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ASYMPTOTIC INTEGRATION OF LINEAR DIFFERENTIAL EQUATIONS SUBJECT TO MILD INTEGRAL CONDITIONS*

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Abstract. Sufficient conditions are given for a linear differential equation of order n to have a solution which behaves asymptotically like a given polynomial of degree $<n$. The integral smallness conditions on the coefficient and forcing functions are stated largely in terms of ordinary (rather than absolute) convergence, and the manner in which the solution behaves like the given polynomial is specified precisely.

1. Introduction and main theorem. We study the behavior as $t\to\infty$ of solutions of the scalar equation

$$x^{(n)}+P_1(t)x^{(n-1)}+\cdots+P_n(t)x=f(t),\qquad t>0\,(n\geq 2), \tag{1}$$

where $P_1,\cdots,P_n$, f, and x may be complex-valued. We regard (1) as a perturbation of the equation

$$y^{(n)}=0, \tag{2}$$

and give conditions which imply that (1) has a solution x_0 which behaves for large t like a given polynomial p of degree $<n$. Although this problem has already received much attention, we believe that our results are of interest because we specify bounds on the differences $x_0^{(r)}-p^{(r)}$ $(0\leq r\leq n-1)$ more precisely than is usually the case, and our integral smallness conditions on $P_1,\cdots,P_n$, and f are stated largely in terms of improper integrals which may converge conditionally rather than absolutely, as is usually required.

The main theorem is stated and proved in §2. Section 3 contains corollaries and examples. Section 4 is an appendix which contains the proof of a lemma used in §2.

2. The main theorem. Throughout this section, p is a given polynomial of degree $<n$. For convenience below, we rewrite (1) as

$$x^{(n)}+Mx=f, \tag{3}$$

where

$$Mx=\sum_{k=1}^{n}P_kx^{(n-k)},$$

and introduce the new unknown

$$h=x-p. \tag{4}$$

Since $p^{(n)}=0$, it is obvious that x is a solution of (3) (and therefore of (1)) if and only if h is a solution of

$$h^{(n)}=-Mh-g, \tag{5}$$

where

$$g=-f+Mp=-f+\sum_{k=1}^{n}P_kp^{(n-k)}. \tag{6}$$

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Thus, g may be regarded as a measure of the extent to which p, a solution of the unperturbed equation (2), fails to be a solution of the perturbed equation (1).

The following is our main theorem.

THEOREM 1. *Let* $P_1,\cdots,P_n$ *and* f *be continuous on* $(0,\infty)$, *and let* g *be as defined in* (6), *where* p *is a given polynomial of degree* $<n$. *Suppose the integral* $\int^{\infty} t^{n-m-1}g(t)\,dt$ *converges, and*

$$\int_t^{\infty} s^{n-m-1}g(s)\,ds = O(\phi(t)), \tag{7}$$

where m *is an integer in* $\{0,1,\cdots,n-1\}$ *and* ϕ *is continuous, positive, and nonincreasing on* $[\overline{T},\infty)$ *for some* $\overline{T}\geq 0$. *Also, if* $m\neq 0$, *suppose* $t^{\gamma}\phi(t)$ *is nondecreasing on* $[\overline{T},\infty)$ *for some* $\gamma<1$. *Assume also that*

$$\int^{\infty}|P_1(t)|\,dt<\infty, \tag{8}$$

and that the integrals $\int^{\infty}P_k(t)\,dt$ $(2\leq k\leq n)$ *converge and satisfy*

$$\int_t^{\infty}P_k(s)\,ds = o(t^{-k+1}), \qquad 2\leq k\leq n. \tag{9}$$

Finally, suppose also that

$$\int_t^{\infty}s^{k-2}\phi(s)\left|\int_s^{\infty}P_k(\lambda)\,d\lambda\right|ds = o(\phi(t)), \qquad 2\leq k\leq n. \tag{10}$$

Then (1) *has a solution* x_0 *such that*

$$x_0^{(r)}(t) = p^{(r)}(t) + O(\phi(t)t^{m-r}), \qquad 0\leq r\leq n-1. \tag{11}$$

Moreover, if (7) *holds with* "O" *replaced by* "o", *then so does* (11).

