GLOBAL EXISTENCE THEOREMS FOR SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS WITH PRESCRIBED ASYMPTOTIC BEHAVIOUR

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ABSTRACT

Most known conditions implying that a nonlinear differential equation $y^{(n)}+f(t,y)=0$ has solutions such that $\lim_{t\to\infty} t^{-k}y(t)=c\neq 0$ are local in that the solutions are guaranteed to exist only for sufficiently large t. This paper presents conditions ensuring that the solutions exist on a given interval and have the prescribed asymptotic behaviour. Some of the integral smallness conditions on f permit conditional convergence.

There are many known sufficient conditions for a nonlinear scalar equation

$$y^{(n)}+f(t,y)=0 \tag{1}$$

to have solutions which behave like solutions of $x^{(n)}=0$ as $t\to\infty$. However, in almost all results of this kind it is required that f satisfy certain conditions on a given interval $[t_0,\infty)$, while the solution with the desired properties is guaranteed to exist only on an interval $[T,\infty)$, where T is some sufficiently large number in $[t_0,\infty)$. The following is a typical theorem of this kind.

THEOREM 0. *Suppose that*

$$f\colon [t_0,\infty)\times(0,\infty)\longrightarrow(-\infty,\infty) \quad \textit{and} \quad F\colon [t_0,\infty)\times(0,\infty)\longrightarrow[0,\infty)$$

are continuous, that F is non-decreasing with respect to y for each t, and that $|f(t,y)|\leqslant F(t,y)$. Let k be an integer, $0\leqslant k\leqslant n-1$, and suppose that

$$\int_{t_0}^{\infty} t^{n-k-1}F(t,at^k)\,dt<\infty$$

for some constant $a>0$. Then (1) *has a solution y_0 which is defined on some interval $[T,\infty)$, with $T\geqslant t_0$, and satisfies the asymptotic condition*

$$\lim_{t\to\infty} t^{-k}y_0(t)=c,$$

where c is a positive constant.

This is a 'local' existence theorem in that y_0 is guaranteed to exist only for sufficiently large t; that is, in a 'small' neighbourhood of infinity. By a global existence theorem for (1), we mean a theorem which guarantees the existence of a solution of (1), with specified properties, on a *given* interval $[t_0,\infty)$.

Nehari [**1**] and Noussair and Swanson [**2**] have considered the question of global existence of solutions of the semilinear second order equation

$$y''+yg(t,y)=0,$$

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and have given, for example, conditions which imply that this equation has a bounded positive solution on a given interval $[t_0, \infty)$. The question of global existence of solutions of higher order equations with specified asymptotic behaviour appears to have been virtually ignored.

Here we let n be arbitrary (but at least 2), and give two main theorems, one of which includes more specific estimates of the order of convergence in (2), and permits conditional convergence of some of the improper integrals that occur in the conditions on f. Both proofs use the Schauder–Tychonoff theorem to obtain y_0 as a fixed point of the operator T defined by

$$Ty(t) = \begin{cases} c + \displaystyle\int_t^\infty \frac{(t-s)^{n-1}}{(n-1)!} f(s, y(s))\, ds & \text{if } k = 0, \\ ct^k + \displaystyle\int_{t_0}^t \frac{(t-\tau)^{k-1}}{(k-1)!}\, d\tau \int_\tau^\infty \frac{(\tau-s)^{n-k-1}}{(n-k-1)!} f(s, y(s))\, ds & \text{if } 1 \leqslant k \leqslant n-1. \end{cases} \tag{3}$$

In each case the domain of T is an appropriate closed convex subset Y of $C[t_0, \infty)$, which is given the standard topology of uniform convergence on finite intervals. (We write $y_j \to y$ to denote this convergence.)

In the following we assume that $t_0 \geqslant 0$ and $k \in \{0, 1, \dots, n-1\}$.

