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Asymptotic Integration of $y^{(n)}+P(t)y^{\gamma}=f(t)$ under Mild Integral Smallness Conditions

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This paper gives sufficient conditions for the equation

$$y^{(n)}+P(t)y^{\gamma}=f(t) \qquad (n\geq 2) \tag{1}$$

to have solutions which behave like polynomials of degree $<n$ as $t\rightarrow\infty$. This question has been investigated by many authors, but, to our knowledge, always under integral smallness conditions on P and f which require absolute convergence of all improper integrals in question. Also, most authors have assumed that $\gamma>0$. (For exceptions to this, see [1], [2], and [4].) Here we allow γ to be any real number. Moreover, our integral smallness conditions require only ordinary (i.e., perhaps conditional) convergence, except for a condition on P in Theorem 2 which does require absolute convergence, but is still considerably weaker than the usual condition.

We assume throughout that P and f are real-valued and continuous on $(0, \infty)$ and that γ is real. When we say that an improper integral converges, we mean that it may converge conditionally, unless, of course, it is clear that the integrand is nonnegative.

Theorem 1. *Suppose ν and m are integers, $0\leq\nu\leq m\leq n-1$, and let α be a nonnegative number such that*

$$\nu-\alpha<m; \tag{2}$$

moreover, suppose that $\alpha<1$ if $\nu\neq 0$. Assume that the integrals

$$\int^{\infty} t^{n-\nu-1+\alpha+m\gamma}P(t)dt \tag{3}$$

and

$$\int^{\infty} t^{n-\nu-1+\alpha}f(t)dt \tag{4}$$

converge. Let

$$q(t)=\sum_{j=\nu}^{m} a_j\frac{t^j}{j!}. \tag{5}$$

where $a_\nu, \cdots, a_m$ *are given real constants, with* $a_m>0$. *Then* (1) *has a solution* y_0, *defined for sufficiently large* t, *such that*

$$y_0^{(r)}(t)=p^{(r)}(t)+o(t^{\nu-r-\alpha}), \qquad 0\leq r\leq n-1. \tag{6}$$

The following two lemmas will be useful in proving Theorem 1 as well as Theorem 2, which is stated below.

Lemma 1. *Suppose* $u \in C[t_0, \infty)$ *for some* $t_0\geq 0$ *and let* a *and* b *be constants,* $0\leq a\leq b$. *Suppose also that* $\int^\infty t^b u(t)dt$ *converges, and define*

$$\rho_0(t)=\sup_{T\geq t}\left|\int_T^\infty s^b u(s)ds\right|.$$

Then

$$\left|\int_t^\infty (s-t_1)^a u(s)ds\right|\leq 2\rho_0(t)t^{a-b}, \qquad t\geq t_1\geq t_0. \tag{7}$$

Proof. Let $U(t)=\int_t^\infty s^b u(s)ds$, and note that

$$|U(t)|\leq\rho_0(t). \tag{8}$$

Now,

$$(s-t_1)^a u(s)=-\left(1-\frac{t_1}{s}\right)^a s^{a-b}U'(s),$$

so integrating by parts yields

$$\int_t^{\bar{t}}(s-t_1)^a u(s)ds=-\left(1-\frac{t_1}{s}\right)^a s^{a-b}U(s)\Big|_t^{\bar{t}}+\int_t^{\bar{t}} U(s)\frac{d}{ds}\left[\left(1-\frac{t_1}{s}\right)^a s^{a-b}\right]ds. \tag{9}$$

But

$$\frac{d}{ds}\left[\left(1-\frac{t_1}{s}\right)^a s^{a-b}\right]=\left(1-\frac{t_1}{s}\right)^a\frac{d}{ds}(s^{a-b})+s^{a-b}\frac{d}{ds}\left[\left(1-\frac{t_1}{s}\right)^a\right],$$

where the first product is nonpositive and the second is positive if $s>t_1$. This enables us to let $\bar{t}\to\infty$ in (9) and infer that

$$\begin{aligned}\left|\int_t^\infty (s-t_1)^a u(s)ds\right|&\leq\left(1-\frac{t_1}{t}\right)^a t^{a-b}|U(t)|-\int_t^\infty\left(1-\frac{t_1}{s}\right)^a|U(s)|\frac{d}{ds}(s^{a-b})ds\\&\quad+\int_t^\infty|U(s)|s^{a-b}\frac{d}{ds}\left(1-\frac{t_1}{s}\right)^a ds\\&\leq 2\rho_0(t)t^{a-b}.\end{aligned}$$

(see (8)). This completes the proof of Lemma 1.

