Global existence of solutions of mixed sublinear-superlinear differential equations

Takaši Kusano and William F. Trench
(Received November 16, 1985)

1. Introduction

Let $L_n$ be the general disconjugate operator

$$L_n = \frac{1}{p_n(t)} \frac{d}{dt} \frac{1}{p_{n-1}(t)} \frac{d}{dt} \cdots \frac{1}{p_1(t)} \frac{d}{dt} \frac{1}{p_0(t)} \quad (n \geq 2),$$

where $p_i : [0, \infty) \to (0, \infty), 0 \leq i \leq n$, are continuous. Let (1) be in canonical form $[10]$ at $\infty$; i.e.,

$$\int_0^\infty p_i(t)dt = \infty, \quad 1 \leq i \leq n - 1.$$

Consider the mixed sublinear-superlinear differential equation

$$L_n y + [a(t)|y|^\alpha + b(t)|y|^{\beta}] \text{sgn} y = 0, \quad t > 0,$$

where

$$0 < \alpha < 1, \quad \beta > 1,$$

and $a, b : [0, \infty) \to \mathbb{R}$ are continuous.

In Theorem 1 we give conditions which imply that (3) has a solution $\hat{y}$ which is defined on the entire interval $[0, \infty)$ and behaves as $t \to \infty$ like a prescribed solution of the unperturbed equation

$$L_n y = 0, \quad t > 0.$$

This type of global existence problem for equations of the form $L_n y + f(t, y) = 0$ has recently been studied by the present authors $[9]$. The theory developed in $[9]$ covers the sublinear case ($b(t) \equiv 0$) as well as the superlinear case ($a(t) \equiv 0$), but not the mixed sublinear-superlinear case ($a(t) \neq 0$ and $b(t) \neq 0$). In this paper a device is presented which enables us to demonstrate the existence of global solutions of the mixed sublinear-superlinear equation (3).

By means of a similar device we find conditions which imply that the mixed sublinear-superlinear elliptic equation

$$\Delta u + \varphi(|x|)u^\alpha + \psi(|x|)u^\beta = 0, \quad x \in \mathbb{R}^N, \quad N \geq 3,$$
has a positive entire (i.e., defined for all \( x \) in \( R^N \)) solution \( \hat{u} \) such that
\[
\lim_{|x| \to \infty} |x|^{N-2}\hat{u}(x) = c > 0.
\]
Here \( x = (x_1, \ldots, x_N) \), \( |x| \) is the Euclidean length of \( x \), \( \Delta \) is the \( N \)-dimensional Laplacian,
\[
(7) \quad 0 < \lambda < 1, \quad \mu > 1,
\]
and \( \varphi, \psi : [0, \infty) \to (0, \infty) \) are continuous. The existence of decaying positive entire solutions for second order semilinear elliptic equations has been the subject of recent intensive investigations; see, e.g. [1–8]. However, the results obtained previously are not applicable to (6) in the mixed sublinear-superlinear case (7).

2. Ordinary differential equations

We define the iterated integrals
\[
I_0 = 1,
\]
\[
I_j(t, s; q_1, q_2, \ldots, q_1) = \int_s^t q_j(r)I_{j-1}(r, s; q_{j-1}, \ldots, q_1)dr, \quad s, \quad i \geq 0, \quad j \geq 1,
\]
where \( q_1, q_2, \ldots \) are locally integrable on \([0, \infty)\), and put
\[
(8) \quad y_i(t) = p_0(t)I_i(t, 0; p_1, \ldots, p_{i-1}), \quad 1 \leq i \leq n,
\]
\[
(9) \quad z_i(t) = p_i(t)I_{n-i}(t, 0; p_{n-1}, \ldots, p_i), \quad 1 \leq i \leq n.
\]
It is easily seen that \( y_i(t), 1 \leq i \leq n \), form a fundamental system for (5) on \([0, \infty)\), and that \( z_i(t), 1 \leq i \leq n \), are similarly related to the formal adjoint equation
\[
L^*_n z = \frac{1}{p_0(t)} \frac{d}{dt} \frac{1}{p_1(t)} \frac{d}{dt} \cdots \frac{1}{p_{n-1}(t)} \frac{d}{dt} \frac{d}{dt} z = 0.
\]
We also introduce the notation:
\[
\hat{J}_i(t; Q) = \int_t^\infty p_i(s)I_{n-i}(t, s; p_0, \ldots, p_{n-1})Q(s)ds,
\]
and
\[
\begin{align*}
J_1(t; Q) &= p_0(t)\hat{J}_1(t; Q) & \text{if } & i = 1, \\
J_2(t; Q) &= p_0(t)I_1(t, 0; p_1, \hat{J}_1(\cdot; Q)) & \text{if } & i = 2, \\
J_i(t; Q) &= p_0(t)I_{i-1}(t, 0; p_1, \ldots, p_{i-2}, p_{i-1}\hat{J}_{i-1}(\cdot; Q)) & \text{if } & 3 \leq i \leq n.
\end{align*}
\]
LEMMA 1. If \( Q \in C[0, \infty) \) and
\[
\left(10\right) \quad \int_{0}^{\infty} z(t)|Q(s)| ds < \infty,
\]
then \( J_f(t; Q) \) is defined for \( t \geq 0 \) and
\[
\left(11\right) \quad L_{a} J_f(t; Q) = - Q(t);
\]
moreover,
\[
\left(12\right) \quad |J_f(t; Q)| \leq y(t) \int_{0}^{\infty} z(s)|Q(s)| ds, \quad t \geq 0,
\]
and
\[
\left(13\right) \quad J_f(t; Q) = o(y(t)), \quad t \to \infty.
\]

