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LINEAR PERTURBATIONS OF A CONSTANT COEFFICIENT DIFFERENTIAL EQUATION SUBJECT TO MILD INTEGRAL SMALLNESS CONDITIONS

WILLIAM F. TRENCH, Philadelphia

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1. INTRODUCTION

We consider the equation

$$x^{(n)} + [a_1 + p_1(t)]\, x^{(n-1)} + \ldots + [a_n + p_n(t)]\, x = f(t)\,, \quad t > 0\,, \tag{1}$$

assuming throughout that $a_1, \ldots, a_n$ are complex constants and $p_1, \ldots, p_n, f$ are complex-valued and continuous on $(0, \infty)$. We give conditions implying that if λ_m is a simple zero of

$$Q(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_n\,,$$

then (1) has a solution x_m which behaves asymptotically like $c\mathrm{e}^{\lambda_m t}$.

We use "o" and "O" in the standard way to denote behavior as $t \to \infty$. The following theorem is due to Dunkel [1]; see also Hartman [2; Thm 17.2, p. 316].

Theorem 1. *Suppose that*

$$\int^\infty t^q |p_k(t)|\, \mathrm{d}t < \infty\,, \quad 1 \leqq k \leqq n\,, \tag{2}$$

for some $q \geqq 0$, that λ_m is a simple zero of $Q(\lambda)$, and that if λ_j is any other zero of $Q(\lambda)$, then $\operatorname{Re}(\lambda_j - \lambda_m) \neq 0$. Then the equation

$$x^{(n)} + [a_1 + p_1(t)]\, x^{(n-1)} + \ldots + [a_n + p_n(t)]\, x = 0\,, \quad t > 0\,, \tag{3}$$

has a solution x_m such that

$$x_m^{(r)}(t) = (\lambda_m^r + o(t^{-q}))\, \mathrm{e}^{\lambda_m t}\,, \quad 0 \leqq r \leqq n - 1\,. \tag{4}$$

Šimša [3] has recently given conditions which imply that (3) has a fundamental system $x_1, x_2, \ldots, x_n$ which satisfies (4) for $1 \leqq m \leqq n$. His proof easily implies the following result for a *given* m in $\{1, \ldots, n\}$.

Theorem 2. *Suppose that $Q(\lambda)$ has simple roots $\lambda_j = \mu_j + \mathrm{i}\nu_j$ $(1 \leqq j \leqq n)$, and* *that*

$$(5) \qquad \int^{\infty} |p_1(t)|\, dt < \infty\,.$$

Suppose also that for some integer m $(1 \leqq m \leqq n)$ *and nonnegative constants* q *and* ϱ, *the integrals*

$$(6) \qquad \int^{\infty} t^q\, p_k(t)\, e^{[\varrho + i(v_m - v_j)]t}\, dt\,, \quad 1 \leqq k \leqq n$$

converge (*perhaps conditionally*), *for* j *in*

$$(7) \qquad S = \{j \mid j = m \text{ or } \operatorname{Re}(\lambda_j - \lambda_m) + \varrho = 0\}\,.$$

Finally, suppose that at least one of the following is true:
(i) $\varrho > 0$; (ii) $q \geqq 1$; *or*

$$(8) \qquad \int^{\infty} t^{-q} \left| \int_t^{\infty} s^q\, p_k(s)\, ds \right| dt < \infty\,, \quad 2 \leqq k \leqq n\,.$$

Then (3) *has a solution* x_m *such that*

$$x_m^{(r)}(t) = (\lambda_m^r + o(e^{-\varrho t} t^{-q}))\, e^{\lambda_m t}\,, \quad 0 \leqq r \leqq n - 1\,.$$

Except for the assumption that *all* zeros of $Q(\lambda)$ be distinct (which is required for Šimša's stronger result), Theorem 2 is a considerable extension of Theorem 1, since under assumptions (i) and (ii) there are no integral smallness conditions on $p_2, \ldots, p_n$ which require absolute convergence, and (8) is weaker than (2) for $2 \leqq k \leqq n$, while (5) is weaker than (2) with $k = 1$ and $q > 0$.