Remark. Under the stated assumptions on ϕ it is clear that if $\lim_{t\to\infty}\phi(t)>0$, then it may as well be assumed that $\phi=1$. In this case, of course, (7) holds with "O" replaced by "o," and therefore so does (11).

By way of motivation, we first outline the proof of Theorem 1.

From the remarks preceding the statement of Theorem 1, x_0 is a solution of (1) which satisfies (11) if and only if

$$x_0 = p + h_0$$

(see (4)), where h_0 is a solution of (5) such that

$$h_0^{(r)}(t) = O(\phi(t)t^{m-r}), \qquad 0\leq r\leq n-1. \tag{12}$$

We will show that (5) has a solution with these properties by exhibiting h_0 as the fixed point of a contraction mapping on the Banach space $H(t_0)$ of functions h in $C^{(n-1)}[t_0,\infty)$ such that

$$h^{(r)}(t) = O(\phi(t)t^{m-r}), \qquad 0\leq r\leq n-1, \tag{13}$$

with norm

$$\|h\| = \sup_{t\geq t_0}\left\{(\phi(t))^{-1}\sum_{r=0}^{n-1}t^{r-m}|h^{(r)}(t)|\right\}. \tag{14}$$

The contraction mapping will be obtained by converting (5) to an integral equation whose form is dictated by the integrability conditions that we have imposed. To guarantee that the mapping which we define in fact has the contraction property, we must assume that t_0 is sufficiently large, and the fixed point (function) h_0 is at first defined only on $[t_0, \infty)$. However, this presents no difficulty, since our assumptions clearly guarantee the continuability of any solution of (5) over $(0, \infty)$.

With g as in (6), let

$$G(t)=\int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!} g(s)\,ds \quad \text{if } m=0, \tag{15}$$

or

$$G(t)=\int_{t_0}^t \frac{(t-\lambda)^{m-1}}{(m-1)!}\,d\lambda \int_\lambda^\infty \frac{(\lambda-s)^{n-m-1}}{(n-m-1)!} g(s)\,ds \quad \text{if } m=1,\cdots,n-1, \tag{16}$$

and notice that our integrability condition on g implies that the improper integral in (15) or (16) converges, by Dirichlet's theorem for improper integrals.

Now define the transformation L by

$$(Lh)(t)=\int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!}(Mh)(s)\,ds \quad \text{if } m=0, \tag{17}$$

or by

$$(Lh)(t)=\int_{t_0}^t \frac{(t-\lambda)^{m-1}}{(m-1)!}\,d\lambda \int_\lambda^\infty \frac{(\lambda-s)^{n-m-1}}{(n-m-1)!}(Mh)(s)\,ds \quad \text{if } m=1,\cdots,n-1. \tag{18}$$

We will show that the mapping $\mathfrak{T}$ defined by

$$\mathfrak{T}h=G+Lh \tag{19}$$

maps $H(t_0)$ into itself, and is a contraction mapping if t_0 is sufficiently large. It will then follow that $\mathfrak{T}$ has a fixed point (function) h_0 in $H(t_0)$ such that

$$\mathfrak{T}h_0=h_0. \tag{20}$$

If $m=0$, then (15), (17), (19), and (20) imply that h_0 satisfies the integral equation

$$h_0(t)=\int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!}[Mh_0(s)+g(s)]\,ds. \tag{21}$$

If $m=1,\cdots,n-1$, then (16), (18), (19), and (20) imply that h_0 satisfies the integral equation

$$h_0(t)=\int_{t_0}^t \frac{(t-\lambda)^{m-1}}{(m-1)!}\,d\lambda \int_\lambda^\infty \frac{(\lambda-s)^{n-1}}{(n-m-1)!}[Mh_0(s)+g(s)]\,ds. \tag{22}$$

In either case, routine differentiation shows that h_0 satisfies (5). Since $h_0 \in H(t_0)$, it automatically satisfies (12).

From these observations it should be clear that the proof reduces to showing that the mapping $\mathfrak{T}$ is a contraction mapping of $H(t_0)$ into itself if t_0 is sufficiently large, and that (12) can be replaced by

$$h_0^{(r)}(t)=o(\phi(t)t^{m-r}), \qquad 0\le r\le n-1, \tag{23}$$

if (7) holds with "O" replaced by "o." The following lemma is crucial for this proof.

