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FUNCTIONAL PERTURBATIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS*

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Abstract. Conditions are given which imply that the functional differential equation

$$(r(t)x'(t))'+q(t)x(t)=f(t,x(g(t)))$$

has a solution $\bar{x}$ which behaves for large t in a precisely defined way like a given solution $\bar{y}$ of the ordinary differential equation

$$(r(t)y')'+q(t)y=0.$$

It is not assumed that $g(t)-t$ is sign-constant, and $f(t,u)$ need only be defined and continuous on a subset of the (t,u) plane which is near the curve $u=\bar{y}(g(t))$ in an appropriate sense for large t. The integral smallness conditions on $f(t,u)$ permit some of the improper integrals in question to converge conditionally. Separate treatments are given for the cases where the unperturbed equation is oscillatory or nonoscillatory. The results are new even in the case where $g(t)=t$.

1. Introduction. We present conditions implying that the functional differential equation

$$(r(t)x'(t))'+q(t)x(t)=f(t,x(g(t))) \tag{1}$$

has a solution $\bar{x}$ which behaves for large t like a given solution $\bar{y}$ of the ordinary differential equation

$$(r(t)y')'+q(t)y=0, \qquad t>a. \tag{2}$$

We give specific estimates of $\bar{x}-\bar{y}$ as $t\to\infty$. We do not require $g(t)-t$ to be sign constant, and the perturbing function $f=f(t,u)$ need be defined and continuous only on a subset of the (t,u) plane near the curve $u=\bar{y}(g(t))$ for large t, in a sense made precise below. We believe that our results are new even if $g(t)=t$. Our integral smallness conditions on the function $f(t,\bar{y}(g(t)))$ require only ordinary (i.e., perhaps conditional) convergence; however, we do impose conditions which imply absolute convergence of certain integrals involving differences

$$f(t,x(g(t)))-f(t,\bar{y}(g(t))), \tag{3}$$

where x is a function near $\bar{y}$ in an appropriate sense. Since forcing functions (i.e., terms in $f(t,u)$ which are independent of u) obviously cancel out of (3), this means that our integral smallness conditions on them always allow conditional convergence; however, this is not the only way in which possibly conditional convergence enters into our hypotheses. Accordingly, all integrability assumptions below should be interpreted as allowing conditional convergence, except when the integrands in question are obviously nonnegative. Moreover, to avoid repetition, it is to be understood that whenever we write an improper integral in stating an assumption, we are assuming that it converges.

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Since the asymptotic theory of (1) depends critically on whether (2) is oscillatory or nonoscillatory, we consider these two cases separately in §§2 and 3. Some of our results in §3 are related to results of Kusano and Naito [3] and Kusano and Onose [4]. Hallam [2] obtained related results, valid when (2) is either oscillatory of nonoscillatory, for the case where $r=1$ and $g(t)=t$.

To avoid repetition, we state here that three proofs below demonstrate the existence of a solution $\bar{x}$ of (1), with prescribed asymptotic properties, as a fixed point of a mapping $\mathscr{T}$ defined on a closed convex subset D of the Frechet space $C[\tau_0,\infty)$ (for some $\tau_0 \geqq a$), with the topology of uniform convergence on compact intervals. In this context we write

$$D\lim_{k\to\infty} x_k = x \tag{4}$$

to mean that $\{x_k\}$ is a sequence of functions in D which converges uniformly to x on compact subintervals of $[\tau_0,\infty)$.

The proof in each case consists of establishing the following:

(i) $\mathscr{T}(D)\subset D$.

(ii) $\mathscr{T}$ is continuous; that is, (4) implies that

$$D\lim_{k\to\infty}\mathscr{T}x_k=\mathscr{T}x.$$

(iii) There is a continuous positive function ψ such that

$$|(\mathscr{T}x)'(t)|\leqq\psi(t), \qquad x\in D, \quad t\geqq\tau_0. \tag{5}$$

The last inequality implies that the function in $\mathscr{T}(D)$ are equicontinuous on compact intervals. Since it will be clear in all cases that the functions in D are uniformly bounded on compact intervals, this and (i) imply that $\mathscr{T}(D)$ has compact closure, by the Arzela–Ascoli theorem. The Schauder–Tykhonov fixed point theorem will then imply that $\mathscr{T}\bar{x}=\bar{x}$ for some $\bar{x}$ in D, and routine differentiation (which we omit) will show that $\bar{x}$ satisfies (1) on some interval (t_0,∞), with $t_0\geqq\tau_0$. We will call such a function a solution of (1).

All quantities are assumed to be real. The following assumption applies throughout.

Assumption A. The functions r,q, and g are continuous on $[a,\infty)$, $r>0$, and

$$g(t)\geqq a, \qquad \lim_{t\to\infty} g(t)=\infty. \tag{6}$$

The functions y_1 and y_2 are solutions of (2) such that

$$y_2'y_1-y_2y_1'=\frac{1}{r}, \tag{7}$$

and

$$\bar{y}=c_1y_1+c_2y_2 \qquad (c_1,c_2=\text{constants}) \tag{8}$$

is a given solution of (2). The function ϕ is positive, continuous, and nonincreasing on $[a,\infty)$, and either

$$\lim_{t\to\infty}\phi(t)=0 \quad \text{or } \phi=1. \tag{9}$$

2. Perturbations of an oscillatory equation. In this section,

$$z=\left(y_1^2+y_2^2\right)^{1/2}. \tag{10}$$

Our proofs here make no use of the assumption that (2) is oscillatory, so our results apply even if it is nonoscillatory; however, in the latter case, better results are obtained in §3.

