Reprinted from

PROCEEDINGS

Equadiff 6

Proceedings of the International Conference
on Differential Equations and their Applications
held in Brno, Czechoslovakia, Aug. 26–30, 1985

Edited by J. Vosmanský and M. Zlámal

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LINEAR PERTURBATIONS OF GENERAL DISCONJUGATE EQUATIONS

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Suppose that $p_1,\ldots,p_{n-1},\ q \in C[a,\infty]$, $p_i > 0$, and

(1) $\int^\infty p_i dt = \infty, \quad 1 \le i \le n-1,$

and define the quasi-derivatives

(2) $L_0x = x; \quad L_r x = \frac{1}{p_r}(L_{r-1}x)', \ 1 \le r \le n$

(with $p_n = 1$). We will give conditions which imply that the equation

(3) $L_n u + q(t)u = 0$

has solutions which behave as $t \to \infty$ like solutions of the equation $L_n x = 0$.

Let $I_0 = 1$ and

$$I_j(t,s;\, q_j,\ldots,q_i) = \int_s^t q_j(w) I_{j-1}(w,s;q_{j-1},\ldots,q_i)dw, \ j \ge 1.$$

Then a principal system [2] for $L_n x = 0$ is given by

$$x_i(t) = I_{i-1}(t,a;p_1,\ldots,p_{i-1}), \quad 1 \le i \le n\ ;$$

in fact,

(4) $$L_r x_i(t) = \begin{cases} I_{i-r-1}(t,a;p_{r+1},\ldots,p_{i-1}), & 0 \le r \le i-1\ , \\ 0, & i \le r \le n-1\ . \end{cases}$$

We also define

$$y_i(t) = I_{n-i}(t,a;p_{n-1},\ldots,p_i), \ 1 \le i \le n,$$

and

(5) $$d_{ir}(t) = \begin{cases} L_r x_i(t), & 0 \le r \le i-1, \\ 1/I_{r-i+1}(t,a;p_r,\ldots,p_i), & i \le r \le n\ . \end{cases}$$

We give sufficient conditions for (3) to have a solution u_i such that

(6) $L_r u_i = L_r x_i + o(d_{ir}) \quad (t \to \infty),\ 0 \le r \le n-1,$

for some given i in $\{1,\ldots,n\}$. This formulation of the question is due to Fink and Kusano, and the best previous result on this question is the following special case of a theorem obtained by them in [1].

THEOREM 1. *If*

(7) $$\int^{\infty} x_i y_i |q| ds < \infty,$$

then (3) *has a solution* u_i *which satisfies* (6).

Our results require less stringent integrability conditions. We need the following lemma from [4].

LEMMA 1. *Suppose that* $Q \in C[t_0,\infty)$ *for some* $t_0 \geq a$, *that* $\int^{\infty} y_i Q dt$ *converges (perhaps conditionally), and that*

$$\sup_{\tau \geq t} \left| \int_{\tau}^{\infty} y_i Q ds \right| \leq \psi(t), \quad t \geq t_0,$$

where ψ *is nonincreasing and continuous on* $[t_0 \infty)$. *Define*

$$K(t;Q) = \int_t^{\infty} I_{n-i}(t,s;p_i,\ldots,p_{n-1}) Q(s) ds,$$

and, for $t \geq t_0$, *let*

$$J(t;Q) = K(t;Q) \quad \text{if } i = 1;$$

or

$$J(t;Q) = \int_{t_0}^{t} p_1(s) K(s;Q) ds = I_1(t,t_0;p_1 K(\ ;Q)) ds$$

if $i = 2$; *or*

$$J(t;Q) = I_{i-1}(t,t_0;p_1,\ldots,p_{i-1} K(\ ;Q))$$

if $3 \leq i \leq n$.

Then

(8) $$L_n J(t;Q) = -Q(t), \; t \geq t_0,$$

and

$$|L_r J(\ ;Q)| \leq \begin{cases} \Psi(t_0) d_{ir}(t), & 0 \leq r \leq i-2, \\ 2\Psi(t) d_{ir}(t), & i-1 \leq r \leq n-1, \end{cases} \quad t \geq t_0;$$

moreover, if $\lim_{t\to\infty} \Psi(t) = 0$, *then also*

$$L_r(J(t;Q)) = o(d_{ir}(t)), \quad 0 \leq r \leq i-2.$$

The following assumption applies throughout.