Šimša [4] has given an example showing that some additional assumption such as (8) must be imposed to obtain the conclusion of Theorem 2 with $\varrho = 0$ and $0 \leqq \leqq q < 1$; however, we will show below that (8) can be weakened. (Very recently, Šimša [5] has obtained results for this case without assuming (8); however, they do not seem to be directly related to the results that we present here.)

2. THE MAIN THEOREM

It is convenient to state the following assumption separately from our main theorem.

Assumption A. *Let*

$$(9) \qquad Q(\lambda) = (\lambda - \lambda_1)^{d_1} \ldots (\lambda - \lambda_L)^{d_L}\,,$$

where

$$\lambda_j = \mu_j + i v_j\,, \quad 1 \leqq j \leqq L,$$

are distinct, and

$$\mu_1 \leqq \mu_2 \leqq \ldots \leqq \mu_L\,.$$

Let m be a fixed integer in $\{1, \ldots, L\}$ *such that* $d_m = 1$. *Let* ϱ *be a nonnegative constant, and suppose that* $d_j = 1$ *if*

$$\mu_j - \mu_m + \varrho = 0 . \tag{10}$$

Let N be the unique integer in $\{1, \ldots, L\}$ *such that*

$$\mu_j - \mu_m + \varrho < 0 , \quad 1 \leqq j \leqq N - 1 \tag{11}$$

(*which is vacuous if* $N = 1$) *and*

$$\mu_j - \mu_m + \varrho \geqq 0 , \quad N \leqq j \leqq L . \tag{12}$$

Let ϕ *be positive and nonincreasing on* $(0, \infty)$ *and, if* $N \geqq 2$, *let* $e^{\alpha t}\phi(t)$ *be nondecreasing for t sufficiently large* (*say* $t \geqq T_1$) *for some* α *such that*

$$0 < \alpha < \mu_m - \mu_{N-1} - \varrho .$$

Finally, let c be a given constant, and define

$$g(t) = f(t) - c e^{\lambda_m t} \sum_{k=1}^{n} \lambda_m^{n-k} p_k(t) . \tag{13}$$

Notice that λ_j need not be a simple root of $Q(\lambda)$ except for those j's (if any) that satisfy (10); thus, if $\varrho > 0$, then λ_m itself need not be simple.

Improper integrals occurring in hypotheses below are assumed to converge, and the convergence may be conditional, except, of course, where the integrands are necessarily nonnegative.

The following is our main result.

Theorem 3. *Suppose that Assumption* A *holds and*

$$\int_t^\infty g(s)\, e^{(\varrho - \mu_m - i\nu_j)s}\, ds = O(\phi(t)) \tag{14}$$

(*see* (13)) *for j in S* (*see* (7)). *Suppose also that*

$$\int_t^\infty |p_1(s)|\, \phi(s)\, ds = o(\phi(t)) \tag{15}$$

and

$$\int_t^\infty \left| \int_s^\infty p_k(\lambda)\, d\lambda \right| \phi(s)\, ds = o(\phi(t)) , \quad 2 \leqq k \leqq n . \tag{16}$$

Then (1) *has a solution* x_m *such that*

$$x_m^{(r)}(t) = (c\lambda_m^r + O(e^{-\varrho t}\phi(t))\, e^{\lambda_m t} , \quad 0 \leqq r \leqq n - 1 . \tag{17}$$

Moreover, if "O" *can be replaced by* "o" *in* (14), *then*

$$x_m^{(r)}(t) = (c\lambda_m^r + o(e^{-\varrho t}\phi(t)))\, e^{\lambda_m t} , \quad 0 \leqq r \leqq n - 1 . \tag{18}$$

Following Šimša [3], we use the Banach contraction principle to prove Theorem 3. It is convenient to introduce the new dependent variable

$$h(t) = x(t) - ce^{\lambda_m t},$$

in terms of which (1) becomes

$$Q(D)\,h = g - Mh \tag{19}$$

(see (9) and (13)), with

$$Mh = \sum_{k=1}^{n} p_k h^{(n-k)}. \tag{20}$$

Now suppose that $t_0 \geqq 0$ and let $B(t_0)$ be the Banach space of functions h in $C^{(n-1)}[t_0, \infty)$ such that

$$h^{(r)}(t) = O(e^{(\mu_m - \varrho)t}\phi(t)), \quad 0 \leqq r \leqq n-1,$$

with norm

$$\|h\| = \sup_{t \geqq t_0} \{e^{(\varrho - \mu_m)t}(\phi(t))^{-1} \sum_{r=0}^{n-1} |h^{(r)}(t)|\}. \tag{21}$$