THEOREM 1. *Suppose*

$$\overline{\lim_{t\to\infty}}\,(\phi(t))^{-1}\left|\int_t^\infty y_i(s)f(s,\bar{y}(g(s)))\,ds\right|=\alpha_i<\infty \tag{11}$$

and

$$\overline{\lim_{t\to\infty}}\,(\phi(t))^{-1}\int_t^\infty |y_i(s)|\sigma(s)\,ds=\beta_i<\infty \tag{12}$$

for $i=1,2$, *where* σ *is positive and continuous on* $[a,\infty)$. *Suppose further that there are constants* $T\geqq a$ *and*

$$M>K=\left[(\alpha_1+\beta_1)^2+(\alpha_2+\beta_2)^2\right]^{1/2} \tag{13}$$

such that f is continuous and

$$|f(t,u)-f(t,\bar{y}(g(t))|\leqq\sigma(t) \tag{14}$$

on the set

$$\Omega=\left\{(t,u)\,\middle|\,t\geqq T,\,|u-\bar{y}(g(t))|\leqq M\phi(g(t))z(g(t))\right\}. \tag{15}$$

Then (1) *has a solution* $\bar{x}$ *such that*

$$\overline{\lim_{t\to\infty}}\,[\phi(t)z(t)]^{-1}|\bar{x}(t)-\bar{y}(t)|\leqq K. \tag{16}$$

Proof. Let

$$\nu_i(t)=\int_t^\infty |y_i(s)|\sigma(s)\,ds+\sup_{\tau\geqq t}\left|\int_\tau^\infty y_i(s)f(s,\bar{y}(g(s)))\,ds\right| \tag{17}$$

and

$$v=\left(\nu_1^2+\nu_2^2\right)^{1/2}; \tag{18}$$

then (11), (12), and (13) imply that

$$\overline{\lim_{t\to\infty}}\,(\phi(t))^{-1}v(t)\leqq K, \tag{19}$$

and that there is a $\tau_0\geqq T$ such that

$$v(t)\leqq M\phi(t),\qquad t\geqq\tau_0. \tag{20}$$

Let

$$D=\left\{x\in C[\tau_0,\infty)\,\middle|\,|x(\tau)-\bar{y}(\tau)|\leqq M\phi(\tau)z(\tau),\tau\geqq\tau_0\right\} \tag{21}$$

(recall (10)), and choose $t_0 \geqq \tau_0$ so that

(22) $$g(t) \geqq \tau_0, \qquad t \geqq t_0$$

(recall (6)). Then (21), (22), and our assumptions on f imply that $f(t, x(g(t)))$ is continuous on $[t_0, \infty)$ if x is in D. Moreover, since

$$\int_t^\infty y_i(s) f(s, x(g(s)))\, ds = \int_t^\infty y_i(s) f(s, \bar{y}(g(s)))\, ds$$
$$+ \int_t^\infty y_i(s)[f(s, x(g(s))) - f(s, \bar{y}(g(s)))]\, ds,$$

(14) and (17) imply that

(23) $$\left| \int_t^\infty y_i(s) f(s, x(g(s)))\, ds \right| \leqq v_i(t), \qquad t \geqq t_0, \quad x \in D.$$

Now define $\mathscr{T}$ on D by

(24)
$$(\mathscr{T}x)(t) = \begin{cases} \bar{y}(t) + \int_t^\infty [y_2(s)y_1(t) - y_1(s)y_2(t)] f(s, x(g(s)))\, ds, & t \geqq t_0, \\ \bar{y}(t) + \int_{t_0}^\infty [y_2(s)y_1(t) - y_1(s)y_2(t)] f(s, x(g(s)))\, ds, & \tau_0 \leqq t < t_0. \end{cases}$$

(The second line is vacuous if $\tau_0 > t_0$.) From (23) and Schwarz's inequality,

(25) $$|(\mathscr{T}x)(t) - \bar{y}(t)| \leqq z(t) v(t), \qquad t \geqq \tau_0$$

(to see this for $\tau_0 \leqq t \leqq t_0$, note that v is nonincreasing), which, with (20), implies that $\mathscr{T}(D) \subset D$.

If $D \lim_{k \to \infty} x_k = x$, then

$$\left| \int_t^\infty y_i(s)[f(s, x_k(g(s))) - f(s, x(g(s)))]\, ds \right|$$
$$\leqq \int_{t_0}^\infty |y_i(s)| |f(s, x_k(g(s))) - f(s, x(g(s)))|\, ds, \qquad t \geqq t_0,$$

where the integrand on the right converges pointwise to zero as $k \to \infty$, and is bounded for all k by $2|y_i(s)|\sigma(s)$ (recall (14)); hence, (12) and Lebesgue's dominated convergence theorem imply that the integral on the right approaches zero as $k \to \infty$. Therefore, if $\varepsilon > 0$ there is an N such that

(26) $$\left| \int_t^\infty y_i(s)[f(s, x_k(g(s))) - f(s, x(g(s)))]\, ds \right| < \varepsilon, \qquad t \geqq t_0, \quad k \geqq N,$$

for $i = 1, 2$. From this, (24), and Schwarz's inequality,

$$|(\mathscr{T}x_k)t) - (\mathscr{T}t)(x)| \leqq \varepsilon\sqrt{2}\, z(t), \qquad t \geqq \tau_0, \quad k \geqq N.$$

This implies that $D \lim_{k \to \infty} \mathscr{T}x_k = \mathscr{T}x$.

By differentiating (24), we see from (18), (20), (23), Schwarz's inequality, and the monotonicity of ϕ that (5) holds, with

$$\psi=|\bar{y}'|+\left[(y_1')^2+(y_2')^2\right]^{1/2}M\phi,$$

and this completes the verification of (i), (ii), and (ii) of §1. Therefore, $\mathscr{T}$ has a fixed point (function) $\bar{x}$ in D. From (24),

$$\bar{x}(t)=\bar{y}(t)+\int_t^\infty [y_2(s)y_1(t)-y_1(s)y_2(t)]f(s,\bar{x}(g(s)))\,ds, \qquad t\geqq t_0, \tag{27}$$

so $\bar{x}$ satisfies (1) on (t_0,∞). Setting $x=\bar{x}$ in (25) and recalling that $\bar{x}=\mathscr{T}\bar{x}$, we see that

$$|\bar{x}(t)-\bar{y}(t)|\leqq z(t)v(t), \qquad t\geqq t_0,$$

so (19) implies (16). This completes the proof.