LEMMA 1. *Let ϕ, m, and γ be as in Theorem 1, and suppose $w\in C[t_0,\infty)$ for some $t_0\ge\bar{T}$. Suppose also that $\int^\infty t^{n-m-1}w(t)\,dt$ converges, and*

$$\int_t^\infty s^{n-m-1}w(s)\,ds=O(\phi(t)), \tag{24}$$

and define

$$\rho(t)=\sup_{\tau\ge t}\left|(\phi(\tau))^{-1}\int_\tau^\infty s^{n-m-1}w(s)\,ds\right|. \tag{25}$$

Then the function v defined by

$$v(t)=\int_t^\infty\frac{(t-s)^{n-1}}{(n-1)!}w(s)\,ds \quad \text{if } m=0 \tag{26}$$

or by

$$v(t)=\int_{t_0}^t\frac{(t-\lambda)^{m-1}}{(m-1)!}d\lambda\int_\lambda^\infty\frac{(\lambda-s)^{n-m-1}}{(n-m-1)!}w(s)\,ds \quad \text{if } m=1,2,\cdots,n-1, \tag{27}$$

is in $C^{(n)}[t_0,\infty)$, and it satisfies the inequalities

$$|v^{(r)}(t)|\le\frac{\rho(t_0)\phi(t)t^{m-r}}{(n-m-1)!\prod_{j=1}^{m-r}(j-\gamma)}, \qquad 0\le r\le m-1, \tag{28}$$

$$|v^{(m)}(t)|\le\frac{\rho(t)\phi(t)}{(n-m-1)!}, \tag{29}$$

and

$$|v^{(r)}(t)|\le\frac{2\rho(t)\phi(t)t^{m-r}}{(n-r-1)!}, \qquad m+1\le r\le n-1. \tag{30}$$

Moreover, if

$$\lim_{t\to\infty}\rho(t)=0, \tag{31}$$

then

$$v^{(r)}(t)=o(\phi(t)t^{m-r}), \qquad 0\le r\le n-1. \tag{32}$$

We leave the proof of this lemma for the appendix (§4). Since the lemma would be essentially trivial under the stronger assumption that $\int^\infty t^{n-m-1}|w(t)|\,dt<\infty$, it is important to observe that we are not assuming this. Notice that the lemma implies that the function v defined by (26) or (27) is in $H(t_0)$.

Proof of Theorem 1. First notice that, because of (7), Lemma 1 with $w=g$ implies that G, as defined by (15) or (16), is in $H(t_0)$ for any $t_0>0$. The next step, then, is to show that Lh (see (17) or (18)) is defined and in $H(t_0)$ whenever $h\in H(t_0)$. We start by showing that the improper integral in (17) or (18) converges if $h\in H(t_0)$. To this end, we first consider the integral

$$J(t;h)=\int_t^\infty s^{n-m-1}(Mh)(s)\,ds=\sum_{k=1}^{n}\int_t^\infty s^{n-m-1}P_k(s)h^{(n-k)}(s)\,ds. \tag{33}$$

We will show that the integrals in this sum converge, and estimate them. In the following, let $s\geq t\geq t_0$.

From (14),

$$\left|s^{n-m-1}P_1(s)h^{(n-1)}(s)\right|\leq\|h\||P_1(s)|\phi(s).$$

Therefore, (8) and the monotonicity of ϕ imply that the first integral on the right of (33) converges, and that

$$\left|\int_t^\infty s^{n-m-1}P_1(s)h^{(n-1)}(s)\,ds\right|\leq\|h\|\phi(t)\int_t^\infty|P_1(s)|\,ds. \tag{34}$$

If $2\leq k\leq n$, then integration by parts yields

$$\begin{aligned}\int_t^\infty s^{n-m-1}P_k(s)h^{(n-k)}(s)\,ds&=t^{n-m-1}h^{(n-k)}(t)\int_t^\infty P_k(\lambda)\,d\lambda\\&\quad+\int_t^\infty\left[s^{n-m-1}h^{(n-k)}(s)\right]'\left(\int_s^\infty P_k(\lambda)\,d\lambda\right)ds.\end{aligned} \tag{35}$$

To justify this, observe that

$$\lim_{t\to\infty}t^{n-m-1}h^{(n-k)}(t)\int_t^\infty P_k(\lambda)\,d\lambda=0,$$

because of (9) and (13), and the integral on the right of (35) converges absolutely because of the convergence of the integral in (10) and the inequality

$$\left|\left[s^{n-m-1}h^{(n-j)}(s)\right]'\right|\leq(n-m)\|h\|s^{k-2}\phi(s), \tag{36}$$

which follows from (14) and straightforward manipulation.