Clearly, if (19) has a solution h_m in $B(t_0)$, then the function

$$x_m(t) = ce^{\lambda_m t} + h_m(t) \tag{22}$$

satisfies (1) on $[t_0, \infty)$ (and can be continued as a solution of (1) over $(0, \infty)$), and has the asymptotic behavior (17). We will now define a transformation which we will show to be a contraction of $B(t_0)$ if t_0 is sufficiently large, whose fixed point (function) h_m satisfies (19) on $[t_0, \infty)$.

To this end, let $A_1(t), \ldots, A_L(t)$ be the unique polynomials such that $\deg A_j < d_j$ $(1 \leqq j \leqq L)$ and

$$\sum_{j=1}^{L} [A_j(t)\, e^{\lambda_j t}]^{(r)}\big|_{t=0} = \delta_{r,n-1}, \quad 0 \leqq r \leqq n-1,$$

and define the associated polynomials

$$A_{jr}(t) = e^{-\lambda_j t}[A_j(t)\, e^{\lambda_j t}]^{(r)}, \quad 0 \leqq r \leqq n-1, \quad 1 \leqq j \leqq L.$$

Then

$$\deg A_{jr} = \deg A_j < d_j, \quad 0 \leqq r \leqq n-1, \quad 1 \leqq j \leqq L,$$

and the standard variation of parameters argument shows that if $w \in C[t_0, \infty)$ and

$$v(t; w) = \sum_{j=1}^{N-1} \int_{t_0}^{t} A_j(t-\tau)\, e^{\lambda_j(t-\tau)}\, w(\tau)\, d\tau - \tag{23}$$

$$- \sum_{j=N}^{L} \int_{t}^{\infty} A_j(t-\tau)\, e^{\lambda_j(t-\tau)} w(\tau)\, d\tau$$

(where the first sum is vacuous if $N = 1$), then

(24)
$$v^{(r)}(t; w) = \sum_{j=1}^{N-1} \int_{t_0}^{t} A_{jr}(t - \tau)\, e^{\lambda_j(t-\tau)}\, w(\tau)\, d\tau - \sum_{j=N}^{L} \int_{t}^{\infty} A_{jr}(t - \tau)\, e^{\lambda_j(t-\tau)}\, w(\tau)\, d\tau\,, \quad 0 \leqq r \leqq n - 1\,,$$

and

(25)
$$Q(D)\, v(t; w) = w\,,$$

provided that the improper integrals in (23) and (24) converge. This prompts us to consider the transformation $\mathscr{T}$ defined by

$$\mathscr{T} h = G - \mathscr{L} h\,,$$

where

(26)
$$G(t) = v(t; g)$$

and

(27)
$$(\mathscr{L} h)(t) = v(t; Mh)\,;$$

thus,

$$(\mathscr{T} h)(t) = v(t; g - Mh)\,,$$

and (25) with $w = g - Mh$ implies that

$$Q(D)\, \mathscr{T} h = g - Mh\,.$$

Therefore, h_m satisfies (19) if $\mathscr{T} h_m = h_m$.

We assume henceforth that $t_0 > 0$ or, if $N \geqq 2$, that $t_0 \geqq T_1$, so that $e^{\alpha t}\phi(t)$ is nondecreasing on $[t_0, \infty)$. (See Assumption A.) The proof of Theorem 3 reduces to showing that $\mathscr{T}$ is a contraction mapping of $B(t_0)$ into itself provided that t_0 is sufficiently large, since this implies that there is and h_m in $B(t_0)$ such that $\mathscr{T} h_m = h_m$. We will do this by showing that

(28)
$$G \in B(t_0)\,,$$

(29)
$$\mathscr{L}(B(t_0)) \subset B(t_0)\,,$$

and that there is a positive function σ on $(0, \infty)$ such that

(30)
$$\lim_{t \to \infty} \sigma(t) = 0$$

and

(31)
$$\|\mathscr{L} h\| \leqq \sigma(t_0)\, \|h\|\,.$$

The following lemma is needed for these proofs.

