem 1.

We now consider the case where $m = \nu$ and $\alpha = 0$, so that (2) does not hold; that is, we will give sufficient conditions for (1) to have a solution $y_0$ which satisfies

\[ y_0^{(r)}(t) = \begin{cases} (a_0 + o(1))t^{\nu-r}/(\nu-r)! , & 0 \leq r \leq \nu , \\ o(t^{\nu-r}) , & \nu + 1 \leq r \leq n-1 . \end{cases} \]

A digression is needed to formulate this condition.

**Lemma 3.** Suppose $u \in C[t_0, \infty)$ for some $t_0 \geq 0$ and $\int_{t_0}^{\infty} t^{k-r}u(t)\,dt$ converges. Define

\[ I_0(t; u) = u(t) \]

and

\[ I_j(t; u) = \int_{t}^{\infty} \frac{(s-t)^{j-1}}{(j-1)!} u(s)\,ds, \quad 1 \leq j \leq k. \]
Then the integrals (40) converge and satisfy the inequalities

\begin{equation}
|I_j(t; u)| \leq \frac{2\delta(t)t^{j-k}}{(j-1)!}, \quad 1 \leq j \leq k,
\end{equation}

where

\[ \delta(t) = \sup_{r \geq t} \left| \int_r^\infty s^{j-1} u(s) ds \right| . \]

The integrals

\begin{equation}
\int_t^\infty t^{j-1} I_j(t; u) dt, \quad 0 \leq j \leq k-1,
\end{equation}

all converge, and if this convergence is absolute for some \( j_0 \) in \( \{0, 1, \ldots, k-1\} \), then it is absolute for \( j_0 \leq j \leq k-1 \).

**Proof.** The convergence of the integrals (40) and inequality (41) follow from Lemma 1. Since

\begin{equation}
I_j'(t; u) = -I_{j-1}(t; u), \quad 1 \leq j \leq k-1,
\end{equation}

integration by parts yields

\[ \int_{t_1}^{t_2} t^{j-1} I_j(t; u) dt = \frac{t^{k-j}}{k-j} I_j(t; u) \bigg|_{t_1}^{t_2} + \frac{1}{k-j} \int_{t_1}^{t_2} t^{j-1} I_{j-1}(t; u) dt, \]

so (41) and the assumed convergence of

\[ \int_t^\infty t^{j-1} I_j(t; u) dt = \int_t^\infty t^{j-1} u(t) dt \]

imply that (42) converges, by finite induction. If

\begin{equation}
\int_t^\infty t^{j-1} |I_j(t; u)| dt < \infty
\end{equation}

for some \( j < k-1 \), then

\begin{equation}
\int_t^\infty |I_j(s; u)| ds = o(t^{-k+j+1}),
\end{equation}

and

\[
\int_{t_1}^{t_2} t^{k-j-1} \left( \int_t^\infty |I_j(s; u)| ds \right) dt = \frac{t^{k-j-1}}{k-j-1} \int_t^{t_2} |I_j(s; u)| ds \bigg|_{t_1}^{t_2} + \frac{1}{k-j-1} \int_{t_1}^{t_2} t^{j-1} |I_j(t; u)| dt.
\]
Now (44) and (45) imply that
\[
\int_{t}^{\infty} t^{k-j-\frac{3}{2}} \left( \int_{t}^{\infty} |I_{j}(s; u)| \, ds \right) < \infty,
\]
which in turn implies that
\[
\int_{t}^{\infty} t^{k-j-\frac{3}{2}} |I_{j+1}(t; u)| \, dt < \infty,
\]
since
\[
|I_{j+1}(t; u)| \leq \int_{t}^{\infty} |I_{j}(s; u)| \, ds.
\]
(See (43) with \( j \) replaced by \( j + 1 \).) This completes the proof of Lemma 1.

If \( 1 \leq j_{0} \leq k - 1 \), there are functions \( u \) such that
\[
\int_{t}^{\infty} t^{k-j-\frac{3}{2}} |I_{j}(t; u)| \, dt \begin{cases} = \infty & \text{if } 0 \leq j \leq j_{0} - 1, \\ < \infty & \text{if } j_{0} \leq j \leq k - 1. \end{cases}
\]

For example, the function
\[
u(t) = t^{-k} \sin t
\]
satisfies this condition with \( j_{0} = 1 \). A rather tedious argument involving repeated integration by parts shows that the function
\[
u(t) = t^{-k} \cos ((\log t)^{\nu+1})
\]
satisfies (46) if \( j_{0}^{-1} < \alpha < (j_{0} - 1)^{-1} \).