This proves that $J(t;h)$ converges. Moreover, from (10), (14), (34), (35), and (36),

$$|J(t;h)|\leq\|h\|\phi(t)\sigma(t), \tag{37}$$

where

$$\begin{aligned}\sigma(t)=\int_t^\infty|P_1(\lambda)|\,d\lambda&+\sum_{j=2}^{n}t^{k-1}\left|\int_t^\infty P_j(\lambda)\,d\lambda\right|\\&+(n-m)(\phi(t))^{-1}\sum_{j=2}^{n}\int_t^\infty s^{j-2}\phi(s)\left|\int_s^\infty P_j(\lambda)\,d\lambda\right|ds.\end{aligned}$$

Now we can apply Lemma 1 with $w=Mh$ and $v=Lh$. (Compare (26) and (27) with (17) and (18).) Then (25) becomes

$$\rho(t)=\sup_{\tau\geq t}(\phi(\tau))^{-1}|J(\tau;h)|,$$

which, with (37), implies that

$$\rho(t)\leq\|h\|\sup_{\tau\geq t}\sigma(\tau)=o(1). \tag{38}$$

Now (28), (29), and (30) with $w=Mh$ and $v=Lh$ imply that $Lh\in H(t_0)$ and

$$\|Lh\|\leq K\|h\|\sup_{\tau\geq t_0}\sigma(\tau),$$

where K is a universal constant.

Since G is also in $H(t_0)$, the transformation $\mathfrak{T}$ defined in (19) also maps $H(t_0)$ into itself. Moreover, if $h_1,h_2\in H(t_0)$, then

$$\|\mathfrak{T}h_1-\mathfrak{T}h_2\|=\|L(h_1-h_2)\|\leq K\|h_1-h_2\|\sup_{\tau\geq t_0}\sigma(\tau).$$

Therefore, $\mathfrak{T}$ is a contraction mapping if t_0 is so large that

$$\sup_{\tau\geq t_0}\sigma(\tau)<1/K,$$

which we now assume. (Recall that $\sigma(t)=o(1)$.) Consequently, $\mathfrak{T}$ has a fixed point (function) h_0 which satisfies

$$h_0=G+Lh_0, \tag{39}$$

which can also be written out as (21) if $m=0$, or as (22) if $m=1,\cdots,n-1$. Since (38) implies (31), Lemma 1 with $w=Mh_0$ and $v=Lh_0$ implies that

$$(Lh_0)^{(r)}(t)=o(\phi(t)t^{m-r}),\qquad 0\leq r\leq n-1. \tag{40}$$

Moreover, if we can replace "O" by "o" in (7), then Lemma 1 implies that

$$G^{(r)}(t)=o(\phi(t)t^{m-r}),\qquad 0\leq r\leq n-1. \tag{41}$$

But (39), (40), and (41) imply (23); that is, in this case we can replace "O" by "o" in (11). This completes the proof of Theorem 1.

3. Corollaries and examples. There are applications of Theorem 1 in which (8) is the only integral smallness condition on functions appearing in (1) which requires absolute convergence. The following corollary illustrates this.

COROLLARY 1. *Theorem* 1 *remains valid if* (10) *is replaced by*

$$\int_t^\infty \frac{\phi(s)}{s}\,ds=O(\phi(t)). \tag{42}$$

Proof. If (42) holds, then (9) implies (10).

The following corollary is of interest if (42) does not hold.

COROLLARY 2. *Theorem* 1 *remains valid if* (10) *is replaced by*

$$\int^\infty s^{k-2}\left|\int_s^\infty P_k(\lambda)\,d\lambda\right|ds<\infty,\qquad 2\leq k\leq n. \tag{43}$$

Proof. Since ϕ is nonincreasing, (43) implies (10).