Lemma 2. *Suppose $u \in C[t_0, \infty)$ for some $t_0 \geq 0$ and*

$$\int^{\infty} t^{n-\nu-1+\alpha} u(t)dt$$

converges, where ν and α are as in Theorem 1. Define

$$w(t)=\int_t^{\infty} \frac{(t-s)^{n-1}}{(n-1)!} u(s)ds \qquad \text{if } \nu=0, \tag{10}$$

or

$$w(t)=\int_{t_0}^{t} \frac{(t-\lambda)^{\nu-1}}{(\nu-1)!} d\lambda \int_{\lambda}^{\infty} \frac{(\lambda-s)^{n-\nu-1}}{(n-\nu-1)!} u(s)ds \qquad \text{if } 1 \leq \nu \leq n-1, \tag{11}$$

and let

$$\rho(t)=\sup_{T \geq t} \left| \int_T^{\infty} s^{n-\nu-1+\alpha} u(s)ds \right|.$$

Then $w \in C^{(n)}[t_0, \infty)$,

$$|w^{(r)}(t)| \leq \frac{2\rho(t) t^{\nu-r-\alpha}}{(n-r-1)!}, \qquad \nu \leq r \leq n-1, \tag{12}$$

and, if $\nu \geq 1$,

$$|w^{(r)}(t)| \leq \frac{2\rho(t_0) t^{\nu-r-\alpha}}{(n-\nu-1)! \prod_{j=1}^{\nu-r} (j-\gamma)}, \qquad 0 \leq r \leq \nu-1. \tag{13}$$

Moreover,

$$w^{(r)}(t)=o(t^{\nu-r-\alpha}), \qquad 0 \leq r \leq n-1. \tag{14}$$

Proof. From Lemma 1 with

$$a=n-r-1, \quad b=n-\nu-1+\alpha, \quad t=t_1,$$

$$\left| \int_t^{\infty} (t-s)^{n-r-1} u(s)ds \right| \leq 2\rho(t) t^{\nu-r-\alpha}, \qquad \nu \leq r \leq n-1. \tag{15}$$

This implies that w as defined by (10) or (11) is in $C^{(n)}[t_0, \infty)$ and satisfies (12) and (14) for $\nu \leq r \leq n-1$. Therefore, the proof is complete if $\nu=0$. If $0 \leq r \leq \nu-1$, then

$$|w^{(r)}(t)| \leq \frac{2}{(\nu-r-1)!\,(n-\nu-1)!} \int_{t_0}^{t} (t-\lambda)^{\nu-r-1} \lambda^{-\alpha} \rho(\lambda) d\lambda \tag{16}$$

from (11) and (15) (the latter with $r=\nu$). Since ρ is nonincreasing, we may replace

$\rho(\lambda)$ by $\rho(t_0)$ here, then replace t_0 by zero in the lower limit of integration (recall that $\alpha<1$), and integrate repeatedly by parts to obtain (13).

From (16),

$$|w^{(r)}(t)|t^{-\nu+r+\alpha}\leq\frac{2t^{\alpha-1}}{(\nu-r-1)!\,(n-\nu-1)!}\int_{t_0}^{t}\rho(\lambda)\lambda^{-\alpha}d\lambda,$$

which implies (14) for $0\leq r\leq\nu-1$. $\left(\text{If}\int^{\infty}\rho(\lambda)\lambda^{-\alpha}d\lambda<\infty\right.$, this is obvious; if $\int^{\infty}\rho(\lambda)\lambda^{-\alpha}d\lambda=\infty$, it follows from l'Hospital's rule. Here again we have used the assumption that $\left.\alpha<1.\right)$ This completes the proof of Lemma 2.