**Theorem 2.** Let \( \nu \) be an integer in \( \{0, 1, \ldots, n-1\} \) and suppose the integrals
\[
\int_{0}^{\infty} t^{n-1+\nu(j-1)} P(t) \, dt
\]
and
\[
\int_{0}^{\infty} t^{n-j-1} f(t) \, dt
\]
converge. Suppose also that
\[
\int_{0}^{\infty} t^{\nu(j-1)} |I_{n-1}(t; P)| \, dt < \infty \quad \text{if } \gamma \geq 1,
\]
or that
(49) \[ \int |I_{n-1}(t; Q)| \, dt < \infty \quad \text{if} \quad r < 1, \]

where

(50) \[ Q(t) = t^{\nu(r-1)} P(t). \]

Let \( a_s \) be an arbitrary positive constant. Then (1) has a solution \( y_0 \) which is defined for sufficiently large \( t \) and satisfies (39).

**Proof.** For \( t_0 \geq 0 \), let \( H(t) \) be the Banach space of functions \( h \) in \( C^{(n-1)}[t_0, \infty) \) such that

\[ h^{(r)}(t) = 0(t^{r-1}), \quad 0 \leq r \leq n-1, \]

with norm

\[ \| h \| = \sup \left\{ \sum_{r=0}^{n-1} t^{-r} |h^{(r)}(t)| \right\}, \]

and let \( H_0(t_0) \) be as in (18). It is convenient here to write

(51) \[ u(t) = \frac{a_s}{\nu!} t^r + h(t), \quad h \in H_0(t_0). \]

Since \( a_s > 0 \), there are constants \( M, \lambda \) and \( T_0 \) such that

(52) \[ u(t) \geq \frac{1}{2} \frac{a_s}{\nu!} t^r \]

and

(53) \[ |u^{(r)}(t)| \leq \lambda t^{r-1}, \quad 0 \leq r \leq n-1, \]

if (51) holds and \( t \geq t_0 \geq T_0 \), which we assume henceforth. As in the proof of Theorem 1, we will show that \( T \) as defined by (22) and (23) or (24) is a contraction mapping of \( H_0(t_0) \) into itself if \( t_0 \) is sufficiently large; therefore, we first consider the integral

(54) \[ F(t; h) = \int_t^\infty s^{n-r-1} P(s)(u(s))r \, ds \]

(recall (51)), which is the appropriate analog of (25). We must consider two cases, depending upon \( r \).

**Case 1.** Suppose \( r \geq 1 \). Then (43) and repeated integration by parts yields

(55) \[ \int_t^\infty s^{n-r-1} P(s)(u(s))r \, ds \]

\[ = - \sum_{j=1}^{n-1} I_j(s; P)[s^{n-r-1}(u(s))^r]^{(j-1)} |_t^\infty + \int_t^\infty I_{n-1}(s; P)[s^{n-r-1}(u(s))^r]^{(n-1)} \, ds. \]
From the formula of Faa di Bruno [3] for the derivatives of a composite function,

\[ \frac{d^l}{ds^l} u^r = \sum_{k=1}^{r} (\gamma)^{(k)} u^{r-k} \sum_{k_1, \ldots, k_l} \frac{k!}{k_1! \cdots k_l!} \left( \frac{u'}{1!} \right)^{k_1} \left( \frac{u''}{2!} \right)^{k_2} \cdots \left( \frac{u^{(l)}}{l!} \right)^{k_l} \]

if \( l = 1, 2, \ldots \), where

\[ (\gamma)^{(k)} = \gamma(\gamma-1) \cdots (\gamma-k+1) \]

and \( \sum_{k} \) is over all partitions of \( k \) as a sum of nonnegative integers,

\[ k_1 + k_2 + \cdots + k_l = k \]

such that

\[ k_1 + 2k_2 + \cdots + lk_l = l. \]