THEOREM 1. *Suppose that f is continuous and satisfies an inequality of the form*

$$|f(t, y)| \leqslant \phi(t)\, F(y) \quad \text{for } (t, y) \in [t_0, \infty) \times (0, \infty), \tag{4}$$

where $\phi: [t_0, \infty) \to [0, \infty)$ and $F: (0, \infty) \to [0, \infty)$ are continuous,

$$F(1) = 1, \tag{5}$$

and

$$\frac{1}{(n-k-1)!} \int_{t_0}^\infty t^{n-k-1} \phi(t)\, F(t^k)\, dt = M < \infty. \tag{6}$$

Suppose also that F satisfies one of the following conditions:

(C_1) *F is non-decreasing and $\lim_{y \to 0^+} F(y)/y = 0$,*
(C_2) *F is non-decreasing and $\lim_{y \to \infty} F(y)/y = 0$,*
(C_3) *F is non-increasing.*

In addition, if $1 \leqslant k \leqslant n-1$, let

$$F(yz) \leqslant F(y)\, F(z) \quad \text{if } y, z > 0. \tag{7}$$

Now let θ and c be positive constants, with $0 < \theta < 1$. Then (1) has a solution y_0 on $[t_0, \infty)$ which belongs to

$$Y = \{y \in C[t_0, \infty) : |y(t) - ct^k| \leqslant c\theta t^k,\ t \geqslant t_0\} \tag{8}$$

and satisfies (2), provided c is sufficiently small if (C_1) holds, or sufficiently large if (C_2) or (C_3) holds.

Proof. Restrict c as follows.

(i) If (C_1) holds, let c be sufficiently small so that

$$F[(1+\theta)\, c] \leqslant c\theta k!/M. \tag{9}$$

(ii) If (C_2) holds, let c be sufficiently large so that (9) holds.

(iii) If (C_3) holds, let c be sufficiently large so that

$$F[(1-\theta)c] \leqslant c\theta k!/M. \tag{10}$$

If $y \in Y$, then (4), (5), (7) (if $1 \leqslant k \leqslant n-1$), (8), and the appropriate choice of c as in (9) or (10) imply that

$$|f(t, y(t))| \leqslant \phi(t) F(t^k) c\theta k!/M, \quad t \geqslant t_0. \tag{11}$$

From this and (6), the improper integral in (3) converges; hence $Ty \in C[t_0, \infty)$.

If $1 \leqslant k \leqslant n-1$, then

$$\int_{t_0}^{t} (t-s)^{k-1}\, ds \leqslant t^k/k \tag{12}$$

(since $t_0 \geqslant 0$); therefore, (3) implies that

$$|Ty(t) - ct^k| \leqslant \frac{t^k}{k!} \int_{t_0}^{\infty} \frac{\tau^{n-k-1}}{(n-k-1)!} |f(\tau, y(\tau))|\, d\tau, \quad t \geqslant t_0, \tag{13}$$

which is also valid if $k = 0$. Now (6), (8), (11) and (13) imply that $Ty \in Y$; that is,

$$T(Y) \subset Y. \tag{14}$$

Now suppose that $\{y_j\} \subset Y$ and $y_j \to y$. From (3) and (12),

$$|Ty_j(t) - Ty(t)| \leqslant \frac{t^k}{k!(n-k-1)!} \int_{t_0}^{\infty} \tau^{n-k-1} |f(\tau, y_j(\tau)) - f(\tau, y(\tau))|\, d\tau, \quad t \geqslant t_0. \tag{15}$$

The integrand on the right-hand side converges pointwise to zero as $j \to \infty$, and is dominated by

$$2\tau^{n-k-1} \phi(\tau) F(\tau^k) c\theta k!/M \tag{16}$$

(recall (11)); hence, (6) and Lebesgue's dominated convergence theorem imply that the integral in (15) approaches zero as $j \to \infty$. This implies that $Ty_j \to Ty$; hence, T is continuous.