Taking $\phi=1$ in Theorem 1, so that obviously $\alpha_i=\beta_i=0$, yields the following corollary.

COROLLARY 1. *Suppose the integrals*

$$\int^\infty y_i(t)f(t,\bar{y}(g(t)))\,dt, \qquad i=1,2, \tag{28}$$

converge, and

$$\int^\infty z(t)\sigma(t)\,dt<\infty \tag{29}$$

with σ positive and continuous on $[a,\infty)$. Suppose also that there are constants $T\geqq a$ and $M>0$ such that f is continuous and satisfies (14) *on the set*

$$\bar{\Omega}=\{(t,u)\,|\,t\geqq T, |u-\bar{y}(g(t))|\leqq Mz(g(t))\}. \tag{30}$$

Then (1) *has a solution $\bar{x}$ such that*

$$\bar{x}(t)=\bar{y}(t)+o(z(t)). \tag{31}$$

Remark 1. Although (9) was not used in the proof of Theorem 1, it imposes no loss of generality, since Theorem 1 without (9) is easily shown to be equivalent to Corollary 1 if $\lim_{t\to\infty}\phi(t)>0$.

Remark 2. If, *in addition* to the assumptions of Corollary 1, the stronger integral conditions (11) and (12) hold (with $i=1,2$), then it is routine to verify that the solution $\bar{x}$ which satisfies (31) actually satisfies the stronger condition (16). However, this does not mean that Theorem 1 is only a trivial extension of Corollary 1. The hypotheses of Theorem 1 with $\lim_{t\to\infty}\phi(t)=0$ do not imply those of Corollary 1, since the set Ω in (15) on which (14) is required to hold in Theorem 1 is then smaller than the set $\bar{\Omega}$ in (30). Put another way, the hypotheses of Theorem 1 in this case imply the hypotheses of the Schauder–Tykhonov theorem for the subset D of $C[\tau_0,\infty)$ as defined by (21), but not for the larger subset $\bar{D}$ which would result if ϕ were replaced by one in (21). Example 2 below will illustrate this point.

Remarks similar to these apply to other results which follow.

THEOREM 2. *Suppose* (11) *holds with $i=1,2$. Let λ be nonnegative and continuous on $[a,\infty)$, and*

$$\overline{\lim_{t\to\infty}}\,(\phi(t))^{-1}\int_t^\infty |y_i(s)|\lambda(s)z(g(s))\phi(g(s))\,ds=b_i, \qquad i=1,2, \tag{32}$$

where

$$b_1^2+b_2^2<1. \tag{33}$$

Suppose further that there are constants $T \geqq a$ and $M>0$ such that

$$(\alpha_1+Mb_1)^2+(\alpha_2+Mb_2)^2<M^2 \tag{34}$$

and f is continuous and satisfies the inequality

$$|f(t,u)-f(t,\bar{y}(g(t)))| \leqq \lambda(t)|u-\bar{y}(g(t))| \tag{35}$$

on the set Ω in (15). *Then* (1) *has a solution $\bar{x}$ such that*

$$\overline{\lim_{t\to\infty}}\,[\phi(t)z(t)]^{-1}|\bar{x}(t)-\bar{y}(t)| \leqq \left[(\alpha_1+Mb_1)^2+(\alpha_2+Mb_2)^2\right]^{1/2}$$

Proof. (Note that (34) holds if M is sufficiently large, because of (33).) If f satisfies (35) on Ω, then it also satisfies (14) on Ω, with

$$\sigma(t)=M\lambda(t)z(g(t))\phi(g(t)).$$

This and (32) imply (12) with $\beta_i=Mb_i$, and then (34) implies (13). Hence, Theorem 1 implies the conclusion.

COROLLARY 2. *Suppose the integrals* (28) *converge, and*

$$\int^{\infty} z(t)\lambda(t)z(g(t))\,dt<\infty,$$

where λ is continuous and positive on $[a,\infty)$. Suppose also that there are constants $T \geqq a$ and $M>0$ such that f is continuous and satisfies (35) *on the set $\bar{\Omega}$ in* (30). *Then* (1) *has a solution $\bar{x}$ which satisfies* (31).

We now apply our results to the equation

$$(r(t)x'(t))'+q(t)x(t)=p(t)(x(g(t)))^{\gamma}+h(t), \tag{36}$$

which has the form of a generalized Emden–Fowler equation, but is unusual in that (2) may be oscillatory. (In §3 we consider (36) in the case where (2) is nonoscillatory.)

THEOREM 3. *Suppose $p,\ h \in C[a,\infty)$, and γ is positive and rational, with odd denominator. Suppose further that the integrals*

$$\int^{\infty} y_i(t)h(t)\,dt, \qquad i=1,2 \tag{37}$$

converge, and that

$$\int^{\infty} z(t)|p(t)|(z(g(t)))^{\gamma}\,dt<\infty. \tag{38}$$

Then (36) *has a solution $\bar{x}$ such that*

$$\bar{x}(t)=\bar{y}(t)+o(z(t)).$$

Proof. For (36), the function f in (1) is

$$f(t,u)=p(t)u^{\gamma}+h(t), \tag{39}$$

which is continuous on $\bar{\Omega}$ in (30) for any $T \geqq a$ and $M > 0$. Moreover, if $(t,u) \in \bar{\Omega}$, then

$$|u| \leqq |\bar{y}(g(t))| + Mz(g(t)) \leqq (C+M)z(g(t)), \tag{40}$$

where $C = (c_1^2 + c_2^2)^{1/2}$. (See (8) and (10).) Since obviously

$$\begin{aligned}|f(t,u) - f(t,\bar{y}(g(t)))| &\leqq |p(t)|\left[|u|^\gamma + (\bar{y}(g(t)))^\gamma\right] \\ &\leqq |p(t)|\left[|u|^\gamma + C(z(g(t)))^\gamma\right],\end{aligned}$$

(40) implies (14) for $(t,u) \in \bar{\Omega}$, with

$$\sigma(t) = |p(t)|(z(g(t)))^\gamma\left[(C+M)^\gamma + C^\gamma\right],$$

Therefore (38) implies (29). Since (37) and (38) also imply that the integrals (28) converge, Corollary 1 implies the conclusion.

