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TOEPLITZ SYSTEMS ASSOCIATED WITH THE PRODUCT OF A FORMAL LAURENT SERIES AND A LAURENT POLYNOMIAL*

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Abstract. A method is proposed for solving linear algebraic systems with Toeplitz matrices generated by $T(z) = C(z)\Phi(z)$, where $C(z)$ is a Laurent polynomial and $\Phi(z)$ is a formal Laurent series, and a convenient method is available for solving systems with Toeplitz matrices generated by $\Phi(z)$. Special cases of the method provide $O(n)$ procedures for solving $n \times n$ systems with banded or rationally generated Toeplitz matrices. The latter do not require recursion with respect to n.

Key words. Toeplitz systems, banded Toeplitz matrices, rationally generated Toeplitz matrices

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1. Introduction. To motivate the problem considered here, let $\{x_j\}$ be a wide-sense stationary time series (possibly complex-valued) with zero mean and covariance $E(x_i\bar{x}_j) = \phi_{i-j}$. If

$$y_j = \sum_{l=0}^{p} b_l x_{j-l}, \qquad -\infty < j < \infty,$$

then $\{y_j\}$ has zero mean and covariance $E(y_i\bar{y}_j) = t_{i-j}$, where

$$t_i = \sum_{l=-p}^{p} c_l \phi_{i-l}, \tag{1}$$

with

$$c_l = \sum_{\nu=0}^{p-l} \bar{b}_\nu b_{\nu+l}, \qquad 0 \leqq l \leqq p,$$

and

$$c_l = \sum_{\nu=0}^{p+l} b_\nu \bar{b}_{\nu-l}, \qquad -p \leqq l \leqq -1.$$

Minimum variance estimation problems concerning the time series $\{y_j\}$ require solutions of the systems

$$T_n X = Y, \tag{2}$$

where T_n is the $n \times n$ Toeplitz matrix

$$T_n = (t_{i-j})_{i,j=1}^{n}. \tag{3}$$

(See, e.g., [16, pp. 20–23].) Definition (1) suggests that if we have an efficient way to solve the systems

$$\Phi_m U = V, \tag{4}$$

where

$$\Phi_m = (\phi_{i-j})_{i,j=1}^{m}, \tag{5}$$

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then it should be possible to exploit it in solving (2). Here we propose a method that does this; however, since our results are not restricted to systems with positive definite Hermitian Toeplitz matrices, we first formulate the situation more generally.

Let

$$\Phi(z) = \sum_{j=-\infty}^{\infty} \phi_j z^j$$

be a formal Laurent series, and let

$$C(z) = \sum_{j=-q}^{p} c_j z^j \tag{6}$$

be a Laurent polynomial, with

$$p, q \geqq 0, \quad p+q=k \geqq 1, \quad c_p c_{-q} \neq 0. \tag{7}$$

Now define

$$T(z) = C(z)\Phi(z) = \sum_{j=-\infty}^{\infty} t_j z^j,$$

so that

$$t_i = \sum_{l=-q}^{p} c_l \phi_{i-l}. \tag{8}$$

We are still interested in solving (2).

There are many algorithms for solving Toeplitz systems that take advantage of their special simplicity. (See, e.g., [3], [8], [11], [12], [17] and [18]—by no means a complete list.) However, most require assumptions that are not met by all Toeplitz matrices, and some are stable only for certain classes of Toeplitz matrices. (In this connection, see [2].) Our results should be useful if there is a convenient algorithm for dealing with the matrices generated by $\Phi(z)$ which does not apply to those generated by $T(z)$. This could be so, for example, if the former are Hermitian, symmetric, triangular, or positive definite, or if there is a convenient explicit formula for their inverses, while the latter do not exhibit the desirable property. Our results provide a way to transfer the burden of computation in solving (2) to a problem involving Φ_{n+k} and the banded matrix

$$C_{n+k} = (c_{i-j})_{i,j=1}^{n+k} \tag{9}$$

(cf. (7)). The method also entails the solution of a $k \times k$ system. Since there are several algorithms for solving banded Toeplitz systems (see, e.g., [1], [4], [9], [10], [13], and [14]), this procedure should be useful if n is large compared with k. Moreover, we also formulate a procedure that avoids using any of the previously published algorithms for solving banded Toeplitz systems and—as a by-product—provides a new method for this purpose; however, for reasons of stability, this method requires some knowledge of the locations of the zeros of $C(z)$. The method also provides an $O(n)$ procedure for solving (2) when T_n is generated by a rational function. (See § 4.)