From Leibniz's formula for the derivatives of a product

\[ [s^{n-v-1} u(s)]^{(j-1)} = \sum_{l=0}^{j-1} \binom{j-1}{l} (s^{n-v-1})^{(j-l-1)} [(u(s))']^{(l)} \]

From (52), (53), (56) and (59), it can be shown that

\[ [s^{n-v-1} (u(s))']^{(j-1)} \leq B_i s^{n-v+j+\gamma-1}, \quad 1 \leq j \leq n-1, \]

for some constant \( B_i \). (To verify this it is important to invoke (57) and (58).) However, from Lemma 1 and the convergence of (47),

\[ |I_i(s; P)| \leq \frac{2\delta(s) s^{n-v+j+\gamma-1}}{(j-1)!}, \quad 1 \leq j \leq n, \]

where

\[ \delta(t) = \sup_{T \geq t} \left| \int_T^s s^{n-1+v(\gamma-1)} P(s) ds \right|. \]

From (60) and (61), we can let \( t \to \infty \) in (55) to obtain

\[ F(t; h) = \sum_{j=1}^{n-1} I_j(t; P) [t^{n-v-1} (u(t))]^{(j-1)} + \int_t^\infty I_{n-j}(s; P) [s^{n-v-1} (u(s))]^{(n-j)} ds, \]

where the integral on the right converges absolutely because of (48) and (60) with \( j = n \).

Now suppose
\[ u(t) = \frac{a_r}{\nu!} t^r + h(t), \quad i = 1, 2. \]

By applying the mean value theorem to the function
\[ G_i(x_0, x_1, \ldots, x_i) = \sum_{k=1}^{i} \frac{(t^k - x)^{(k-1)}}{k!} \sum_{k} \frac{k!}{k_1! \ldots k_i!} \left( \frac{x_1^{k_1}}{1!} \right) \left( \frac{x_2^{k_2}}{2!} \right) \cdots \left( \frac{x_i^{k_i}}{i!} \right) \]
(see (56), (57) and (58)), and then using estimates similar to those which led to (60), it can be shown that
\[
|s^{n-\nu-1}[(u_t(s))'-a_t(s)]^{(j-1)}| \leq C_j \| h_1 - h_2 \| s^{n-j+\nu(r-1)} , \quad 1 \leq j \leq n,
\]
where \( C_1, \ldots, C_n \) are constants. This, (61) and (63) imply that
\[
|F(t; h_1) - F(t; h_2)| \leq \| h_1 - h_2 \| \left( K_t \delta(t) + C_n \int_t^s s^{\nu(r-1)} |I_{n-1}(s; P)| ds \right)
\]
where \( K_t \) is a constant.

Case 2. Suppose \( \nu < 1 \). Then we rewrite (54) as
\[
F(t; h) = \int_t^s s^{n-\nu-1} Q(s)(u(s))' ds
\]
(see (50)), and proceed as in Case 1, to obtain
\[
F(t; h) = \sum_{j=1}^{n-1} I_j(t; Q) [t^{n-\nu-1}(u(t))']^{(j-1)} + \int_t^s I_{n-1}(s; Q) [s^{n-\nu-1}(u(s))']^{(n-1)} ds,
\]
where the integral on the right converges absolutely because of (49), and
\[
|F(t; h_1) - F(t; h_2)| \leq \| h_1 - h_2 \| \left( \hat{K}_t \delta(t) + \hat{C}_n \int_t^s |I_{n-1}(s; Q)| ds \right),
\]
where \( \hat{K}_t \) and \( \hat{C}_n \) are constants, and \( \delta \) is as in (62).

Now that we have shown that \( F(\cdot; h) \) satisfies a Lipschitz condition with respect to \( h \) for all real \( \nu \), the rest of the proof is similar to the part of the proof of Theorem 1 which follows (34).

**Remark.** If \( \nu \) is rational with odd denominator, so that \( y^\nu \) is real-valued for \( y < 0 \), then only trivial modifications of the proofs given above show that the conclusions of Theorems 1 and 2 are also valid if \( a_m < 0 \) or \( a_n < 0 \), respectively. A similar comment applies if (1) is replaced by
\[
y^{(m)} + P(t)|y|^\nu \operatorname{sgn} y = f(t),
\]
without restrictions on (real) \( \nu \).
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References


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