Lemma 1. *Suppose that u is complex-valued and continuous on $[t_0, \infty)$ and the integral*

$$U(t) = \int_{t}^{\infty} u(s)\, ds$$

converges for $t \geqq t_0$. *Denote*

$$\psi(t) = \sup_{\tau \geqq t} \left| \int_\tau^\infty u(s)\, ds \right| . \tag{32}$$

Let A be a polynomial, and suppose that γ *is a complex constant, with* $\mathrm{Re}(\gamma) = \xi$.

(i) *If* $\xi > 0$, *then*

$$\left| \int_t^\infty A(t-s)\, e^{\gamma(t-s)} u(s)\, ds \right| \leqq K_1\, \psi(t), \quad t \geqq t_0, \tag{33}$$

where K_1 *is a constant which depends only on* γ *and A.*

(ii) *If* $\xi < 0$ *and there is an* α *such that* $0 < \alpha < -\xi$ *and* $e^{\alpha t}\psi(t)$ *is nondecreasing on* $[t_0, \infty)$, *then*

$$\left| \int_{t_0}^t A(t-s)\, e^{\gamma(t-s)} u(s)\, ds \right| \leqq K_2\, \psi(t), \quad t \geqq t_0, \tag{34}$$

where K_2 *is a constant which depends only on* α, γ, *and A.*

Proof. (i) Integrating by parts yields

$$\int_t^\infty A(t-s)\, e^{\gamma(t-s)} u(s)\, ds = A(0)\, U(t) - \tag{35}$$
$$- \int_t^\infty [A(t-s)\, e^{\gamma(t-s)}]'\, U(s)\, ds .$$

From (32),

$$|[A(t-s)\, e^{\gamma(t-s)}]'\, U(s)| \leqq \psi(t)\, B(s-t)\, e^{\xi(t-s)}, \quad s \geqq t, \tag{36}$$

where B is a polynomial with nonnegative coefficients determined by γ and the coefficients of A; therefore, (32) and (35) imply (33), with

$$K_1 = |A(0)| + \int_0^\infty e^{-\xi\tau} B(\tau)\, d\tau .$$

(ii) Integration by parts yields

$$\int_{t_0}^t A(t-s)\, e^{\gamma(t-s)} u(s)\, ds = A(t-t_0)\, e^{\gamma(t-t_0)}\, U(t_0) - A(0)\, U(t) - \tag{37}$$
$$- \int_{t_0}^t [A(t-s)\, e^{\gamma(t-s)}]'\, U(s)\, ds .$$

Our assumptions regarding α imply that

$$\psi(t_0) \leqq e^{\alpha(t-t_0)} \psi(t), \quad t \geqq t_0 ;$$

hence,

$$|A(t-t_0)\, e^{\gamma(t-t_0)} U(t_0)| = |A(t-t_0)|\, e^{\xi(t-t_0)} \psi(t_0) \leqq \tag{38}$$
$$\leqq |A(t-t_0)|\, e^{(\xi+\alpha)(t-t_0)} \psi(t), \quad t \geqq t_0 .$$

With B as in (36), the assumption regarding α also implies that

$$(39) \qquad \left|\int_{t_0}^{t} [A(t-t_0)\,e^{\gamma(t-s)}]'\,U(s)\,ds\right| \leqq \int_{t_0}^{t} B(t-s)\,e^{\xi(t-s)}\psi(s)\,ds \leqq$$

$$\leqq \psi(t)\int_{t_0}^{t} B(t-s)\,e^{(\xi+\alpha)(t-s)}\,ds\,.$$

Now (32), (37), (38), and (39) imply (34), with

$$K_2 = |A(0)| + \int_0^\infty B(\tau)\,e^{(\xi+\alpha)\tau}\,d\tau + \sup_{\tau\geqq 0}|B(\tau)|\,e^{(\xi+\alpha)\tau}\,.$$

This completes the proof of Lemma 1.