Proof of Theorem 1. For $t_0\geq0$, let $H(t_0)$ be the Banach space of functions h in $C^{(n-1)}[t_0,\infty)$ such that

$$h^{(r)}(t)=0(t^{\nu-r-\alpha}),\qquad 0\leq r\leq n-1,$$

with norm

$$\|h\|=\sup_{t\geq t_0}\left\{t^{-\nu+\alpha}\sum_{r=0}^{n-1}t^r|h^{(r)}(t)|\right\}. \tag{17}$$

For $M>0$, let

$$H_M(t_0)=\{h\in H(t_0)\,|\,\|h\|\leq M\}. \tag{18}$$

Since $\nu\leq m$ and $a_m>0$ in (5), there are constants M, λ, and T_0 such that if

$$t_0\geq T_0\quad\text{and}\quad h\in H_M(t_0), \tag{19}$$

then

$$q(t)+h(t)\geq\frac{1}{2}a_m\frac{t^m}{m!} \tag{20}$$

and

$$|q^{(r)}(t)+h^{(r)}(t)|\leq\lambda t^{m-r},\qquad 0\leq r\leq n-1, \tag{21}$$

for all $t\geq t_0$. (From (20), $(q+h)^\gamma$ is defined and real-valued on $[t_0,\infty)$ if (19) holds.) We assume henceforth that h, h_1, and h_2 are in $H_M(t_0)$ for some $t_0\geq T_0$. The constants appearing in estimates that follow may depend upon T_0, but they do not depend upon t_0, h, h_1, h_2, etc.. We assume that $t\geq t_0$ throughout.

We will show that the transformation

$$\hat{h}=\mathscr{T}h \tag{22}$$

defined by

$$\hat{h}(t)=\int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!}[-f(s)+P(s)(q(s)+h(s))^\gamma]ds \qquad \text{if } \nu=0 \tag{23}$$

or by

$$\hat{h}(t)=\int_{t_0}^t \frac{(t-\lambda)^{\nu-1}}{(\nu-1)!}d\lambda\int_\lambda \frac{(\lambda-s)^{n-\nu-1}}{(n-\nu-1)!}[-f(s)+P(s)(q(s)+h(s))^\gamma]ds \qquad \text{if } 1\leq\nu\leq n-1, \tag{24}$$

is a contraction mapping of $H_M(t_0)$ into itself if t_0 is sufficiently large. To this end, we first study the integral

$$F(t;h)=\int_t^\infty s^{n-\nu-1+\alpha}P(s)(q(s)+h(s))^\gamma ds. \tag{25}$$

The convergence of this integral follows easily from Dirichlet's test and the convergence of (3); nevertheless, we will write out the details of the proof, because they will be useful in obtaining estimates that we need below. Let

$$\Phi(t)=\int_t^\infty s^{n-\nu-1+\alpha+m\gamma}P(s)ds,$$

which exists, because (3) converges. If $\tau\geq t$, then

$$\begin{aligned}
&\int_t^\tau s^{n-\nu-1+\alpha}P(s)(q(s)+h(s))^\gamma ds\\
&=-\int_t^\tau \Phi'(s)[s^{-m}(q(s)+h(s))]^\gamma ds\\
&=-\Phi(s)[s^{-m}(q(s)+h(s))]^\gamma\Big|_t^\tau\\
&\quad+\gamma\int_t^\tau \Phi(s)[s^{-m}(q(s)+h(s))]^{\gamma-1}[(s^{-m}q(s))'+(s^{-m}h(s))']ds.
\end{aligned} \tag{26}$$

Now,

$$(s^{-m}q(s))'=0(s^{-2}) \tag{27}$$

and

$$|(s^{-m}h(s))'|\leq(m+1)Ms^{\nu-m-1-\alpha}. \tag{28}$$

(See (17) and (18).) Since $\Phi(t)=o(1)$, the last two inequalities together with (2), (20) and (21) enable us to let $\tau\rightarrow\infty$ in (26) to obtain

$$\begin{aligned}
F(t;h)&=\Phi(t)[t^{-m}(q(t)+h(t))]^\gamma\\
&\quad+\gamma\int_t^\infty \Phi(s)[s^{-m}(q(s)+h(s))]^{\gamma-1}[(s^{-m}q(s))'+(s^{-m}h(s))']ds,
\end{aligned} \tag{29}$$

where the integral on the right converges absolutely.