ASSUMPTION A. Let $\int^{\infty} y_i x_i q ds$ converge (perhaps conditionally), and suppose that

(9) $$E(t) = \int_t^{\infty} y_i x_i q ds = O(\varphi(t))$$

with φ nonincreasing on $[a,\infty)$, and

(10) $$\lim_{t\to\infty} \varphi(t) = 0 .$$

If $t_0 > a$, let $B(t_0)$ be the set of functions h such that $L_0h,\ldots,L_{n-1}h \in C[t_0,\infty)$ and

$$L_rh = \begin{cases} O(d_{ir}), & 0 \le r \le i-2, \\ & \qquad\qquad t \ge t_0 , \\ O(\varphi d_{ir}), & i-1 \le r \le n-1 , \end{cases}$$

with norm $\|\ \|$ defined by

(11) $$\|h\| = \sup_{t\ge t_0} \max \left\{ \frac{|L_rh(t)|}{\varphi(t_0)d_{ir}(t)}\ (0\le r\le i-2),\ \frac{|L_rh(t)|}{2\varphi(t)d_{ir}(t)}\ (i-1\le r\le n-1) \right\}$$

Then Lemma 1 with $Q = qv$ and $\Psi = K\varphi$ implies the following lemma.

LEMMA 2. *If* $v \in C[t_0,\infty)$ *and*

$$\left| \int_t^\infty y_i qv\,ds \right| \le K\varphi(t), \quad t \ge t_0 ,$$

then

$$J(\ ;qv) \in B(t_0)$$

and

$$\|J(\ ;qv)\| \le K .$$

Now define the transformation T by

(12) $$(Th)(t) = J(t;qx_i) + J(t;qh) .$$

Lemma 2 and Assumption A imply that $J(\ ;qx_i) \in B(t_0)$ for all $t_0 > a$. We need only impose further conditions which will imply that $\int^\infty y_i qh\,ds$ converges (perhaps conditionally) if $h \in B(t_0)$, and that

$$\left| \int_t^\infty y_i qh\,ds \right| \le \|h\|\sigma(t;t_0)\varphi(t), \quad t \ge t_0 ,$$

where σ does not depend on h, and

(13) $$\sup_{t\ge t_0} \sigma(t;t_0) = \theta < 1$$

if t_0 is sufficiently large. Lemma 2 will then imply that T is a contraction mapping of $B(t_0)$ into itself, and therefore that there is an h_i in $B(t_0)$ such that $Th_i = h_i$. It will then follow from (8) and (12) that $u_i = x_i + h_i$ is a solution of (3). Moreover, Lemma 3 with $Q = qu_i$ will imply that

(14) $$L_ru_i - L_rx_i = \begin{cases} o(d_{ir}), & 0 \le r \le i-2 \\ O(\varphi d_{ir}), & i-1 \le r \le n-1 . \end{cases}$$

The next lemma can be obtained from (9) and integration by parts.

ee [3] for the proof of the special case where $p_1 = \cdots = p_n = 1$.

LEMMA 3. *Let*

$$H_0 = y_i q;\; H_j(t) = \int_t^\infty p_{j-1} H_{j-1} ds,\quad 1 \le j \le i \quad (p_0 = 1). \tag{15}$$

Then (9) implies that

$$H_j = O(\varphi / L_{j-1} x_i),\quad 1 \le j \le i, \tag{16}$$

nd that the integrals

$$\int^\infty p_j (L_j x_i) H_j ds,\quad 0 \le j \le i-1, \tag{17}$$

ll converge. Moreover, if the convergence is absolute for some $j = k$ ith $0 \le k \le i-2$, then it is absolute for $k \le j \le i-1$.

THEOREM 2. *If*

$$\overline{\lim_{t\to\infty}}\, (\varphi(t))^{-1} \int_t^\infty p_{i-1} |H_{i-1}| \varphi ds = A < \frac{1}{2}, \tag{18}$$

then (3) has a solution u_i which satisfies (14).