We now turn to the proof of (28). We must show that $G \in C^{(n-1)}[t_0, \infty)$ and that

$$G^{(r)}(t) = O(e^{(\mu_m-\varrho)t}\phi(t))\,, \quad 0 \leqq r \leqq n-1\,.$$

Because of (24) with $w = g$ (see also (26)), this will follow if we show that for $0 \leqq r \leqq n-1$,

$$(40) \qquad \int_{t_0}^{t} A_{jr}(t-s)\,e^{\lambda_j(t-s)}g(s)\,ds = O(e^{(\mu_m-\varrho)t}\phi(t))\,, \quad 1 \leqq j \leqq N-1\,,$$

and

$$(41) \qquad \int_{t}^{\infty} A_{jr}(t-s)\,e^{\lambda_j(t-s)}g(s)\,ds = O(e^{(\mu_m-\varrho)t}\phi(t))$$

if $N \leqq j \leqq L$. To this end, notice that

$$(42) \qquad e^{\lambda_j(t-s)}g(s) = e^{(\lambda_m-\varrho)t}e^{(\lambda_j-\lambda_m+\varrho)(t-s)}[e^{(\varrho-\lambda_m)s}g(s)]\,.$$

Since the integral (14) converges with $j = m$, we can infer (40) for $1 \leqq j \leqq N-1$ (recall (11) and our condition on α in Assumption A) and (41) for those j's such that $N \leqq j \leqq L$ and strict inequality holds in (12), from Lemma 1 with $A = A_{jr}$, $\gamma = \lambda_j - \lambda_m + \varrho$, $u(t) = g(t)\,e^{(\varrho-\lambda_m)t}$, and $\psi(t) = O(\phi(t))$. If the equality holds in (12), then A_{jr} = constant (by Assumption A) and (42) reduces to

$$e^{\lambda_j(t-s)}g(s) = e^{(\mu_m-\varrho+i\nu_j)t}[e^{(\varrho-\mu_m-i\nu_j)s}g(s)]\,,$$

so (14) implies (41). This proves (28).

The next lemma will be used to prove (29) and to establish the existence of the function σ satisfying (30) and (31).

Lemma 2. *Suppose that the integrals in (15) and (16) converge, that $h \in B(t_0)$, and that β is a real constant. Then the functions*

$$W_k(t; h) = \int_t^\infty p_k(s)\,e^{(\varrho-\mu_m+i\beta)s}h^{(n-k)}(s)\,ds\,, \quad 1 \leqq k \leqq n\,,$$

are defined on $[t_0, \infty)$, *and they satisfy the inequalities*

(43) $$|W_1(t; h)| \leqq \|h\| \int_t^\infty |p_1(s)| \, \phi(s) \, \mathrm{d}s$$

and

(44) $$|W_k(t; h)| \leqq \|h\| \left[\phi(t) \left| \int_t^\infty p_k(\lambda) \, \mathrm{d}\lambda \right| + \right.$$
$$\left. + \, (1 + |\varrho - \mu_m + \mathrm{i}\beta|) \int_t^\infty \left| \int_s^\infty p_k(\lambda) \, \mathrm{d}\lambda \right| \phi(s) \, \mathrm{d}s \right], \quad 2 \leqq k \leqq n .$$

Proof. The existence of $W_1(t; h)$ and (43) follow from (21) and the assumed existence of the integral on the right side of (43). If $2 \leqq k \leqq n$, integration by parts yields

$$\int_t^T p_k(s) \, \mathrm{e}^{(\varrho - \mu_m + \mathrm{i}\beta)s} h^{(n-k)}(s) \, \mathrm{d}s =$$
$$= - \left[\mathrm{e}^{(\varrho - \mu_m + \mathrm{i}\beta)s} h^{(n-k)}(s) \int_s^\infty p_k(\lambda) \, \mathrm{d}\lambda \right] \Big|_t^T +$$
$$+ \int_t^T [\mathrm{e}^{(\varrho - \mu_m + \mathrm{i}\beta)s} h^{(n-k)}(s)]' \left(\int_s^\infty p_k(\lambda) \, \mathrm{d}\lambda \right) \mathrm{d}s ,$$

and routine estimates based on (21) imply (44), given the assumed convergence of the integrals on the right side. This completes the proof of Lemma 2.