Routine estimates based on (3), (4), (5), (6) and (11) show that if $y \in Y$, then

$$|(Ty)'(t)| \leqslant \begin{cases} \dfrac{c\theta}{M} \displaystyle\int_t^{\infty} \frac{s^{n-2}}{(n-2)!} \phi(s)\, ds & \text{if } k = 0, \\ kc(1+\theta)\, t^{k-1} & \text{if } 1 \leqslant k \leqslant n-1; \end{cases} \tag{17}$$

hence, the family $\{(Ty)' : y \in Y\}$ is equibounded on finite intervals. This implies that $T(Y)$ is equicontinuous on finite intervals, and since it is obviously uniformly bounded on such intervals because of (14) and the definition of Y, the Arzela–Ascoli theorems imply that $T(Y)$ has compact closure. Now the Schauder–Tychonoff theorem guarantees that $Ty_0 = y_0$ for some y_0 in Y, and it is routine to verify that y_0 satisfies (1) and (2). This completes the proof.

REMARK 1. Only minor modifications of this proof are required to show that the conclusions of Theorem 1 remain valid for $k = 0$ if the stated monotonicity conditions on F hold only for sufficiently small y if (C_1) holds, or for sufficiently large y if (C_2) or (C_3) hold.

EXAMPLE 1. Consider the equation

$$y^{(n)} + q(t)\, y^{\gamma} [\log(1+y)]^{\delta} \sin y = 0, \quad t \geqslant t_0, \tag{18}$$

where γ and δ are constant such that $\delta \geqslant 0$ and $\gamma+\delta \neq 1$. The function $f(t, y)$ in (18) satisfies (4) with $\phi(t) = |q(t)|$ and $F(y) = y^{\gamma+\delta}$. Theorem 1 therefore implies that if

$$\int_{t_0}^{\infty} t^{n-1+k(\gamma+\delta-1)} |q(t)| \, dt < \infty,$$

then (18) has, for suitably chosen c, a solution y_0 which is defined on $[t_0, \infty)$ and satisfies (2).

EXAMPLE 2. Consider the equation

$$\Delta^m u + q(|x|) u^\gamma = 0 \tag{19}$$

in an exterior domain

$$\Omega_a = \{x \in \mathbb{R}^3 : |x| \geqslant a\},$$

where $a > 0$, Δ is the Laplacian in $\mathbb{R}^3$, $m \geqslant 1$ is an integer, $\gamma\ (\neq 1)$ is a constant, and $q : [a, \infty) \to \mathbb{R}$ is continuous. If z is a solution of the ordinary differential equation

$$\left[t^{-2} \frac{d}{dt} t^2 \frac{d}{dt}\right]^m z + q(t) z^\gamma = 0, \quad t \geqslant a, \tag{20}$$

then the function $u(x) = z(|x|)$ is a spherically symmetric solution of (19) in Ω_a. But (20) can be rewritten as

$$t^{-1} \frac{d^{2m}}{dt^{2m}} (tz) + q(t) z^\gamma = 0$$

which, with $y = tz$, is equivalent to

$$y^{(2m)} + t^{1-\gamma} q(t) y^\gamma = 0. \tag{21}$$

Theorem 1 implies that if

$$\int_a^{\infty} t^{2m-k+\gamma(k-1)} |q(t)| \, dt < \infty$$

for some k in $\{0, 1, \ldots, 2m-1\}$, then (21) has a solution y_0 on $[a, \infty)$ which satisfies (2) for suitable positive c. Hence, (19) has a spherically symmetric solution u_0 on Ω_a such that

$$\lim_{|x| \to \infty} u(x) |x|^{-k+1} = c.$$

REMARK 2. Theorem 1 can be generalized, for example, by assuming that

$$|f(t, y)| \leqslant \sum_{i=1}^{N} \phi_i(t) F_i(y) \quad \text{for } (t, y) \in [t_0, \infty) \times (0, \infty)$$