COROLLARY 3. *Theorem* 1 *remains valid if* (10) *is replaced by*

$$\int^\infty t^{k-1}|P_k(t)|\,dt<\infty,\qquad 2\leq k\leq n. \tag{44}$$

Proof. We will show that (44) implies (43). If (44) holds, then the function

$$Q_k(t)=\int_t^\infty|P_k(s)|\,ds$$

is defined on $(0, \infty)$, and

$$Q_k(t) = o(t^{-k+1}). \tag{45}$$

Integration by parts yields

$$\int_{t_1}^{t_2} s^{k-2} Q_k(s)\,ds = \frac{1}{k-1} s^{k-1} Q_k(s)\Big|_{t_1}^{t_2} - \frac{1}{k-1} \int_{t_1}^{t_2} s^{k-1} |P_k(s)|\,ds.$$

From (44) and (45), we can let $t_2 \to \infty$ here and conclude that

$$\int^{\infty} s^{k-2} Q_k(s)\,ds < \infty.$$

Therefore, (43) holds, since

$$\left| \int_t^{\infty} P_k(\lambda)\,d\lambda \right| \le Q_k(t).$$

To see that (43) is weaker than (44), notice that the function

$$P_k(t) = t^{-k+1/2} \sin t$$

satisfies (43), but not (44).

Example 1. Hartman [1, p. 315] has shown that if $P_1, \cdots, P_n \in C(0, \infty)$, and

$$\int^{\infty} t^{k-1+\alpha} |P_k(t)|\,dt < \infty, \qquad 1 \le k \le n, \tag{46}$$

for some $\alpha > 0$, then the homogeneous equation

$$x^{(n)} + P_1(t) x^{(n-1)} + \cdots + P_n(t) x = 0 \tag{47}$$

has a fundamental system $x_0, x_1, \cdots, x_{n-1}$ such that

$$x_\nu^{(r)}(t) = \begin{cases} t^{\nu-r}[1 + o(t^{-\alpha})]/(\nu-r)!, & 0 \le r \le \nu, \\ o(t^{\nu-r-\alpha}), & \nu+1 \le r \le n-1. \end{cases} \tag{48}$$

The author [2] showed that this conclusion remains valid with (46) replaced by the assumption that

$$\int^{\infty} t^{\alpha} |P_1(t)|\,dt < \infty$$

and the integrals

$$\int^{\infty} t^{k-1+\alpha} P_k(t)\,dt, \qquad 2 \le k \le n,$$

converge, perhaps conditionally. The same conclusion can be obtained under the still weaker assumptions that

$$\int^{\infty} |P_1(t)|\,dt < \infty$$

and

$$\int_t^{\infty} P_k(s)\,ds = o(t^{-k+1-\alpha}), \qquad 1 \le k \le n. \tag{49}$$

To see this, let ν be any integer in $\{0,1,\cdots,n-1\}$ and let $p(t)=t^\nu/\nu!$. Then the function g in (6) becomes

$$g(t)=\sum_{k=n-\nu}^{n} P_k(t)\frac{t^{\nu-n+k}}{(\nu-n+k)!},$$

and (49) implies that

$$\int_t^\infty s^{n-m-1}g(s)\,ds=o(\phi(t)),$$

with $m=\max\{0,\nu-[\alpha]\}$ and $\phi(t)=t^{\nu-m-\alpha}$. Since (49) implies (43), Corollary 2 implies that (47) has a solution x_ν which satisfies (48).

Corollary 2 also implies that if (49) holds only with "O" (rather than "o") on the right, then the stated conclusion also holds with "O" rather than "o" on the right of (48).