THEOREM 4. *Suppose $p, h \in C[a,\infty)$ and $\gamma \geqq 1$ is rational, with odd denominator. Suppose also that*

$$\lim_{t\to\infty} \phi(t) = 0, \tag{41}$$

$$\overline{\lim_{t\to\infty}}\,(\phi(t))^{-1}\left|\int_t^\infty y_i(s)\left[p(s)(\bar{y}(g(s)))^\gamma + h(s)\right]ds\right| = \alpha_i < \infty, \qquad i = 1,2,$$

and

$$\overline{\lim_{t\to\infty}}\,(\phi(t))^{-1}\int_t^\infty |y_i(s)p(s)|(z(g(s)))^\gamma\phi(g(s))\,ds = B_i < \infty, \qquad i=1,2. \tag{42}$$

Finally, suppose that

$$\gamma C^{\gamma-1}\left(B_1^2 + B_2^2\right)^{1/2} < 1$$

and $M > 0$ satisfies the inequality

$$\left(\alpha_1 + M\gamma C^{\gamma-1}B_1\right)^2 + \left(\alpha_2 + M\gamma C^{\gamma-1}B_2\right)^2 < M^2.$$

Then (36) *has a solution $\bar{x}$ such that*

$$\overline{\lim_{t\to\infty}}\,[\phi(t)z(t)]^{-1}|\bar{x}(t) - \bar{y}(t)| \leqq \left[\left(\alpha_1 + M\gamma C^{\gamma-1}B_1\right)^2 + \left(\alpha_2 + M\gamma C^{\gamma-1}B_2\right)^2\right]^{1/2}$$

Proof. Again, f as in (39) is continuous on Ω in (15) for any $T \geqq a$ and $M > 0$. As in (40),

$$|u| \leqq [C + M\phi(g(t))]z(g(t)), \qquad (t,u) \in \Omega.$$

Therefore, the mean value theorem implies (35) with $\lambda(t) = \gamma|p(t)|[C + M\phi(g(t))]^{\gamma-1}$. This, (9), (41), and (42) imply (32) with $b_i = \gamma C^{\gamma-1}B_i$, and this for any M. Now Theorem 2 implies the conclusion.

In the following examples, we take $y_1(t) = \cos t$ and $y_2(t) = \sin t$.

Example 1. Suppose $p, h \in [a,\infty)$ and γ is as in Theorem 3. Suppose further that

$$\lim_{t\to\infty} h(t) = 0, \qquad \int^\infty |h'(t)|\,dt < \infty, \tag{43}$$

and $\int^\infty |p(t)|\,dt<\infty$. Then Theorem 3 implies that the equation

$$x''(t)+x(t)=p(t)(x(g(t)))^\gamma+h(t)$$

has a solution $\bar{x}$ such that

$$\bar{x}(t)=c_1\cos t+c_2\sin t+o(1) \tag{44}$$

for any given constants c_1 and c_2. (Notice that (43) and Dirichlet's theorem imply the convergence of the integrals (37).)

Example 2. It is straightforward to verify that the equation

$$x''+x=\frac{x^2}{t} \tag{45}$$

satisfies the hypotheses of Theorem 4 with $g(t)=t$, $\phi(t)=1/t$, and B_1, $B_2\leqq 1$. Therefore, (45) has a solution $\bar{x}$ such that

$$\bar{x}(t)=c_1\cos t+c_2\sin t+O\left(\frac{1}{t}\right),$$

provided $c_1^2+c_2^2<\frac{1}{8}$. Notice that even though its conclusion would only be of the weaker form (44) anyway, Theorem 3 does not apply here, since (45) does not satisfy (38). This illustrates the point raised in Remark 2.

3. Perturbations of a nonoscillatory equation. If (2) is nonoscillatory, then it has a fundamental system which satisfies the following assumption on some semi-infinite interval, which we take—without loss of generality—to be $[a,\infty)$.

Assumption B. The functions y_1 and y_2 of Assumption A are also positive on $[a,\infty)$ and, if

$$\rho=\frac{y_2}{y_1} \tag{46}$$

then

$$\lim_{t\to\infty}\rho(t)=\infty. \tag{47}$$

Also, in all of the following, either (a) $i=2$ and $j=1$, or (b) $i=1$ and $j=2$. In Case (b) there is a number $\mu<1$ such that $\phi\rho^\mu$ is nondecreasing.

Assumptions A and B apply throughout the remainder of the paper.

Note that

$$\rho'=\frac{1}{ry_1^2}>0, \tag{48}$$

from (7) and (46).

The following lemma will be used to prove Theorem 5.

LEMMA 1. *Suppose $F\in C[t_0,\infty)$ for some $t_0\geqq a$ and $\int^\infty y_2(t)F(t)\,dt$ converges. Let*

$$\nu(t)=\sup_{\tau\geqq t}\left|\int_\tau^\infty y_2(s)F(s)\,ds\right|.$$

Then

(49) $$\left|\int_{t_1}^{\infty}[y_2(s)y_1(t)-y_1(s)y_2(t)]F(s)\,ds\right|\leqq \nu(t)y_1(t), \qquad t_0\leqq t\leqq t_1,$$

and

(50) $$\left|\int_{t}^{\infty}y_1(s)F(s)\,ds\right|\leqq 2\nu(t)/\rho(t), \qquad t\geqq t_0.$$

Proof. With $U(t)=\int_t^\infty y_2(s)F(s)\,ds$, integration by parts yields

(51) $$\int_{t_1}^{\infty}[y_2(s)y_1(t)-y_1(s)y_2(t)]F(s)\,ds$$
$$=U(t_1)y_1(t)\left[1-\frac{\rho(t)}{\rho(t_1)}\right]+y_2(t)\int_{t_1}^{\infty}\frac{\rho'(s)}{\rho^2(s)}U(s)\,ds,$$

where the integral on the right converges absolutely because of (47), (48), and the boundedness of U. Since $|U(s)|\leqq\nu(t_1)$ if $s\geqq t_1$, (51) and the monotonicity of ρ imply (49).