where, for each i $(1 \leqslant i \leqslant N)$, the functions ϕ_i and F_i satisfy the hypotheses imposed on ϕ and F in Theorem 1. (However, either (C_1) must hold for all i, or (C_2) or (C_3) must hold for all i.) A prototype of such equations is

$$y^{(n)} + \sum_{i=1}^{N} q_i(t) y^{\gamma_i} = 0, \tag{22}$$

where all $\gamma_i < 1$ or all $\gamma_i > 1$. If this is so and

$$\int_{t_0}^{\infty} t^{n-k-1+k\gamma_i} |q_i(t)| \, dt < \infty, \quad 1 \leqslant i \leqslant N,$$

then (22) has solutions on $[t_0, \infty)$ which satisfy (2) for suitable positive c.

The next theorem applies in some situations where Theorem 1 does not.

THEOREM 2. *Let ψ be continuous and non-increasing on $[t_0, \infty)$. If $1 \leqslant k \leqslant n-1$, suppose also that*

$$\sup_{t \geqslant t_0} (t^k \psi(t))^{-1} \int_{t_0}^{t} (t-s)^{k-1} \psi(s)\, ds = \alpha/k < \infty. \tag{23}$$

Let c and ρ be constants, with $\rho > 0$. Suppose that f is continuous on the set

$$S = \{(t, y) \colon t \geqslant t_0, |y - ct^k| \leqslant \rho\psi(t)\, t^k\} \tag{24}$$

and there satisfies the inequality

$$|f(t, y) - f(t, ct^k)| \leqslant w(t, |y - ct^k|), \tag{25}$$

where $w(t, \lambda)$ is continuous and positive on the set

$$S_1 = \{(t, \lambda) \colon t \geqslant t_0, 0 \leqslant \lambda \leqslant \rho\psi(t)\, t^k\}$$

and non-decreasing in λ for each t. Suppose that the integral

$$\int_{t_0}^{\infty} t^{n-k-1} f(t, ct^k)\, dt$$

converges (perhaps conditionally), and

$$\int_{t_0}^{\infty} t^{n-k-1} w(t, \rho\psi(t)\, t^k)\, dt < \infty. \tag{26}$$

Finally, suppose that the function

$$\sigma(t) = \int_{t}^{\infty} s^{n-k-1} w(s, \rho\psi(s)\, s^k)\, ds + \sup_{\tau \geqslant t} \left| \int_{\tau}^{\infty} s^{n-k-1} f(s, cs^k)\, ds \right| \tag{27}$$

satisfies the inequality

$$\sigma(t) \leqslant A\psi(t), \tag{28}$$

where

$$A \leqslant \begin{cases} (n-1)!\,\rho & \text{if } k = 0, \\ k!(n-k-1)!\,\rho/\alpha & \text{if } 1 \leqslant k \leqslant n-1. \end{cases} \tag{29}$$

Then (1) has a solution y_0 on $[t_0, \infty)$ which belongs to

$$Y = \{y \in C[t_0, \infty) \colon |y(t) - ct^k| \leqslant \rho\psi(t)\, t^k\}, \tag{30}$$

and satisfies the asymptotic condition

$$y_0(t) = (c + O(\psi(t)))\, t^k. \tag{31}$$

If $\lim_{t \to \infty} \psi(t) > 0$, then (31) can be replaced by (2).

REMARK 3. Before proving Theorem 2, we comment on the role of ψ. There are situations (see Example 3, below), where (26) may fail to hold with $\psi = 1$, but (26) and (28) hold for some ψ such that $\lim_{t \to \infty} \psi(t) = 0$. In these cases (31) is more precise than (2), but a comment is in order concerning (23), which is needed for technical reasons if $1 \leqslant k \leqslant n-1$: it is, roughly speaking, a reasonable restriction on how rapidly $\psi(t) \to 0$ as $t \to \infty$. Thus, it is easy to show that (23) holds if $t^\gamma \psi(t)$ is eventually non-decreasing for some $\gamma < 1$, but not if $t^\gamma \psi(t)$ is eventually non-increasing for some

$\gamma \geqslant 1$. However, if the latter is the case, then it would be reasonable (although not always possible if $\gamma = 1$) to restate the theorem with k and $\psi(t)$ replaced by $k-1$ and $\psi_1(t) = t\psi(t)$, respectively.