We now turn to the proof of (29). Lemma 2 implies that if $h \in B(t_0)$ and β is a real constant, then

$$\left| \int_t^\infty \mathrm{e}^{(\varrho - \mu_m - \mathrm{i}\beta)s} M \, h(s) \, \mathrm{d}s \right| \leqq \|h\| \, \sigma(t; \beta)$$

(see (20)), where

(45) $$\sigma(t; \beta) = \int_t^\infty |p_1(s)| \, \Phi(s) \, \mathrm{d}s + \phi(t) \sum_{k=2}^n \left| \int_t^\infty p_k(s) \, \mathrm{d}s \right| +$$
$$+ \, (1 + |\varrho - \mu_m + \mathrm{i}\beta|) \sum_{k=2}^n \int_t^\infty \left| \int_s^\infty p_k(\lambda) \, \mathrm{d}\lambda \right| \phi(s) \, \mathrm{d}s .$$

Therefore, since

$$\mathrm{e}^{\lambda_j(t-s)} M \, h(s) = \mathrm{e}^{(\lambda_m - \varrho)t} \mathrm{e}^{(\lambda_j - \lambda_m + \varrho)(t-s)} [\mathrm{e}^{(\varrho - \lambda_m)s} M \, h(s)] .$$

Assumption A and Lemma 1 with $A = A_{jr}$, $\gamma = \lambda_j - \lambda_m + \varrho$, $u(t) = \mathrm{e}^{(\varrho - \lambda_m)t} M \, h(t)$, and $\psi(t) = \|h\| \sup_{\tau \geqq t} \sigma(\tau; 0)$ imply that there is a constant K (independent of h and t_0) such that

(46) $$\left| \int_{t_0}^t A_{jr}(t - s) \, \mathrm{e}^{\lambda_j(t-s)} M \, h(s) \, \mathrm{d}s \right| \leqq K \|h\| \, \mathrm{e}^{(\mu_m - \varrho)t} \sup_{\tau \geqq t} \sigma(\tau; 0) ,$$
$$0 \leqq r \leqq n - 1 , \quad 1 \leqq j \leqq N - 1 ,$$

$$(47)\quad \left|\int_t^\infty A_{jr}(t-s)\, e^{\lambda_j(t-s)} M\, h(s)\, ds\right| \leqq K\|h\|\, e^{(\mu_m-\varrho)t} \sup_{\tau \geqq t} \sigma(\tau; 0)\,, \quad 0 \leqq r \leqq n-1\,,$$

If $N \leqq j \leqq L$ and (10) does not holds. On the other hand, if (10) holds, then

$$e^{\lambda_j(t-s)} M\, h(s) = e^{(\mu_m-\varrho+i\nu_j)t}\left[e^{(\varrho-\mu_m-i\nu_j)s} M h(s)\right]$$

and A_{jr} = constant (Assumption A), so we can choose K so that (47) also holds if $\sigma(\tau; 0)$ is replaced by $\sigma(\tau; \nu_j)$. Since (15), (16), and (45) imply that $\sigma(t; \beta) = o(\phi(t))$ for all real β, (24) with $w = Mh$, (27), (46), and (47) imply that

$$(48)\quad |(\mathscr{L}h)^{(r)}(t)| \leqq \|h\|\, e^{(\mu_m-\varrho)t} \phi(t)\, \sigma(t)/n\,, \quad 0 \leqq r \leqq n-1\,, \quad t \geqq t_0\,,$$

where σ satisfies (30). Since (21) and (48) imply (31), we now conclude that $\mathscr{T}$ has a fixed point h_m in $B(t_0)$ if t_0 is sufficiently large; hence x_m as defined in (22) satisfies (1) and (17). To deduce the improved estimate (18) in the case where (14) is replaced by

$$(49)\quad \int_t^\infty g(s)\, e^{(\varrho-\mu_m-i\nu_j)s}\, ds = o(\Phi(t))\,,$$

it suffices to show that

$$(50)\quad h_m^{(r)}(t) = o(e^{(\mu_m-\varrho)t}\phi(t))\,, \quad 0 \leqq r \leqq n-1\,,$$

in this case. Since $h_m = G - \mathscr{L}h_m$, we see from (30) and (48) (with $h = h_m$) that (50) will follow if

$$(51)\quad G^{(r)}(t) = o(e^{(\mu_m-\varrho)t}\phi(t))\,, \quad 0 \leqq r \leqq n-1\,.$$