2. Derivation of the method. We emphasize that we are not proposing to produce a complete algorithm here. Rather, we are assuming that an algorithm is already available for solving the system (4), where $m = n + p + q = n + k$ henceforth, and we wish to indicate how this can be exploited to solve (2).

Let $\mathscr{F}$ be the underlying field. From (5), (8), and (9),

$$C_m\Phi_m = \begin{bmatrix} [p\times p] & [p\times n] & [p\times q] \\ [n\times p] & T_n & [n\times q] \\ [q\times p] & [q\times n] & [q\times q] \end{bmatrix},$$

where T_n is as in (3) and the other blocks have the indicated dimensions. Therefore, an n-vector X satisfies (2) if and only if

$$C_m\Phi_m \begin{bmatrix} 0_p \\ X \\ 0_q \end{bmatrix} = \begin{bmatrix} U_0 \\ Y \\ V_0 \end{bmatrix},$$

where 0_p and 0_q are zero vectors of dimensions p and q, respectively, $U_0 \in \mathscr{F}^p$, and $V_0 \in \mathscr{F}^q$. For our purposes, it is convenient to view this in the manner stated in the following now obvious lemma.

LEMMA 1. *The system* (2) *has a solution for a given Y if and only if there are vectors U_0 in $\mathscr{F}^p$ and V_0 in $\mathscr{F}^q$ such that the system*

$$C_m\Phi_m G = \begin{bmatrix} U_0 \\ Y \\ V_0 \end{bmatrix} \tag{10}$$

has a solution G of the form

$$G = \begin{bmatrix} 0_p \\ X \\ 0_q \end{bmatrix}, \tag{11}$$

in which case X satisfies (2).

Now let $\mathscr{W}$ be the subspace of $\mathscr{F}^m$ consisting of vectors

$$W = [w_{-p+1}, \cdots, w_{n+q}]^t$$

whose components satisfy the homogeneous difference equation

$$\sum_{l=-q}^{p} c_l w_{i-l} = 0, \qquad 1 \leqq i \leqq n, \tag{12}$$

and let

$$W_j = [w_{-p+1}^{(j)}, \cdots, w_{n+q}^{(j)}]^t, \qquad 1 \leqq j \leqq k, \tag{13}$$

form a basis for $\mathscr{W}$. Let

$$F = [f_{-p+1}, \cdots, f_{n+q}]^t \tag{14}$$

be a vector in $\mathscr{F}^m$ whose components satisfy the nonhomogeneous difference equation

$$\sum_{l=-q}^{p} c_l f_{i-l} = y_i, \qquad 1 \leqq i \leqq n. \tag{15}$$

From the definition of C_m, (12) is equivalent to

$$C_m W_j = \begin{bmatrix} U_j \\ 0_n \\ V_j \end{bmatrix}, \qquad 1 \leqq j \leqq k, \tag{16}$$

and (15) is equivalent to

$$C_m F = \begin{bmatrix} U \\ Y \\ V \end{bmatrix}, \tag{17}$$

where $U, U_1, \cdots, U_k$ are in $\mathscr{F}^p$, 0_n is the zero vector in $\mathscr{F}^n$, and $V, V_1, \cdots, V_k$ are in $\mathscr{F}^q$.

There is no doubt about the existence of F and $W_1, \cdots, W_k$; in fact, there are many ways to choose them. We will discuss this in § 3.