Proof of Theorem 2. If $y \in Y$ and $t_1 \geqslant t \geqslant t_0$, we can write

$$\int_t^{t_1} s^{n-k-1} f(s, y(s))\, ds = \int_t^{t_1} s^{n-k-1} f(s, cs^k)\, ds + \int_t^{t_1} s^{n-k-1} [f(s, y(s)) - f(s, cs^k)]\, ds. \tag{32}$$

Since (25) and (30) imply that

$$|f(s, y(s)) - f(s, cs^k)| \leqslant w(s, \rho\psi(s)\, s^k),$$

our integrability conditions imply that we can let $t_1 \to \infty$ in (32) and infer from (27) that

$$\left| \int_t^{\infty} s^{n-k-1} f(s, y(s))\, ds \right| \leqslant \sigma(t), \quad t \geqslant t_0, \tag{33}$$

where the integral may converge conditionally. If $F(t)$ denotes this integral, then integration by parts shows that if $0 \leqslant k \leqslant n-2$, then

$$\int_t^{\infty} (t-s)^{n-k-1} f(s, y(s))\, ds = \int_t^{\infty} F(s) \frac{d}{ds}\left(\frac{t}{s} - 1\right)^{n-k-1} ds,$$

and it is easy to see from this, (33), and the monotonicity of σ that

$$\left| \int_t^{\infty} (t-s)^{n-k-1} f(s, y(s))\, ds \right| \leqslant \sigma(t). \tag{34}$$

This inequality also holds if $k = n-1$, since then it reduces to (33).

From (3), (28) and (34),

$$|Ty(t) - c| \leqslant \frac{\sigma(t)}{(n-1)!} \leqslant \frac{A\psi(t)}{(n-1)!} \quad \text{if } k = 0 \tag{35}$$

or, if $1 \leqslant k \leqslant n-1$,

$$\begin{aligned} |Ty(t) - ct^k| &\leqslant \frac{1}{(k-1)!\,(n-k-1)!} \int_{t_0}^{t} (t-s)^{k-1} \sigma(s)\, ds \\ &\leqslant \frac{A}{(k-1)!\,(n-k-1)!} \int_{t_0}^{t} (t-s)^{k-1} \psi(s)\, ds \\ &\leqslant \frac{A\alpha t^k \psi(t)}{k!(n-k-1)!} \qquad \text{(see (23))}. \end{aligned} \tag{36}$$

Therefore, (29) and (30) imply that $Ty \in Y$ (see (30)), which verifies (14).

The proof that T is continuous is the same as in Theorem 1, except that the majorizing function (16) is now replaced by

$$2\tau^{n-k-1} w(\tau, \rho\psi(\tau)\, \tau^k)$$

(see (25) and (30)). It is straightforward to verify from (3) and (34) that if $y \in Y$, then

$$|(Ty)'(t)| \leqslant \left[k|c| + \frac{\sigma(t_0)}{(k-1)!\,(n-k-1)!} \right] t^{k-1}, \quad t \leqslant t_0, \quad \text{if } 1 \leqslant k \leqslant n-1. \tag{37}$$

If $k = 0$, then differentiating (3) yields

$$(Ty)'(t) = \frac{-1}{(n-2)!} \int_t^\infty (t-s)^{n-2} s^{-n+1} F'(s)\,ds,$$

where $F(t)$ is the integral in (33). Integrating by parts yields

$$(Ty)'(t) = \frac{-1}{(n-2)!} (t-s)^{n-2} s^{-n+1} F(s) \Big|_t^\infty + \frac{1}{(n-2)!} \int_t^\infty F(s) \frac{d}{ds}\left[\left(\frac{t}{s}-1\right)^{n-2} \frac{1}{s}\right] ds. \tag{38}$$