The following theorem is the main result of this section.

THEOREM 1. Suppose that
\[
\left(14\right) \quad A = \int_{0}^{\infty} |a(t)| y(t) z(t) dt < \infty
\]
and
\[
\left(15\right) \quad B = \int_{0}^{\infty} |b(t)| y(t) z(t) dt < \infty
\]
for some \( i, 1 \leq i \leq n \), and define
\[
\left(16\right) \quad \gamma = A^{(\beta - 1)/(\beta - 2)} B^{(1 - \alpha)/(\beta - 2)} \left[ \left( \frac{\beta - 1}{1 - \alpha} \right)^{(1 - \alpha)/(\beta - 2)} + \left( \frac{1 - \alpha}{\beta - 1} \right)^{(\beta - 1)/(\beta - 2)} \right].
\]
Suppose also that \( \gamma < 1 \), choose \( \theta \) so that
\[
\left(17\right) \quad \gamma < \frac{\theta}{1 + \theta} < 1,
\]
and define
\[
\left(18\right) \quad c(\theta) = \frac{1}{1 + \theta} \left[ \frac{(1 - \alpha) A}{(\beta - 1) B} \right]^{1/(\beta - 2)}.
\]

Then, if \( c \) is a positive constant sufficiently close to \( c(\theta) \), \( 3 \) has a solution \( \hat{y} \) on \([0, \infty)\) such that
\[
|\hat{y}(t) - cy(t)| \leq \theta cy(t), \quad t \geq 0,
\]
and
\[ \lim_{t \to \infty} \frac{\dot{y}(t)}{y_i(t)} = c. \]  

**Proof.** The set of functions

\[ Y = \{ y \in C[0, \infty) : |y(t) - cy_i(t)| \leq \theta c y_i(t), \quad t \geq 0 \}, \]

where \( c > 0 \) is a constant to be specified later, is a closed convex subset of the Fréchet space \( C[0, \infty) \) of continuous functions on \([0, \infty)\) with the topology of uniform convergence on compact subintervals of \([0, \infty)\). Define the mapping \( \mathscr{F} \) by

\[ \mathscr{F} y(t) = cy_i(t) + J_i(t; [a|y|^s + b|y|^\theta] \text{sgn } y). \]

Now suppose that \( y \in Y \). Then

\[ |a|y|^s + b|y|^\theta| \leq |a|[(c(1+\theta)y_i]^s + |b|[(c(1+\theta)y_i]^\theta; \]

hence (14) and (15) imply (10) with

\[ Q = [a|y|^s + b|y|^\theta] \text{sgn } y. \]

Therefore \( \mathscr{F} y \) is defined on \([0, \infty)\) and satisfies the inequality

\[ |\mathscr{F} y(t) - cy_i(t)| \leq [c^s(1+\theta)^s A + c^\theta(1+\theta)^\theta B] y_i(t), \quad t \geq 0. \]

(see (12) with \( Q \) as in (20).) Consequently, \( \mathscr{F} \) maps \( Y \) into itself if

\[ c^s(1+\theta)^s A + c^\theta(1+\theta)^\theta B \leq \theta c, \]

or, equivalently, if

\[ c^{s-1}(1+\theta)^s A + c^{\theta-1}(1+\theta)^\theta B = f(c) \leq \theta. \]

It is elementary to verify that if (4) holds, then \( f(c) \) assumes the minimum value \( f(c(\theta)) = \gamma(1+\theta) < \theta \)

(cf. (16) and (17)) at the point \( c(\theta) \) defined in (18). Therefore, \( \mathscr{F}(Y) \subset Y \) if \( c \) is sufficiently close to \( c(\theta) \).