We will now show that $F(t; h)$ satisfies a Lipschitz condition with respect to h. Applying the mean value theorem to $G(u)=u^{\gamma}$ and invoking (20) if $\gamma<0$ or (21) with $r=0$ if $\gamma>0$ yields the inequality

$$\begin{aligned} &|[t^{-m}(q(t)+h_1(t))]^{\gamma}-[t^{-m}(q(t)+h_2(t))]^{\gamma}| \\ &\le A_1 t^{-m}|h_1(t)-h_2(t)| \\ &\le A_1 t^{\nu-m-\alpha}\|h_1-h_2\| \end{aligned} \tag{30}$$

(see (17)) for some constant A_1. With $Q_j(s)(j=1, 2)$ defined by

$$Q_j(s)=[s^{-m}(q(s)+h_j(s))]^{\gamma-1}[(s^{-m}q(s))'+(s^{-m}h_j(s))'],$$

applying the mean value theorem to $G(u, v)=u^{\gamma-1}v$ and invoking (20) and (21) yields

$$|Q_1(s)-Q_2(s)|\le A_2 s^{-m-1}|h_1(s)-h_2(s)|+A_3|(s^{-m}h_1(s))'-(s^{-m}h_2(s))'| \tag{31}$$

for suitable constants A_2 and A_3. (Here we have also used (2), (27), and (28) to obtain the first term on the right.) From (17) and (31),

$$|Q_1(s)-Q_2(s)|\le A_4\|h_1-h_2\|s^{\nu-m-1-\alpha} \tag{32}$$

for some constant A_4.

From (2), (29), (30) and (32),

$$|F(t; h_1)-F(t; h_2)|\le A_5\|h_1-h_2\|t^{\nu-m-\alpha}\phi(t), \tag{33}$$

for some constant A_5, with

$$\phi(t)=\sup_{T\ge t}\Phi(T)=o(1). \tag{34}$$

Here we have used (2) again.

The convergence of (4) and (25) imply that the function

$$G(t; h)=\int_t^{\infty} s^{n-\nu-1+\alpha}[-f(s)+P(s)(q(s)+h(s))^{\gamma}]ds$$

is defined on $[t_0, \infty)$. Moreover,

$$\begin{aligned} |G(t; h)|&\le|G(t; 0)|+|G(t; h)-G(t; 0)| \\ &=|G(t; 0)|+|F(t; h)-F(t; 0)|, \end{aligned}$$

so that invoking (33) with $h_1=h$ and $h_2=0$ (and recalling that $\|h\|\le M$) yields

$$|G(t; h)|\le\sigma(t)=A_5Mt^{\nu-m-\alpha}+\sup_{T\ge t}|G(T; 0)|.$$

Now Lemma 2 with $u=-f+P(q+h)^r$ implies that $\hat{h}$ as defined by (10) or (11) is in $H(t_0)$, and that

$$\|\hat{h}\|\leq K\sigma(t_0) \tag{35}$$

for a suitable constant K. Moreover, if $\hat{h}_i=\mathscr{T}h_i$ $(i=1, 2)$, we can apply Lemma 2 with

$$u=P[(q+h_1)^r-(q+h_2)^r],$$

and conclude from (33) that

$$\|\hat{h}_1-\hat{h}_2\|\leq KA_5t_0^{\nu-m-\alpha}\phi(t_0)\|h_1-h_2\|. \tag{36}$$

Since σ and ϕ both decrease to zero as $t\rightarrow\infty$, we can choose t_0 so that

$$K\sigma(t_0)\leq M \tag{37}$$

and

$$KA_5t_0^{\nu-m-\alpha}\phi(t_0)<1. \tag{38}$$

Now (35) and (37) imply that $\mathscr{T}$ maps $H_M(t_0)$ into itself, and (36) and (38) imply that $\mathscr{T}$ is a contraction mapping. Therefore there is a function h_0 in $H_M(t_0)$ such that $h_0=\mathscr{T}h_0$. Since (23) or (24) holds with $\hat{h}=h=h_0$, the function $y_0=q+h_0$ satisfies (1). Moreover, Lemma 2 (specifically, (14)) with $u=w=h_0$ implies that

$$h_0^{(r)}(t)=o(t^{\nu-r-\alpha}),\qquad 0\leq r\leq n-1,$$

and this implies (6). This completes the proof of Theorem 1.