To see that this is so, define

$$\phi_1(t) = \sup_{\tau \geqq t}\left\{\max_{j \in S}\left|\int_\tau^\infty g(s)\, e^{(\varrho-\mu_m-i\nu_j)s}\, ds\right|\right\}$$

(see (7)). Applying the argument used earlier to prove (28), now with ϕ replaced by ϕ_1, shows that

$$G^{(r)}(t) = O(e^{(\mu_m-\varrho)t}\phi_1(t))\,, \quad 0 \leqq r \leqq n-1\,.$$

Since (49) implies that $\phi_1(t) = o(\phi(t))$, this implies (51) and completes the proof of Theorem 1.

3. RELATIONSHIP OF THE MAIN THEOREM WITH ŠIMŠA'S RESULT

We first deduce the following corollary from Theorem 3.

Corollary 1. *Suppose that Assumption* A *and* (15) *hold, and that*

$$(52)\quad \int_t^\infty f(s)\, e^{(\varrho-\mu_m-i\nu_j)s}\, ds = O(\phi(t))$$

and

$$\int_t^\infty p_k(s)\, e^{[\varrho + i(\nu_m - \nu_j)]s}\, ds = O(\phi(t)), \quad 1 \leqq k \leqq n \tag{53}$$

for j in S (see (7)). Then (1) has a solution x_m which satisfies (17), provided that (16), holds. Moreover, (16) holds automatically if any one of the following is true:

(i) $\varrho > 0$; (ii) "O" *can be replaced by* "o" *in* (53) *and*

$$\int_t^\infty \phi^2(s)\, ds = O(\phi(t)); \tag{54}$$

or (iii)

$$\int_t^\infty \phi^2(s)\, ds = o(\phi(t)). \tag{55}$$

Finally, if (52) *and* (53) *hold with* "O" *replaced by* "o", *then x_m satisfies* (18).

Proof. From (13), (52) and (53) imply (14). Therefore, Theorem 3 implies the conclusion if (16) holds. To complete the proof, we need only show that (16) follows from each of (i), (ii), and (iii). Integrating (53) (with $j = m$) by parts shows that

$$\int_t^\infty p_k(s)\, ds = O(e^{-\varrho t}\phi(t)), \quad 2 \leqq k \leqq n, \tag{56}$$

and therefore

$$\int_t^\infty \left| \int_s^\infty p_k(\lambda)\, d\lambda \right| \phi(s)\, ds = O\left(\int_t^\infty e^{-\varrho s}\phi^2(s)\, ds \right), \quad 2 \leqq k \leqq n. \tag{57}$$

The right side of (57) is $o(\phi(t))$ if either $\varrho > 0$ or (55) holds; hence, (i) and (iii) imply (16). To see that (ii) also implies (16), we have only to observe that if (53) holds with "o" on the right, then so does (56) and therefore (57). Given this, (54) implies (16) even if $\varrho = 0$. This proves Corollary 1.

We conclude by showing that Corollary 1 implies Theorem 2. It suffices to show that the integrability conditions of the latter imply those of the former, with $\phi(t) = t^{-q}$. Obviously, (5) implies (15) for any nonincreasing ϕ. Integrating by parts shows that if (6) converges, then

$$\int_t^\infty p_k(s)\, e^{[\varrho + i(\nu_m - \nu_j)]s}\, ds = o(t^{-q}), \quad j \in S,$$

which verifies (53) with "O" replaced by "o". Since

$$\int_t^\infty s^{-2q}\, ds = \frac{t^{-2q+1}}{(2q - 1)} \quad (q > 1/2),$$

(54) holds if $q \geqq 1$. Finally, an argument using integration by parts shows that if (8)

holds for some $q > 0$, then

$$\int^{\infty} \left| \int_s^{\infty} p_k(\lambda)\, d\lambda \right| ds < \infty . \tag{58}$$

(The converse is false.) Obviously, (58) implies (15) for any nonincreasing ϕ.

Since the integrability conditions of Theorem 1 imply those of Theorem 2 with $\varrho = 0$, Corollary 1 also implies Theorem 1.

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Author's address: Department of Mathematics and Computer Science, Drexel University, Philadelphia, Pennsylvania 19104, U.S.A.