It is not difficult to show that \( \mathscr{F} \) is continuous and \( \mathscr{F}(Y) \) is relatively compact with respect to the topology of \( C[0, \infty) \). The Schauder-Tychonoff fixed point theorem then implies that \( \mathscr{F} \) has a fixed point \( \hat{y} \) in \( Y \). Since

\[ \dot{y}(t) = cy_i(t) + J_i(t; [a|\dot{y}|^s + b|\dot{y}|^\theta] \text{sgn } \dot{y}), \]

(11) with \( Q=[a|\dot{y}|^s + b|\dot{y}|^\theta] \text{sgn } \dot{y} \) implies that \( \dot{y} \) satisfies (3), and (13) implies (19). This completes the proof.
REMARK 1. If (17) is replaced by
\[ \gamma = \frac{\theta}{1 + \theta} < 1 \]
(i.e., if \( \theta = \gamma/(1 - \gamma) \)), then the argument given above, with \( c = c(\theta) \), shows that (3) has a solution \( \gamma \) on \([0, \infty)\) such that
\[ |\dot{\gamma}(t) - (1 - \gamma)Ky(t)| \leq \gamma K y(t), \quad t \geq 0, \]
and
\[ \lim_{t \to \infty} \frac{\dot{y}(t)}{y_i(t)} = (1 - \gamma)K, \]
where
\[ K = \left[ \frac{(1 - \alpha)A}{(\beta - 1)B} \right]^{1/(\beta - \alpha)}. \]

REMARK 2. If \( \gamma < 1/2 \), then choosing \( \theta \) so that \( \gamma \leq \theta/(1 + \theta) < 1/2 \) yields a solution \( \gamma \) which is positive on \([0, \infty)\) if \( i = 1 \), or on \((0, \infty)\) if \( 2 \leq i \leq n \).

If \( p_0 = \cdots = p_n = 1 \), then (3) reduces to
\[ y^{(n)} + [a(t)]y^\alpha + b(t)|y|^\beta \text{sgn } y = 0, \]
while (8) and (9) reduce to
\[ y_i(t) = \frac{e^{i-1}}{(i-1)!} , \quad z_i(t) = \frac{e^{i-n}}{(n-i)!} . \]
Theorem 1 implies the following corollary, in which we have taken \( j = i - 1 \) for convenience.

**COROLLARY 1.** Consider the equation (21) with \( a, b: [0, \infty) \to \mathbb{R} \) and \( \alpha, \beta \) as in (4). Suppose that
\[ C = \int_0^\infty t^{n-j-1} + j\alpha |a(t)|dt < \infty \]
and
\[ D = \int_0^\infty t^{n-j-1} + j\beta |b(t)|dt < \infty \]
for some \( j, 0 \leq j \leq n - 1 \). Define
\[ \delta = C^{(\beta - 1)/(\beta - \alpha)}D^{(1-\alpha)/(\beta - \alpha)} \left[ \frac{\beta - 1}{1 - \alpha} \right]^{(1-\alpha)/(\beta - \alpha)} + \frac{1 - \alpha}{(\beta - 1)} \left( \frac{1}{\beta - 1} \right)^{(\beta - 1)/(\beta - \alpha)}. \]
Suppose also that
\[ \delta < j!(n-j-1)!, \]
choose \( \theta \) so that
\[ \frac{\delta}{j!(n-j-1)!} < \frac{\theta}{1+\theta} < 1, \]
and define
\[ c^*(\theta) = \frac{1}{1+\theta} \left[ \frac{(1-\alpha)C}{(\beta-1)D} \right]^{-1/(\beta-\alpha)}. \]

Then, if \( c \) is a positive constant sufficiently close to \( c^*(\theta) \), (21) has a solution \( \mathcal{G} \) on \([0, \infty)\) such that
\[ |\mathcal{G}(t) - ct^j| \leq \theta ct^j, \quad t \geq 0, \]
and
\[ \lim_{t \to \infty} \frac{\mathcal{G}(t)}{t^j} = c. \]

**Remark 3.** If (22) is replaced by
\[ \delta < \frac{j!(n-j-1)!}{2}, \]
then (21) has solutions which are positive on \((0, \infty)\) and satisfy (23) and (24) for suitable \( c > 0 \).