We now consider the case where $m=\nu$ and $\alpha=0$, so that (2) does not hold; that is, we will give sufficient conditions for (1) to have a solution y_0 which satisfies

$$y_0^{(r)}(t)=\begin{cases}(a_\nu+o(1))t^{\nu-r}/(\nu-r)!, & 0\leq r\leq\nu,\\ o(t^{\nu-r}), & \nu+1\leq r\leq n-1.\end{cases} \tag{39}$$

A digression is needed to formulate this condition.

Lemma 3. *Suppose* $u\in C[t_0, \infty)$ *for some* $t_0\geq 0$ *and* $\int^\infty t^{k-1}u(t)dt$ *converges. Define*

$$I_0(t; u)=u(t)$$

and

$$I_j(t; u)=\int_t^\infty \frac{(s-t)^{j-1}}{(j-1)!}u(s)ds,\qquad 1\leq j\leq k. \tag{40}$$

Then the integrals (40) *converge and satisfy the inequalities*

$$|I_j(t;u)|\le\frac{2\delta(t)t^{j-k}}{(j-1)!},\qquad 1\le j\le k, \tag{41}$$

where

$$\delta(t)=\sup_{T\ge t}\left|\int_T^\infty s^{k-1}u(s)ds\right|.$$

The integrals

$$\int^\infty t^{k-j-1}I_j(t;u)dt,\qquad 0\le j\le k-1, \tag{42}$$

all converge, and if this convergence is absolute for some j_0 *in* $\{0, 1, \cdots, k-1\}$, *then it is absolute for* $j_0\le j\le k-1$.

Proof. The convergence of the integrals (40) and inequality (41) follow from Lemma 1. Since

$$I_j'(t;u)=-I_{j-1}(t;u),\qquad 1\le j\le k-1, \tag{43}$$

integration by parts yields

$$\int_{t_1}^{t_2} t^{k-j-1}I_j(t;u)\,dt=\frac{t^{k-j}}{k-j}I_j(t;u)\Big|_{t_1}^{t_2}+\frac{1}{k-j}\int_{t_1}^{t_2} t^{k-j}I_{j-1}(t;u)dt,$$

so (41) and the assumed convergence of

$$\int^\infty t^{k-1}I_0(t;u)dt=\int^\infty t^{k-1}u(t)dt$$

imply that (42) converges, by finite induction. If

$$\int^\infty t^{k-j-1}|I_j(t;u)|\,dt<\infty \tag{44}$$

for some $j<k-1$, then

$$\int_t^\infty |I_j(s;u)|ds=o(t^{-k+j+1}), \tag{45}$$

and

$$\int_{t_1}^{t_2} t^{k-j-2}\left(\int_t^\infty |I_j(s;u)|\,ds\right)dt$$
$$=\frac{t^{k-j-1}}{k-j-1}\int_t^\infty |I_j(s;u)|\,ds\Big|_{t_1}^{t_2}+\frac{1}{k-j-1}\int_{t_1}^{t_2} t^{k-j-1}|I_j(t;u)|\,dt.$$

Now (44) and (45) imply that

$$\int^{\infty} t^{k-j-2}\left(\int_t^{\infty} |I_j(s; u)|\, ds\right) < \infty,$$

which in turn implies that

$$\int^{\infty} t^{k-j-2} |I_{j+1}(t; u)|\, dt < \infty,$$

since

$$|I_{j+1}(t: u)| \leq \int_t^{\infty} |I_j(s; u)|\, ds.$$

(See (43) with j replaced by $j+1$.) This completes the proof of Lemma 1.

If $1 \leq j_0 \leq k-1$, there are functions u such that

$$(46) \qquad \int^{\infty} t^{k-j-1} |I_j(t; u)|\, dt \quad \begin{cases} = \infty & \text{if } 0 \leq j \leq j_0 - 1, \\ < \infty & \text{if } j_0 \leq j \leq k-1. \end{cases}$$

For example, the function

$$u(t) = t^{-k} \sin t$$

satisfies this condition with $j_0 = 1$. A rather tedious argument involving repeated integration by parts shows that the function

$$u(t) = t^{-k} \cos\left((\log t)^{\alpha+1}\right)$$

satisfies (46) if $j_0^{-1} < \alpha < (j_0 - 1)^{-1}$.