We obtain (50) by writing

$$\int_t^\infty y_1(s)F(s)\,ds=U(t)/\rho(t)-\int_t^\infty\frac{\rho'(s)}{\rho^2(s)}U(s)\,ds$$

and applying a similar argument. This completes the proof.

THEOREM 5. *Suppose* (11) *and* (12) *hold and there are constants* $T\geqq a$ *and* $M>0$ *such that* f *is continuous and satisfies* (14) *on the set*

(52) $$\Omega_j=\{(t,u)\,|\,t\geqq T, |u-\bar{y}(g(t))|\leqq M\phi(g(t))\,y_j(g(t))\}.$$

Then: (a) *if* $i=2, j=1$, *and*

(53) $$M>\alpha_2+\beta_2,$$

then (1) *has a solution* $\bar{x}$ *such that*

(54) $$\varlimsup_{t\to\infty}[\phi(t)y_1(t)]^{-1}|\bar{x}(t)-\bar{y}(t)|\leqq\alpha_2+\beta_2.$$

(b) *If* $i=1, j=2$, *and*

(55) $$M>(\alpha_1+\beta_1)/(1-\mu),$$

then (1) *has a solution* $\bar{x}$ *such that*

(56) $$\varlimsup_{t\to\infty}[\phi(t)y_2(t)]^{-1}|\bar{x}(t)-\bar{y}(t)|\leqq\frac{\alpha_1+\beta_1}{1-\mu}.$$

Proof. From (11) and (12),

(57) $$\varlimsup_{t\to\infty}(\phi(t))^{-1}\nu_i(t)\leqq\alpha_i+\beta_i.$$

(See (17).) If $\tau_0 \geqq T$, let

$$D_j = \left\{ x \in C[\tau_0, \infty) \,|\, |\bar{x}(\tau) - \bar{y}(\tau)| \leqq M\phi(\tau) y_j(\tau),\ \tau \geqq \tau_0 \right\}.$$

We now consider Cases (a) and (b) separately.

(a) Choose $\tau_0 \geqq T$ so that

$$\nu_2(t) \leqq M\phi(t), \qquad t \geqq \tau_0, \tag{58}$$

which is possible because of (53) and (57) with $i=2$. Then choose $t_0 \geqq \tau_0$ to satisfy (22). As in the proof of Theorem 1, (23) holds with $i=2$, and, with $\mathscr{T}$ as defined in (24), Lemma 1 implies that

$$|(\mathscr{T}x)(t) - \bar{y}(t)| \leqq \nu_2(t) y_1(t), \qquad t \geqq \tau_0, \quad x \in D_1. \tag{59}$$

This and (58) imply that $\mathscr{T}(D_1) \subset D_1$.

If $D_1 \lim_{k\to\infty} x_k = x$, then the argument given in the proof of Theorem 1 implies that for each $\varepsilon > 0$ there is an N such that (26) holds with $i=2$. This and Lemma 1 with

$$F(s) = f(s, x_k(g(s))) - f(s, x(g(s)))$$

imply that

$$|(\mathscr{T}x_k)(t) - (\mathscr{T}x)(t)| \leqq \varepsilon y_1(t), \qquad t \geqq \tau_0, \quad k \geqq N,$$

so $D_1 \lim_{k\to\infty} \mathscr{T}x_k = \mathscr{T}x$.

Since (23) holds with $i=2$, Lemma 1 (specifically, (50)) implies that

$$\left| \int_t^\infty y_1(s) f(s, x(g(s)))\, ds \right| \leqq \frac{2\nu_2(t)}{\rho(t)}, \qquad t \geqq t_0.$$

Therefore, differentiating (24) and applying routine estimates verifies (5), with

$$\psi = |\bar{y}'| + \nu_2 \left[|y_1'| + 2\frac{|y_2'|}{\rho} \right].$$

Now we conclude that $\mathscr{T}$ has a fixed point (function) $\bar{x}$ which satisfies (27), and therefore (1), on (t_0, ∞). Setting $x = \bar{x}$ in (59) and recalling that $\mathscr{T}\bar{x} = \bar{x}$ yields the inequality

$$|\bar{x}(t) - \bar{y}(t)| \leqq \nu_2(t) y_1(t), \qquad t > t_0,$$

so (57) with $i=2$ implies (54). This completes the proof in Case (a).

(b) Choose $\tau_0 \geqq T$ so that

$$\nu_1(t) \leqq M(1-\mu)\phi(t), \qquad t \geqq \tau_0, \tag{60}$$

which is possible because of (55) and (57) with $i=1$. Then choose $t_0 \geqq \tau_0$ to satisfy (22). Now define $\mathscr{T}$ on D_2 by

$$(\mathscr{T}x)(t) = \begin{cases} \bar{y}(t) - y_1(t) \displaystyle\int_{t_0}^t \rho'(\tau) \left(\int_\tau^\infty y_1(s) f(s, x(g(s)))\, ds \right) d\tau, & t \geqq t_0, \\ \bar{y}(t), \quad \tau_0 \leqq t \leqq t_0, \end{cases} \tag{61}$$

where the second line is vacuous if $\tau_0 > t_0$. (See (46) and (48).) Then

$$(\mathscr{T}x)(t)-\bar{y}(t)=0, \qquad \tau_0 \leqq t \leqq t_0, \tag{62}$$

while (23) with $i=1$ implies that

$$|(\mathscr{T}x)(t)-\bar{y}(t)| \leqq y_1(t)\int_{t_0}^{t} \rho'(\tau)\nu_1(\tau)\,d\tau, \qquad t \geqq t_0. \tag{63}$$