But,

$$\left|\frac{d}{ds}\left[\left(\frac{t}{s}-1\right)^{n-2} \frac{1}{s}\right]\right| \leqslant \frac{1}{s^2} + \frac{1}{t}\frac{d}{ds}\left(1-\frac{t}{s}\right)^{n-2}, \quad t \geqslant s.$$

From this, (33), (38), and the monotonicity of σ,

$$|(Ty)'(t)| \leqslant \frac{2\sigma(t)}{(n-2)!\,t}, \quad t \geqslant t_0, \quad \text{if } k = 0. \tag{39}$$

Now (37) and (39) enable us to complete the verification of the hypotheses of the Schauder–Tychonoff theorem, as (17) did in the proof of Theorem 1. This proves that (1) has a solution y_0 on $[t_0, \infty)$ which satisfies (31). Of course, if $\lim_{t\to\infty} \psi(t) > 0$ (so that we may as well assume that $\psi = 1$), then (31) does not imply (2); however, in this case we may still invoke the first inequalities in (35) and (36), and these imply (2), since obviously $\lim_{t\to\infty} \sigma(t) = 0$. This completes the proof.

As an application of Theorem 2, we improve on a local (near ∞) result obtained in [**3**], under more restrictive assumptions on ψ, for the equation

$$y^{(n)} + P(t)\,y^\gamma = 0. \tag{40}$$

THEOREM 3. *Suppose that* $P \in C[t_0, \infty)$, $\gamma \neq 1$, *and that* ψ *is as in Theorem* 2. *Suppose also that*

$$\int_t^\infty s^{n-1+k(\gamma-1)} P(s)\,ds = O(\psi(t)), \quad t \to \infty, \tag{41}$$

and

$$\int_t^\infty s^{n-1+k(\gamma-1)} |P(s)|\, \psi(s)\,ds = O(\psi(t)), \quad t \to \infty, \tag{42}$$

where the convergence in (41) *may be conditional. Let* θ *and* c *be given, with* $c > 0$ *and* $0 < \theta < 1$. *Then* (40) *has a solution* y_0 *on* $[t_0, \infty)$ *such that*

$$|y_0(t) - ct^k| \leqslant \theta c(\psi(t_0))^{-1} \psi(t) t^k, \quad t \geqslant t_0,$$

provided that $c^{\gamma-1}$ *is sufficiently small.*

Proof. Here $f(t, y) = P(t)\,y^\gamma$, and the mean value theorem implies that if $y > 0$, then

$$f(t,y) - f(t, ct^k) = \gamma P(t)\,\tilde{y}^{\gamma-1}(y - ct^k), \tag{43}$$

where $\tilde{y}$ is between y and ct^k. Let

$$\rho = \theta c(\psi(t_0))^{-1}; \tag{44}$$

then, if $(t, y) \in S$ (see (24)), the monotonicity of ψ implies that

$$(1-\theta)\,ct^k \leqslant \tilde{y} \leqslant (1+\theta)\,ct^k,$$

so (43) implies (25) with

$$w(t, \lambda) = |\gamma| [(1 \pm \theta) c]^{\gamma-1} |P(t)| \lambda,$$

where the $\pm$ is plus if $\gamma > 1$, minus if $\gamma < 1$. Now $\sigma(t)$ in (27) becomes

$$\sigma(t) = c^{\gamma} \left[K \int_t^\infty s^{n-1+k(\gamma-1)} |P(s)| \psi(s)\, ds + \sup_{\tau \geqslant t} \left| \int_\tau^\infty s^{n-1+k(\gamma-1)} P(s)\, ds \right| \right],$$

where

$$K = |\gamma| (1 \pm \theta)^{\gamma-1} \theta(\psi(t_0))^{-1}$$

(recall (44)). Therefore, (41) and (42) imply that $\sigma(t) \leqslant K_1 c^{\gamma}$, where K_1 is independent of c. This verifies (28) with $A = K_1 c^{\gamma}$, and we see from (44) that (29) holds if $c^{\gamma-1}$ is sufficiently small. Now Theorem 2 implies the conclusion.