Theorem 2. *Let* ν *be an integer in* $\{0, 1, \cdots, n-1\}$ *and suppose the integrals*

$$(47) \qquad \int^{\infty} t^{n-1+\nu(\gamma-1)} P(t)\, dt$$

and

$$\int^{\infty} t^{n-\nu-1} f(t)\, dt$$

converge. Suppose also that

$$(48) \qquad \int^{\infty} t^{\nu(\gamma-1)} |I_{n-1}(t; P)|\, dt < \infty \qquad \text{if } \gamma \geq 1,$$

or that

$$\int^{\infty} |I_{n-1}(t; Q)|\, dt < \infty \qquad \textit{if } \gamma < 1, \tag{49}$$

where

$$Q(t) = t^{\nu(\gamma-1)} P(t). \tag{50}$$

Let a_ν be an arbitrary positive constant. Then (1) *has a solution y_0 which is defined for sufficiently large t and satsfies* (39).

Proof. For $t_0 \geq 0$, let $H(t_0)$ be the Banach space of functions h in $C^{(n-1)}[t_0, \infty)$ such that

$$h^{(r)}(t) = 0(t^{\nu-r}), \qquad 0 \leq r \leq n-1,$$

with norm

$$\|h\| = \sup_{t \geq t_0} \left\{ \sum_{r=0}^{n-1} t^{r-\nu} |h^{(r)}(t)| \right\},$$

and let $H_M(t_0)$ be as in (18). It is convenient here to write

$$u(t) = \frac{a_\nu}{\nu!} t^\nu + h(t), \qquad h \in H_M(t_0). \tag{51}$$

Since $a_\nu > 0$, there are constants M, λ and T_0 such that

$$u(t) \geq \frac{1}{2} \frac{a_\nu}{\nu!} t^\nu \tag{52}$$

and

$$|u^{(r)}(t)| \leq \lambda t^{\nu-r}, \qquad 0 \leq r \leq n-1, \tag{53}$$

if (51) holds and $t \geq t_0 \geq T_0$, which we assume henceforth. As in the proof of Theorem 1, we will show that $\mathscr{T}$ as defined by (22) and (23) or (24) is a contraction mapping of $H_M(t_0)$ into itself if t_0 is sufficiently large; therefore, we first consider the integral

$$F(t; h) = \int_t^\infty s^{n-\nu-1} P(s) (u(s))^\gamma ds \tag{54}$$

(recall (51)), which is the appropriate analog of (25). We must consider two cases, depending upon γ.

Case 1. Suppose $\gamma \geq 1$. Then (43) and repeated integration by parts yields

$$\begin{aligned} &\int_t^{\bar{t}} s^{n-\nu-1} P(s) (u(s))^\gamma ds \\ &\qquad = -\sum_{j=1}^{n-1} I_j(s; P) [s^{n-\nu-1}(u(s))^\gamma]^{(j-1)} \Big|_t^{\bar{t}} + \int_t^{\bar{t}} I_{n-1}(s; P) [s^{n-\nu-1}(u(s))^\gamma]^{(n-1)} ds. \end{aligned} \tag{55}$$

From the formula of Faa di Bruno [3] for the derivatives of a composite function,

$$\frac{d^l}{ds^l}u^\gamma=\sum_{k=1}^{l}(\gamma)^{(k)}u^{\gamma-k}\sum_k\frac{k!}{k_1!\cdots k_l!}\left(\frac{u'}{1!}\right)^{k_1}\left(\frac{u''}{2!}\right)^{k_2}\cdots\left(\frac{u^{(l)}}{l!}\right)^{k_l} \tag{56}$$

if $l=1, 2, \cdots$, where

$$(\gamma)^{(k)}=\gamma(\gamma-1)\cdots(\gamma-k+1)$$

and $\sum_k$ is over all partitions of k as a sum of nonnegative integers,

$$k_1+k_2+\cdots+k_l=k \tag{57}$$

such that

$$k_1+2k_2+\cdots+lk_l=l. \tag{58}$$

From Leibniz's formula for the derivatives of a product

$$[s^{n-\nu-1}u(s)]^{(j-1)}=\sum_{l=0}^{j-1}\binom{j-1}{l}(s^{n-\nu-1})^{(j-l-1)}[(u(s))^\gamma]^{(l)}. \tag{59}$$