Example 2. Consider the equation

$$y^{(n)}+\left[t^{-n-\nu+1}\phi(t)\sin t\right]y=t^{-n+1}\phi(t)\cos t, \tag{50}$$

where ν is an integer in $\{0,1,\cdots,n-1\}$ and ϕ is positive and continuously differentiable on $(0,\infty)$, $\phi'\le 0$, and $\lim_{t\to\infty}\phi(t)=0$. Here $P_1=\cdots=P_{n-1}=0$ and

$$\int_t^\infty P_n(s)\,ds=O\left(t^{-n-\nu+1}\phi(t)\right)=o\left(t^{-n+1}\right), \tag{51}$$

which implies (8) and (9), and the function g defined by (6) is

$$g(t)=t^{-n+1}\phi(t)\left[t^{-\nu}p(t)\sin t-\cos t\right],$$

so

$$\int_t^\infty s^{n-1}g(s)\,ds=O(\phi(t)) \tag{52}$$

(and the convergence is conditional if $\int^\infty\phi(t)\,dt=\infty$), provided p is a polynomial of degree $\le\nu$. This implies (7) with $m=0$. Therefore, Theorem 1 implies that if p is any polynomial of degree $\le\nu$, then (50) has a solution x_0 such that

$$x_0^{(r)}(t)=p^{(r)}(t)+O(\phi(t)t^{-r}),\qquad 0\le r\le n-1,$$

provided

$$\int_t^\infty s^{-\nu-1}\phi^2(s)\,ds=o(\phi(t)), \tag{53}$$

since this implies (10), because of (51) and (52). However, (53) obviously holds for any nonincreasing function ϕ if $\nu>0$. If $\nu=0$ it holds, for example, if

$$\phi(t)=(1+\log t)^{-\alpha},$$

with $\alpha>1$.

Example 3. Consider the equation

$$y^{(n)}+\left[t^\alpha\sin(e^t)\right]y=0, \tag{54}$$

where α is an arbitrary real number. By substituting $s=\log u$ it is easy to verify that

$$\int_t^\infty s^\alpha \sin(e^s)\,ds = O(t^\alpha e^{-t}),$$

where the convergence is conditional if $\alpha \geq -1$. Therefore, (54) satisfies (8), (9), and (43). For this equation the function g in (6) is

$$g(t)=t^\alpha p(t)\sin(e^t),$$

so if p is a polynomial of degree $\nu \leq n-1$, then

$$\int_t^\infty s^{n-1} g(s)\,ds = O(t^{n+\alpha+\nu-1}e^{-t}),$$

which implies (7) with $m=0$ and

$$\phi(t)=t^{n+\alpha+\nu-1}e^{-t}.$$

Therefore, Corollary 2 implies that (54) has a solution x_0 such that

$$x_0^{(r)}(t)=p^{(r)}(t)+O(t^{n+\alpha+\nu-r-1}e^{-t}), \qquad 0 \leq r \leq n-1.$$

Example 4. Corollary 1 implies that the equation

$$y^{(n)}+[t^{-n+1/2}\sin t]\,y=\frac{t^{-1/2}\sin t}{(n-1)!}+t^{-1/2}\log t\cos t$$

has a solution x_0 such that

$$x_0^{(r)}(t)=\left[1+O\left(t^{-1/2}\log t\right)\right]t^{n-r-1}/(n-r-1)!, \qquad 0 \leq r \leq n-1.$$

To see this, observe that here $P_1 = \cdots = P_{n-1}=0$ and

$$\int_t^\infty P_n(s)\,ds = O(t^{-n+1/2}),$$

so (8) and (9) hold. With $p(t)=t^{n-1}/(n-1)!$, the function g in (6) is

$$g(t)=-t^{-1/2}\log t\cos t,$$

so

$$\int_t^\infty g(s)\,ds = O\left(t^{-1/2}\log t\right),$$

(with conditional convergence), which verifies (7) with $m=n-1$ and

$$\phi(t)=t^{-1/2}\log t.$$

Since this ϕ satisfies (42), Corollary 1 implies the conclusion.

4. Appendix. Proof of Lemma 1.

Proof. From Dirichlet's theorem, the convergence of the integral in (24) implies that the improper integral in (26) or (27) converges. Therefore, v is well-defined on $[t_0,\infty)$ by (26) or (27), and

$$v^{(r)}(t)=\int_t^\infty \frac{(t-s)^{n-r-1}}{(n-r-1)!}w(s)\,ds, \qquad m \leq r \leq n-1. \tag{55}$$

With

$$Q(t)=\int_t^\infty s^{n-m-1}w(s)\,ds, \tag{56}$$

(55) can be rewritten as

$$v^{(r)}(t)=-\frac{1}{(n-r-1)!}\int_t^\infty\left(\frac{t}{s}-1\right)^{n-r-1}s^{m-r}Q'(s)\,ds, \qquad m\le r\le n-1. \tag{57}$$