If $t \geqq a$, let

$$\hat{\nu}_1(t)=\sup_{\tau \geqq t} \nu_1(\tau)/\phi(\tau). \tag{64}$$

Then, if $t \geqq t_1 \geqq a$,

$$\int_{t_1}^{t} \rho'(\tau)\nu_1(\tau)\,d\tau \leqq \hat{\nu}_1(t_1)\int_{t_1}^{t} \rho'(\tau)\phi(\tau)\,d\tau$$

$$\leqq \hat{\nu}_1(t_1)(\rho(t))^{\mu}\phi(t)\int_{t_1}^{t} \rho'(\tau)(\rho(\tau))^{-\mu}\,d\tau,$$

since $\phi\rho^{\mu}$ is nondecreasing. Since $\mu<1$ and $\rho'>0$, this implies that

$$\int_{t_1}^{t} \rho'(\tau)\nu_1(\tau)\,d\tau \leqq \hat{\nu}_1(t_1)\phi(t)\frac{\rho(t)}{1-\mu}. \tag{65}$$

Setting $t_1=t_0$ here and recalling (60), (63), and (64) shows that

$$|(\mathscr{T}x)(t)-\bar{y}(t)| \leqq M\phi(t)y_2(t), \qquad t \geqq t_0, \quad x \in D_2,$$

which, with (62), implies that $\mathscr{T}(D_2) \subset D_2$.

If $D_2 \lim_{k\to\infty} x_k = x$, the argument used in the proof of Theorem 1 again implies that for each $\varepsilon>0$ there is an N such that (26) holds with $i=1$. This and (61) imply that

$$|(\mathscr{T}x_k)(t)-(\mathscr{T}x)(t)| \leqq \varepsilon y_2(t), \qquad t \geqq \tau_0, \quad k \geqq N,$$

which implies that $D_2 \lim_{k\to\infty} \mathscr{T}x_k = \mathscr{T}x$.

Differentiating (61) and recalling (23) with $i=1$ shows that

$$|(\mathscr{T}x)'(t)| \leqq |\bar{y}'(t)|+|y_1'(t)|\int_{t_0}^{t} \rho'(\tau)\nu_1(\tau)\,d\tau + y_1(t)\rho'(t)\nu_1(t)$$

if $x \in D_2$ and $t \geqq t_0$. Since $\mathscr{T}x=\bar{y}$ on $[\tau_0,t_0]$ for every x in D_2, this is enough to imply the conclusion of (iii) in §1, so $\mathscr{T}$ has a fixed point (function) $\bar{x}$ in D_2. From (61),

$$\bar{x}(t)=\bar{y}(t)-y_1(t)\int_{t_0}^{t} \rho'(\tau)\left(\int_{\tau}^{\infty} y_1(s)f(s,\bar{x}(g(s)))\,ds\right)d\tau, \qquad t \geqq t_0.$$

This function satisfies (1) on (t_0,∞), and, from (63) with $x=\bar{x}(=\mathscr{T}\bar{x})$ and $i=1$,

$$|\bar{x}(t)-\bar{y}(t)| \leqq y_1(t)\int_{t_0}^{t} \rho'(\tau)\nu_1(\tau)\,d\tau.$$

This and (65) imply that

$$|\bar{x}(t)-\bar{y}(t)| \leqq y_1(t)\int_{t_0}^{t_1} \rho'(\tau)\nu_1(\tau)\,d\tau + \hat{\nu}_1(t_1)\phi(t)y_2(t)/(1-\mu) \tag{66}$$

if $t_0 \leqq t_1 \leqq t$. From (47), (48), and our assumption on $\phi\rho^\mu$, $\lim_{t\to\infty}\phi(t)\rho(t)=\infty$; hence, from (66),

$$\overline{\lim_{t\to\infty}}\,[\phi(t)y_2(t)]^{-1}|\bar{x}(t)-\bar{y}(t)| \leqq \hat{\nu}_1(t_1)/(1-\mu)$$

for every $t_1 \geqq t_0$. Letting $t_1 \to \infty$ and recalling (57) (with $i=1$) and (64) therefore implies (56). This completes the proof.

Setting $\phi=1$ in Theorem 5, and noting again that this means that $\alpha_i=\beta_i=0$, yields the following corollary. (Here we reemphasize that Assumption B applies; specifically, that either $i=2$ and $j=1$ or $i=1$ and $j=2$.)

COROLLARY 3. *Suppose* $\int^\infty y_i(t)f(t,\bar{y}(g(t)))\,dt$ *converges and* $\int^\infty y_i(t)\sigma(t)\,dt<\infty$, *where* σ *is positive and continuous on* $[a,\infty)$. *Suppose also that there are constants* $T=a$ *and* $M>0$ *such that* f *is continuous and satisfies* (14) *on the set*

$$\overline{\Omega}_j=\left\{(t,u)\,\middle|\, t\geqq T,\ |u-\bar{y}(g(t))|\leqq My_j(g(t))\right\}.$$

Then (1) *has a solution* $\bar{x}$ *such that* $\bar{x}(t)=\bar{y}(t)+o(y_j(t))$.

COROLLARY 4. *Suppose* $h\in C[a,\infty)$, F *is continuous on* $[a,\infty)\times(0,\infty)$, *and* $|F(t,u)|$ *is either* (i) *nondecreasing in* u *for each* t, *or* (ii) *nonincreasing in* u *for each* t. *Suppose also that*

$$\int^\infty y_i(t)h(t)\,dt \tag{67}$$

converges, and

$$\int^\infty y_i(t)\left|F(t,\delta y_j(g(t)))\right|dt<\infty$$

for some $\delta>0$. *Then the equation*

$$(r(t)x'(t))'+q(t)x(t)=F(t,x(g(t)))+h(t)$$

has a solution $\bar{x}$ *such that*

$$\lim_{t\to\infty}\frac{\bar{x}(t)}{y_j(t)}=c_j, \tag{68}$$

provided $0<c_j<\delta$ *in Case* (i), *or* $c_j>\delta$ *in Case* (ii).