### 3. Partial differential equations

We now consider the elliptic partial differential equation (6), and show that a technique similar to the one used in the preceding section can be applied to give conditions implying that (6) has an entire radially symmetric solution which is positive for all \( x \) in \( \mathbb{R}^N \) and decays to zero as \(|x| \to \infty\).

In the following, we use the notation
\[ \sigma(t) = \max \{t, \ t^{N-1}\}, \quad \rho(t) = \min \{1, \ t^{2-N}\} \quad (\rho(0) = 1). \]

We remind the reader of our earlier stated assumptions on \( \varphi, \psi, \lambda, \mu \) and \( N \), which apply in the following theorem.

**Theorem 2.** Suppose that
\[ \int_0^{\infty} t^{N-1-\lambda(N-2)} \varphi(t) dt < \infty \]
Mixed sublinear-superlinear differential equations

(26) \[ \int_0^\infty t^{N-1-\mu(N-2)}\psi(t)dt < \infty, \]

and put

(27) \[ \Phi = \int_0^\infty \sigma(t)\rho^\lambda(t)\varphi(t)dt, \quad \Psi = \int_0^\infty \sigma(t)\rho^\mu(t)\psi(t)dt, \]

and

(28) \[ \eta = \Phi^{(\mu-1)/(\mu-\lambda)}\Psi^{(1-\lambda)/(\mu-\lambda)} \left[ \left( \frac{1-\lambda}{\mu-1} \right)^{(\mu-1)/(\mu-\lambda)} + \left( \frac{\mu-1}{1-\lambda} \right)^{(1-\lambda)/(\mu-\lambda)} \right]. \]

Then (6) has a radially symmetric solution \( \hat{u} \) which is defined and positive for all \( x \) in \( \mathbb{R}^N \), and has the asymptotic behavior

\[ \lim_{|x| \to \infty} |x|^{N-2}\hat{u}(x) = c(>0), \]

provided that

(29) \[ \eta \leq N - 2. \]

Note that (25) and (26) imply that \( \Phi \) and \( \Psi \) are finite. The proof of this theorem rests on the observation that if \( y = y(t) \) satisfies the ordinary differential equation

(30) \[ (t^{N-1}y')' + t^{N-1}\varphi(t)y^\lambda + t^{N-1}\psi(t)y^\mu = 0, \quad t > 0, \quad y'(0) = 0, \]

then the function \( u(x) = y(|x|) \) is a radially symmetric solution of (6). Therefore, it suffices to show that (30) has a positive solution \( \hat{y} \) on \( [0, \infty) \) such that

(31) \[ \lim_{t \to \infty} t^{N-2}\hat{y}(t) = c > 0. \]

The following lemma will help us to accomplish this.

LEMMA 2. Let \( h: [0, \infty) \to [0, \infty) \) be a continuous function such that

\[ \int_0^\infty t^{N-1}h(t)dt < \infty \]

and define

\[ \mathcal{M}h(t) = \frac{1}{N-2} \left[ \int_0^t \left( \frac{s}{t} \right)^{N-2}sh(s)ds + \int_t^\infty sh(s)ds \right], \quad t \geq 0. \]

Then

\[ [t^{N-1}(\mathcal{M}h)'(t)] = -t^{N-1}h(t), \quad t \geq 0, \]
and

\[(32) \quad I_1(h)\rho(t) \leq \mathcal{M}(t) \leq I_2(h)\rho(t), \quad t \geq 0,\]

where

\[I_1(h) = \frac{1}{N-2} \int_0^\infty \min \{s, s^{N-1}\} h(s)ds,\]
\[I_2(h) = \frac{1}{N-2} \int_0^\infty \max \{s, s^{N-1}\} h(s)ds.\]

Moreover,

\[\lim_{t \to \infty} t^{N-2} \mathcal{M}(t) = \frac{1}{N-2} \int_0^\infty s^{N-1} h(s)ds.\]

We omit the routine proof of this lemma.