Proof. Integration by parts yields

$$\int_t^T y_i q h ds = -\sum_{j=1}^{i-1} H_j (L_{j-1} h)\Big|_t^T + \int_t^T p_{i-1} H_{i-1} (L_{i-1} h) ds \tag{19}$$

if $h \in B(t_0)$ and $2 \le i \le n$; if $i = 1$, then the sum on the right is vacuous and (19) is trivial. (Recall (2) and (15).) Now (5),(9), (11),(18), and Lemma 3 imply that we can let $T \to \infty$ in (19) and infer (13) with

$$\sigma(t;t_0) = \varphi(t_0)(\varphi(t))^{-1} \sum_{j=1}^{i-1} |H_j(t)| L_{j-1} x_i(t) + 2(\varphi(t))^{-1} \int_t^\infty p_{i-1} |H_{i-1}| \varphi ds. \tag{20}$$

From (16), the sum on the right side of (20) is bounded on $[a,\infty)$; hence, (10) and (18) imply (13) for t_0 sufficiently large. This completes the proof.

With $i = 1$, (18) reduces to

$$\overline{\lim_{t\to\infty}}\, (\varphi(t))^{-1} \int_t^\infty y_1 |q| \varphi ds < \frac{1}{2},$$

which is weaker than (7), since $x_1 = 1$. The next two corollaries show that (18) is also weaker than (7) if $2 \le i \le n$.

COROLLARY 1. *If $2 \le i \le n$ and*

$$\int^\infty p_k (L_k x_i)(L_{k-1} x_i)^{-1} \varphi dt < \infty \tag{21}$$

for some k in $\{1,\ldots,i-1\}$, then (3) has a solution u_i which *satisfies* (14).

Proof. From (16),

(22) $$p_k(L_k x_i)|H_k| \le Mp_k(L_k x_i)(L_{k-1}x_i)^{-1}\varphi$$

for some constant M, so (21) implies that (17) with $j = k$ converges absolutely. From the closing sentence of Lemma 3, this means that

$$\int^\infty p_{i-1}|H_{i-1}|ds < \infty ,$$

which obviously implies (18) with $A = 0$.

COROLLARY 2. *If* $2 \le i \le n$ *and*

(23) $$\int_t^\infty p_{i-1}(s)\left(\int_a^s p_{i-1}(w)dw\right)^{-1}\varphi^2(s)ds = o(\varphi(t)),$$

then (3) *has a solution* u_i *which satisfies* (13).

Proof. From (22) with $k = i - 1$ and (4), (23) implies (18) with $A = 0$.

THEOREM 3. *If* $1 \le i \le n - 1$ *and*

(24) $$\overline{\lim_{t\to\infty}}\, (\varphi(t))^{-1}\int_t^\infty \varphi(s)p_i(s)\left(\int_a^s p_i(w)dw\right)^{-1}|H_i(s)|ds = B < \frac{1}{2},$$

then (3) *has a solution which satisfies* (14).

Proof. Lemma 3 and our present assumption enable us to continue the integration by parts in (19) by one more step, to obtain

$$\int_t^\infty y_i qh ds = \sum_{j=1}^{i} H_j(t)L_{j-1}h(t) + \int_t^\infty p_i H_i(L_i h)ds.$$

Because of (5) (with $r = i$) and (11), this yields

$$\sigma(t;t_0) = \varphi(t_0)(\varphi(t))^{-1}\sum_{j=1}^{i-1}|H_j(t)|L_{j-1}x_i(t) + 2H_i(t) +$$
$$+ 2(\varphi(t))^{-1}\int_t^\infty \varphi(s)p_i(s)\left(\int_a^s p_i(w)dw\right)^{-1}|H_i(s)|ds.$$

Now (10) and (16) imply (20) for t_0 sufficiently large. This completes the proof.

COROLLARY 3. *If* $1 \le i \le n - 1$ *and*

(25) $$\int_t^\infty p_i(s)\left(\int_a^\infty p_i(w)dw\right)^{-1}\varphi^2(s)ds = o(\varphi(t)),$$

then (3) *has a solution* u_i *which satisfies* (14).

Proof. From (16) with $j = i$, it follows that (25) implies (24) with B = 0.

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