Proof. It is straightforward to verify that the present assumptions imply those of Corollary 3 with $\bar{y}=c_jy_j$, $f(t,u)=F(t,u)+h(t)$, and $\sigma(t)=2|F(t,\delta y_j(g(t)))|$. In Case (i), choose $M<\min\{c_j,\delta-c_j\}$; in Case II, $M<c_j-\delta$. In either case, let $T=a$.

Kusano and Naito [3] have given necessary and sufficient conditions for the equation

$$(r(t)x'(t))'=f(t,x(g(t))) \tag{69}$$

to have nonoscillatory solutions with specific asymptotic behavior, under the assumption that (69) is sublinear or superlinear (see [3] for definitions of these terms), where $g(t)\leqq t$ and $\lim_{t\to\infty}g(t)=\infty$. Kusano and Onose [4] have obtained analogous results for the case where $g(t)\geqq t$. Corollary 4 essentially contains the sufficiency halves of [3, Thms. 1,2] and [4, Thms. 1,2,5,6]. The reader who wishes to check this should let

$$y_1(t)=1,\qquad y_2(t)=\int_a^t (r(s))^{-1}\,ds\quad\text{if}\int^\infty (r(t))^{-1}\,dt=\infty,$$

or

$$y_1(t)=\int_t^\infty (r(s))^{-1}\,ds, \qquad y_2(t)=1 \quad \text{if} \int^\infty (r(t))^{-1}\,dt<\infty.$$

The next corollary follows easily from Corollary 4. It is perhaps noteworthy in that it deals with the generalized Emden–Fowler equation without the usual assumption that $\gamma>0$. (For other results concerning such equations with arbitrary γ, see [1], [5], [6], [7], and [8].) This observation applies also to Theorems 7 and 8 below.

COROLLARY 5. *Suppose $p,\ h\in C[a,\infty)$, γ is any real number, and c_j is any positive constant. Suppose further that* (67) *converges, and*

$$\int^\infty y_i(t)|p(t)|\big(y_j(g(t))\big)^\gamma\,dt<\infty.$$

Then (36) *has a solution $\bar{x}$ which satisfies* (68).

This corollary essentially contains the sufficiency halves of [4, Corollaries 1, 4].

Theorem 5 implies the next result in much the same way that Theorem 1 implies Theorem 2. We omit the proof.

THEOREM 6. *Suppose* (11) *holds. Let λ be nonnegative and continuous on $[a,\infty)$, and*

$$\overline{\lim_{t\to\infty}}\,(\phi(t))^{-1}\int_t^\infty y_i(s)\lambda(s)\,y_j(g(s))\,\phi(g(s))\,ds=b_i. \tag{70}$$

Suppose further that there are constants $T\geqq a$ and M such that f is continuous and satisfies (35) *on the set Ω_j in* (52). *Then* (1) *has a solution $\bar{x}$ such that*

(a) $\overline{\lim}_{t\to\infty}[\phi(t)y_1(t)]^{-1}|\bar{x}(t)-\bar{y}(t)|\leqq\alpha_2+Mb_2$ *if* $i=1$, $j=1$, $b_2<1$, *and* $M>\alpha_2/(1-b_2)$; *or*

(b) $\overline{\lim}_{t\to\infty}[\phi(t)y_2(t)]^{-1}|\bar{x}(t)-\bar{y}(t)|\leqq(\alpha_1+Mb_1)/(1-\mu)$ *if* $i=1$, $j=2$, $b_1<1-\mu$, *and* $M>\alpha_1/(1-\mu-b_1)$.

We close by applying Theorem 6 to the generalized Emden–Fowler equation (36).

THEOREM 7. *Suppose $p,\ h\in C[a,\infty)$, γ is any real number, and*

$$\overline{\lim_{t\to\infty}}\,(\phi(t))^{-1}\int_t^\infty y_i(s)|p(s)|\big(y_j(g(s))\big)^\gamma\phi(g(s))\,ds=B_i<\infty. \tag{71}$$

Suppose also that

$$\int_t^\infty y_i(s)h(s)\,ds=O(\phi(t))$$

and

$$\int_t^\infty y_i(s)p(s)\big(y_j(g(s))\big)^\gamma\,ds=O(\phi(t)), \tag{72}$$

and let

$$\overline{\lim_{t\to\infty}}\,(\phi(t))^{-1}\left|\int_t^\infty y_i(s)\big[p(s)\big(c_jy_j(g(s))\big)^\gamma+h(s)\big]\,ds\right|=\alpha_i,$$

where c_j is a given positive constant.

(a) *If $i=2$ and $j=1$, suppose also that*

$$|\gamma|c_1^{\gamma-1}B_2<1$$

and

$$M>\alpha_2/\left(1-|\gamma|c_1^{\gamma-1}B_2\right). \tag{73}$$

Then (36) *has a solution* $\bar{x}$ *such that*

$$\overline{\lim_{t\to\infty}}\,[\phi(t)y_1(t)]^{-1}|\bar{x}(t)-c_1y_1(t)|\leqq\alpha_2+M|\gamma|c_1^{\gamma-1}B_2.$$

(b) *If* $i=1$ *and* $j=2$, *suppose also that*

$$|\gamma|c_2^{\gamma-1}B_1<1-\mu$$

and

$$M>\alpha_1\left(1-\mu-|\gamma|c_2^{\gamma-1}B_1\right). \tag{74}$$

Then (36) *has a solution* $\bar{x}$ *such that*

$$\overline{\lim_{t\to\infty}}\,[\phi(t)y_2(t)]^{-1}|\bar{x}(t)-c_2y_2(t)|\leqq\left(\alpha_1+M|\gamma|c_2^{\gamma-1}B_1\right)/(1-\mu).$$