**Proof of Theorem 2.** Let

\[Y = \{y \in C[0, \infty): c_1\rho(t) \leq y(t) \leq c_2\rho(t), \quad t \geq 0\},\]

where \(c_1\) and \(c_2\) are positive constants to be specified later. Notice that \(Y\) is a closed convex subset of \(C[0, \infty)\) with the topology of uniform convergence on compact subintervals. Define the mapping \(\mathcal{F}\) by

\[(33) \quad \mathcal{F}y(t) = \mathcal{M}(\varphi y^k + \psi y^\mu)(t), \quad t \geq 0.\]

If \(y \in Y\), then

\[(34) \quad \mathcal{F}y(t) \leq \mathcal{M}(\varphi c_2^k\rho^k)(t) + \mathcal{M}(\psi c_2^\mu\rho^\mu)(t)
\leq c_2^k I_2(\varphi \rho^k)\rho(t) + c_2^\mu I_2(\psi \rho^\mu)\rho(t), \quad t \geq 0,\]

where we have invoked the second inequality in (32) with \(h = \varphi \rho^k\) and \(h = \psi \rho^\mu\). From (27) and the definition of \(I_2(h)\), (34) can be rewritten as

\[\mathcal{F}y(t) \leq \frac{1}{N-2} [c_2^k \Phi + c_2^\mu \Psi] \rho(t), \quad t \geq 0.\]

Therefore,

\[(35) \quad \mathcal{F}y(t) \leq c_2 \rho(t), \quad t \geq 0,\]

if

\[(36) \quad c_2^{-1} \Phi + c_2^{-1} \Psi = g(c_2) \leq N - 2.\]

A routine computation shows that \(\eta\) as defined in (28) is the minimum value of \(g(c_2)\), and that it is attained when
With this choice of \( c_2 \), (29) implies (36).

Returning now to (33), we note that if \( y \in Y \), then

\[
\mathcal{F} y(t) \geq \mathcal{M}(\varphi c_i^1 \rho^\beta)(t) + \mathcal{M}(\psi c_i^4 \rho^\mu)(t)
\geq c_i^1 I_1(\varphi \rho^\beta) \rho(t) + c_i^4 I_1(\psi \rho^\mu) \rho(t), \quad t \geq 0,
\]

where we have invoked the first inequality in (32) with \( h = \varphi \rho^\lambda \) and \( h = \psi \rho^\mu \). Therefore,

\[
\mathcal{F} y(t) \geq c_i \rho(t), \quad t \geq 0,
\]

if

\[
c_i^{l-1} I_1(\varphi \rho^\beta) + c_i^{k-1} I_1(\psi \rho^\mu) \geq 1.
\]

Since \( \lambda < 1 \), the left side of (38) tends to \( \infty \) as \( c_i \to 0^+ \); therefore, we can certainly choose \( c_i \) so that \( 0 < c_i < c_2 \) and (38) holds.

Now (35) and (37) imply that \( \mathcal{F} \) maps \( Y \) into itself. The continuity of \( \mathcal{F} \) and the relative compactness of \( \mathcal{F}(Y) \) are easily proved. Therefore, \( \mathcal{F} \hat{y} = \hat{y} \) for some \( \hat{y} \) in \( Y \), by the Schauder-Tychonoff theorem. Since

\[
\hat{y} = \mathcal{M}(\varphi \hat{y}^\lambda + \psi \hat{y}^\mu),
\]

Lemma 2, with \( h = \varphi \hat{y}^\lambda + \psi \hat{y}^\mu \) and \( \mathcal{M} h = \hat{y} \), implies that \( \hat{y} \) satisfies (30) and (31). Therefore, the function \( \hat{u}(x) = \hat{y}(|x|) \) satisfies the requirements of the theorem, and the proof is complete.

**Remark 4.** It would be of interest to give conditions which guarantee the existence of decaying positive entire solutions of non-radial equations of the form

\[
\Delta u + \varphi(x)u^\lambda + \psi(x)u^\mu = 0, \quad x \in \mathbb{R}^N,
\]

where \( 0 < \lambda < 1, \mu > 1 \), and \( \varphi, \psi: \mathbb{R}^N \to (0, \infty) \) are locally Hölder continuous.

**References**


Department of Mathematics,
Faculty of Science,
Hiroshima University

and

Department of Mathematics and Computer Science,
Drexel University,
Philadelphia, PA 19104, U. S. A.*

*) The present address of the second author is as follows: Department of Mathematics, Trinity University, San Antonio, Texas 78284, U. S. A.