If $m\le r\le n-2$, integrating (57) by parts yields

$$v^{(r)}(t)=\frac{1}{(n-r-1)!}\int_t^\infty Q(s)\frac{d}{ds}\left[\left(\frac{t}{s}-1\right)^{n-r-1}s^{m-r}\right]ds. \tag{58}$$

But

$$\left|\frac{d}{ds}\left[\left(\frac{t}{s}-1\right)^{n-r-1}s^{m-r}\right]\right|\le t^{m-r}\frac{d}{ds}\left(1-\frac{t}{s}\right)^{n-r-1}+(r-m)s^{m-r-1} \tag{59}$$

if $s\ge t$ and $r\ge m$. Since

$$|Q(s)|\le\rho(t)\phi(t) \quad \text{if } s\ge t \tag{60}$$

(see (25) and (56)), (58) implies (30) for $m+1\le r\le n-2$. If $r=n-1$, integrating (57) by parts yields

$$v^{(n-1)}(t)=t^{m-n+1}Q(t)+(m-n+1)\int_t^\infty s^{m-n}Q(s)\,ds. \tag{61}$$

If $m<n-1$, this and (60) imply (30) with $r=n-1$. Setting $r=m$ in (58) and (59) and invoking (60) yields (29) if $m<n-1$. If $m=n-1$, then (60) and (61) imply (29). If $0\le r\le m-1$, then we can differentiate (27) and substitute (55) with $r=m$ into the result to obtain

$$v^{(r)}(t)=\int_{t_0}^t\frac{(t-\lambda)^{m-r-1}}{(m-r-1)!}v^{(m)}(\lambda)\,d\lambda, \qquad 0\le r\le m-1.$$

Therefore, from (29),

(62)

$$|v^{(r)}(t)|\le\frac{1}{(n-m-1)!(m-r-1)!}\int_{t_0}^t(t-\lambda)^{m-r-1}\rho(\lambda)\phi(\lambda)\,d\lambda, \qquad 0\le r\le m-1.$$

Since ρ is nonincreasing and $t^\gamma\phi(t)$ is nondecreasing, this implies that

$$|v^{(r)}(t)|\le\frac{\rho(t_0)\phi(t)t^\gamma}{(n-m-1)!(m-r-1)!}\int_{t_0}^t(t-\lambda)^{m-r-1}\lambda^{-\gamma}\,d\lambda, \qquad 0\le r\le m-1.$$

Replacing t_0 by zero and integrating repeatedly by parts now yields (28). (Here we need the assumption that $\gamma<1$.)

From (29) and (30), (31) implies (32) for $m\le r\le n-1$. If $0\le r\le m-1$, then (62) and the monotonicity properties of ρ and ϕ imply that

$$|v^{(r)}(t)|\le\frac{t^{m-r-1+\gamma}\phi(t)}{(n-m-1)!(m-r-1)!}\int_{t_0}^t\rho(\lambda)\lambda^{-\gamma}\,d\lambda, \qquad 0\le r\le m-1. \tag{63}$$

But

$$\int_{t_0}^{t}\rho(\lambda)\lambda^{-\gamma}\,d\lambda=\int_{t_0}^{t_1}\rho(\lambda)\lambda^{-\gamma}\,d\lambda+\int_{t_1}^{t}\rho(\lambda)\lambda^{-\gamma}\,d\lambda$$
$$\leq\int_{t_0}^{t_1}\rho(\lambda)\lambda^{-\gamma}\,d\lambda+\rho(t_1)\frac{t^{1-\gamma}-t_1^{1-\gamma}}{1-\gamma}$$

if $t_1>t_0$. This and (63) imply that

$$\overline{\lim_{t\to\infty}}\, t^{-m+r}(\phi(t))^{-1}|v^{(r)}(t)|\leq\frac{\rho(t_1)}{(m-r-1)!(n-m-1)!(1-\gamma)}.$$

Since this holds for all $t_1\geq t_0$, (31) implies (32) for $0\leq r\leq m-1$. This completes the proof of Lemma 1.

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