Proof. Since Corollary 5 implies the conclusions if $\phi=1$, we assume that

$$\lim_{t\to\infty}\phi(t)=0. \tag{75}$$

Choose M to satisfy (73) or (74), whichever is appropriate, and then choose T so that $M\phi(g(t))<c_j$ if $t\geqq T$. (This is possible because of (6) and (75).) With this M and T and $\bar{y}=c_jy_j$, it is easy to show that if (t,u) is in Ω_j as defined in (52), then $0<[c_j-M\phi(g(t))]y_j(g(t))\leqq u\leqq[c_j+M\phi(g(t))]y_j(g(t))$. Therefore, the function f in (39) is continuous on Ω_j, and, by the mean value theorem, satisfies (35), with

$$\lambda(t)=|\gamma p(t)|\left[c_j\pm M\phi(g(t))\right]^{\gamma-1}\left(y_j(g(t))\right)^{\gamma-1}, \tag{76}$$

where the plus applies if $\gamma\geqq1$, the minus if $\gamma<1$. In either case, (6), (71), (75), and (76) imply (70) with $b_i=|\gamma|c_j^{\gamma-1}B_i$. Now Theorem 6 implies the stated conclusion.

THEOREM 8. *Suppose* $p,h\in C[a,\infty)$, γ *is an arbitrary real number, and*

$$\int^{\infty}\frac{\rho'(g(t))}{\rho^2(g(t))}|g'(t)|\,dt<\infty. \tag{77}$$

Suppose also that

$$\overline{\lim_{t\to\infty}}\,(\phi(t))^{-1}\int_t^{\infty}y_2(s)|p(s)|\left(y_2(g(s))\right)^{\gamma-1}y_1(g(s))\phi(g(s))\,ds=B_2<\infty, \tag{78}$$

$$\int_t^{\infty}y_2(s)h(s)\,ds=O(\phi(t)), \tag{79}$$

and

$$E(t)=\int_t^{\infty}y_2(s)p(s)\left(y_2(g(s))\right)^{\gamma}ds=O(\phi(t)). \tag{80}$$

Let c_2 *be a positive constant such that*

$$|\gamma|c_2^{\gamma-1}B_2<1,$$

and let $\bar{y}=c_1y_1+c_2y_2$, where c_1 is arbitrary. Then the quantity

$$\alpha_2=\overline{\lim_{t\to\infty}}(\phi(t))^{-1}\left|\int_t^\infty y_2(s)\left[p(s)(\bar{y}(g(s)))^\gamma+h(s)\right]ds\right| \tag{81}$$

exists and is finite; moreover, $\alpha_2=0$ if (79) and (80) hold with "O" replaced by "o." Furthermore, if

$$M>\alpha_2/\left(1-|\gamma|c_2^{\gamma-1}B_2\right), \tag{82}$$

then (36) has a solution $\bar{x}$ such that

$$\overline{\lim_{t\to\infty}}\left[\phi(t)y_1(t)\right]^{-1}|\bar{x}(t)-c_1y_1(t)-c_2y_2(t)|\leqq\alpha_2+M|\gamma|c_2^{\gamma-1}B_2.$$

Proof. From (80) and integration by parts,

(83)

$$\int_t^\infty y_2(s)p(s)(\bar{y}(g(s)))^\gamma ds$$

$$=-\int_t^\infty E'(s)\left(\frac{\bar{y}(g(s))}{y_2(g(s))}\right)^\gamma ds$$

$$=E(t)\left(\frac{\bar{y}(g(t))}{y_2(g(t))}\right)^\gamma-\gamma c_1\int_t^\infty E(s)\left(\frac{\bar{y}(g(s))}{y_2(g(s))}\right)^{\gamma-1}\frac{\rho'(g(s))}{\rho^2(g(s))}g'(s)\,ds$$

(see (46)), since $\lim_{t\to\infty}E(t)=0$ and

$$\lim_{t\to\infty}\frac{\bar{y}(g(t))}{y_2(g(t))}=c_2.$$

The integral on the right of (83) converges because of (77). From this and (79) it is easy to verify that α_2 in (81) has the stated properties.

Now choose M to satisfy (82), and then choose T so that

$$\bar{y}(g(t))\geqq M\phi(g(t))y_1(g(t)),\qquad t\geqq T.$$

(This is possible even if $\phi=1$, because of (6), (47), and the assumption that $c_2>0$.) With this M and T, it is easy to show that if (t,u) is in Ω_1 as defined in (52), then

$$0<\bar{y}(g(t))-M\phi(g(t))y_1(g(t))\leqq u\leqq\bar{y}(g(t))+M\phi(g(t))y_1(g(t)).$$

Therefore, the function f in (39) is continuous on Ω_1, and, again by the mean value theorem, satisfies (35) with

$$\lambda(t)=|\gamma p(t)|\left[\bar{y}(g(t))\pm M\phi(g(t))y_1(g(t)\right]^{\gamma-1},$$

where the plus applies if $\gamma\geqq1$, the minus if $\gamma<1$. In either case, since the quantity in brackets behaves asymptotically like

$$(c_2y_2(g(t)))^{\gamma-1},$$

(78) implies (70) with $i=2$ and $b_2=|\gamma|c_2^{\gamma-1}B_2$. Now part (a) of Theorem 6 implies the stated conclusion.

Remark 3. Since $\rho' > 0$ and $\lim_{t\to\infty} \rho(t) = \infty$, (77) is automatically satisfied if $g'(t) > 0$ for t sufficiently large.

Remark 4. Theorems 7 and 8 show that integrability conditions involving other than forcing functions may permit conditionally convergent integrals, since (71) does not imply that the integral in (72) converges absolutely if $\lim_{t\to\infty} \phi(t) = 0$, and (78) does not imply that the integral in (80) converges absolutely, even if $\phi = 1$.

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