REMARK 4. Suppose that (41) holds. If

$$\int^\infty s^{n-1+k(\gamma-1)} |P(s)|\, ds < \infty, \tag{45}$$

then (42) holds for any non-increasing ψ, and the conclusion of Theorem 3 can be obtained from Theorem 1 and trivial estimates. Therefore, it is important to observe that if $\lim_{t\to\infty} \psi(t) = 0$, then (41) and (42) do not imply (45); hence, Theorem 3 yields conclusions which do not follow from Theorem 1.

EXAMPLE 3. Suppose that $t_0 > 0$, and consider the equation

$$y^{(n)} + (t^{-n} \sin t) y^{\gamma} = 0. \tag{46}$$

Here

$$\int^\infty t^{n-1} |P(t)|\, dt = \int^\infty t^{-1} |\sin t|\, dt = \infty,$$

so we cannot apply Theorem 1 with $k = 0$. However,

$$\int_t^\infty s^{-1} \sin s\, ds = O(1/t) \quad \text{and} \quad \int_t^\infty s^{-2} |\sin s|\, ds = O(1/t), \quad t \to \infty;$$

hence Theorem 3 (with $\psi(t) = 1/t$) implies that if $0 < \theta < 1$ and $c > 0$, then (46) has a solution y_0 on $[t_0, \infty)$ such that

$$|y_0(t) - c| \leqslant \theta c t_0 / t, \quad t \geqslant t_0,$$

provided $c^{\gamma-1}$ is sufficiently small.

EXAMPLE 4. We now consider the equation

$$y^{(n)} + P(t) e^{y} = 0, \tag{47}$$

where $P \in C[t_0, \infty)$ for some $t_0 > 0$ and

$$\int_{t_0}^\infty t^{n-1} |P(t)| e^{at^k}\, dt = I < \infty$$

for some real a. Theorem 1 does not apply to (47); however, we shall now show that if $\rho > 0$, then there is $a_1 \leqslant a - \rho$ such that if $c \leqslant a_1$, then (47) has a solution y_0 on $[t_0, \infty)$ which satisfies the inequality

$$|y_0(t) - ct^k| \leqslant \rho t^k, \quad t \geqslant t_0,$$

and the asymptotic condition (2). To this end, observe that $f(t,y) = P(t)e^y$ in (47), and therefore the mean value theorem implies that

$$f(t,y)-f(t,ct^k) = P(t)e^{\tilde{y}}(y-ct^k), \tag{48}$$

where $\tilde{y}$ is between y and ct^k. Therefore, if (t,y) is in S as defined by (24) with $\psi = 1$, then (48) implies (25), with

$$w(t,\lambda) = |P(t)|\, e^{(c+\rho)t^k}\lambda.$$

If $c+\rho \leqslant a$, then $\sigma(t)$ in (27) satisfies the inequality

$$\begin{aligned}\sigma(t) &\leqslant \rho\int_{t_0}^{\infty} s^{n-1}|P(s)|\,e^{(c+\rho)s^k}\,ds+\int_{t_0}^{\infty} s^{n-k-1}|P(s)|\,e^{cs^k}\,ds\\ &\leqslant \rho e^{(c-a)t_0^k}[e^{\rho t_0^k}+\rho^{-1}t_0^{-k}]\,I.\end{aligned} \tag{49}$$

Now choose $a_1 \leqslant a-\rho$ such that

$$e^{(a_1-a)t_0^k}[e^{\rho t_0^k}+\rho^{-1}t_0^{-k}]\,I \leqslant k!(n-k-1)!$$

and suppose that $c \leqslant a_1$. Then (49) implies (28) and (29) (with $\alpha = 1$ and $\psi = 1$), and Theorem 2 implies the conclusion.

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