From (52), (53), (56) and (59), it can be shown that

$$|[s^{n-\nu-1}(u(s))^\gamma]^{(j-1)}|\leq B_1s^{n-j+\nu(\gamma-1)}, \qquad 1\leq j\leq n-1, \tag{60}$$

for some constant B_1. (To verify this it is important to invoke (57) and (58).) However, from Lemma 1 and the convergence of (47),

$$|I_j(s; P)|\leq\frac{2\delta(s)s^{-n+j-\nu(\gamma-1)}}{(j-1)!}, \qquad 1\leq j\leq n, \tag{61}$$

where

$$\delta(t)=\sup_{T\geq t}\left|\int_T^\infty s^{n-1+\nu(\gamma-1)}P(s)ds\right|. \tag{62}$$

From (60) and (61), we can let $\bar{t}\to\infty$ in (55) to obtain

$$\begin{aligned}F(t; h)=&\sum_{j=1}^{n-1}I_j(t; P)[t^{n-\nu-1}(u(t))^\gamma]^{(j-1)}\\&+\int_t^\infty I_{n-1}(s; P)[s^{n-\nu-1}(u(s))^\gamma]^{(n-1)}ds,\end{aligned} \tag{63}$$

where the integral on the right converges absolutely because of (48) and (60) with $j=n$.

Now suppose

$$u_i(t)=\frac{a_\nu}{\nu!}t^\nu+h_i(t), \qquad i=1, 2.$$

By applying the mean value theorem to the function

$$G_l(x_0, x_1, \cdots, x_l)=\sum_{k=1}^{l}(\gamma)^{(k)}x_0^{\gamma-k}\sum_k\frac{k!}{k_1!\cdots k_l!}\left(\frac{x_1^{k_1}}{1!}\right)\left(\frac{x_2^{k_2}}{2!}\right)\cdots\left(\frac{x_l^{k_l}}{l!}\right)$$

(see (56), (57) and (58)), and then using estimates similar to those which led to (60), it can be shown that

$$|s^{n-\nu-1}[(u_1(s))^\gamma-(u_2(s))^\gamma]^{(j-1)}|\le C_j\|h_1-h_2\|s^{n-j+\nu(\gamma-1)}, \qquad 1\le j\le n,$$

where $C_1, \cdots C_n$ are constants. This, (61) and (63) imply that

$$|F(t;h_1)-F(t;h_2)|\le\|h_1-h_2\|\left(K_1\delta(t)+C_n\int_t^\infty s^{\nu(\gamma-1)}|I_{n-1}(s;P)|\,ds\right)$$

where K_1 is a constant.

Case 2. Suppose $\gamma<1$. Then we rewrite (54) as

$$F(t;h)=\int_t^\infty s^{n-\nu\gamma-1}Q(s)(u(s))^\gamma ds$$

(see (50)), and proceed as in Case 1, to obtain

$$F(t;h)=\sum_{j=1}^{n-1}I_j(t;Q)[t^{n-\nu\gamma-1}(u(t))^\gamma]^{(j-1)}+\int_t^\infty I_{n-1}(s;Q)[s^{n-\nu\gamma-1}(u(s))^\gamma]^{(n-1)}ds,$$

where the integral on the right converges absolutely because of (49), and

$$|F(t;h_1)-F(t;h_2)|\le\|h_1-h_2\|\left(\hat{K}_1\delta(t)+\hat{C}_n\int_t^\infty|I_{n-1}(s;Q)|\,ds\right),$$

where $\hat{K}_1$ and $\hat{C}_n$ are constants, and δ is as in (62).

Now that we have shown that $F(\cdot\,;h)$ satisfies a Lipschitz condition with respect to h for all real γ, the rest of the proof is similar to the part of the proof of Theorem 1 which follows (34).

Remark. If γ is rational with odd denominator, so that y^γ is real-valued for $y<0$, then only trivial modifications of the proofs given above show that the conclusions of Theorems 1 and 2 are also valid if $a_m<0$ or $a_\nu<0$, respectively. A similar comment applies if (1) is replaced by

$$y^{(n)}+P(t)|y|^\gamma\operatorname{sgn}y=f(t),$$

without restrictions on (real) γ.

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