THEOREM 1. *Let F and $W_1, \cdots, W_k$ be as just defined. Suppose that for each $j = 1, \cdots, k$ the system*

$$\Phi_m \tilde{W}_j = W_j \tag{18}$$

has a solution

$$\tilde{W}_j = \begin{bmatrix} \tilde{U}_j \\ H_j \\ \tilde{V}_j \end{bmatrix}, \tag{19}$$

and that the system

$$\Phi_m \tilde{F} = F \tag{20}$$

has a solution

$$\tilde{F} = \begin{bmatrix} \tilde{U} \\ \tilde{Y} \\ \tilde{V} \end{bmatrix}, \tag{21}$$

where $\{\tilde{U}, \tilde{U}_1, \cdots, \tilde{U}_k\} \subset \mathscr{F}^p$, $\{\tilde{Y}, H_1, \cdots, H_k\} \subset \mathscr{F}^n$, and $\{\tilde{V}, \tilde{V}_1, \cdots, \tilde{V}_k\} \subset \mathscr{F}^q$. Then the system (2) *has a solution if there are constants $a_1, \cdots, a_k$ such that*

$$\begin{bmatrix} \tilde{U} \\ \tilde{V} \end{bmatrix} = a_1 \begin{bmatrix} \tilde{U}_1 \\ \tilde{V}_1 \end{bmatrix} + \cdots + a_k \begin{bmatrix} \tilde{U}_k \\ \tilde{V}_k \end{bmatrix}, \tag{22}$$

in which case the vector

$$X = \tilde{Y} - a_1 H_1 - \cdots - a_k H_k \tag{23}$$

satisfies (2). *Moreover, the converse is true if Φ_m is invertible.*

Proof. For sufficiency, suppose that (22) holds, and let

$$G = \tilde{F} - a_1 \tilde{W}_1 - \cdots - a_k \tilde{W}_k,$$

which is of the form (11) with X as in (23), from (19), (21), and (22). From (18) and (20),

$$C_m \Phi_m G = C_m (F - a_1 W_1 - \cdots - a_k W_k),$$

and so (16) and (17) imply (10), with

$$\begin{bmatrix} U_0 \\ V_0 \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix} - a_1 \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} - \cdots - a_k \begin{bmatrix} U_k \\ V_k \end{bmatrix}.$$

Therefore, Lemma 1 implies that X as defined by (23) satisfies (2).

For the converse, suppose that Φ_m is invertible and (2) has a solution X. Then the vector G in (11) satisfies (10) for some U_0 in $\mathscr{F}^p$ and V_0 in $\mathscr{F}^q$. From (10) and (17),

$$C_m(F - \Phi_m G) = \begin{bmatrix} U - U_0 \\ 0_n \\ V - V_0 \end{bmatrix},$$

so $F - \Phi_m G \in \mathscr{W}$; hence

$$F - \Phi_m G = a_1 W_1 + \cdots + a_k W_k$$

for some scalars $a_1, \cdots, a_k$. From (18) and (20), this can be rewritten as

$$\Phi_m(\tilde{F} - G) = \Phi_m(a_1 \tilde{W}_1 + \cdots + a_k \tilde{W}_k),$$

so

$$\tilde{F} - G = a_1 \tilde{W}_1 + \cdots + a_k \tilde{W}_k,$$

since Φ_m is invertible. Now (11), (19), and (21) imply (22) and (23). This completes the proof.

THEOREM 2. *Suppose that Φ_m is invertible, let $W_1, \cdots, W_k$ be any basis for $\mathscr{W}$, and let Ψ be the $k \times k$ matrix*

$$\Psi = \begin{bmatrix} \tilde{U}_1 & \cdots & \tilde{U}_k \\ \tilde{V}_1 & \cdots & \tilde{V}_k \end{bmatrix},$$

with $\tilde{U}_1, \cdots, \tilde{U}_k$ and $\tilde{V}_1, \cdots, \tilde{V}_k$ as in (19). *Then T_n is invertible if and only if Ψ is invertible.*

Proof. Since Φ_m is invertible, $\tilde{W}_1, \cdots, \tilde{W}_k$ exist; moreover $\tilde{F}$ exists for every choice of F. If Ψ is invertible, then (22) has a solution $a_1, \cdots, a_k$ for every $\hat{U}$ and $\hat{V}$; hence, Theorem 1 implies that (2) has a solution for every Y, and therefore T_n is invertible. For the converse, suppose that Ψ is noninvertible. Then there are constants $b_1, \cdots, b_k$, not all zero, such that

$$b_1 \begin{bmatrix} \tilde{U}_1 \\ \tilde{V}_1 \end{bmatrix} + \cdots + b_k \begin{bmatrix} \tilde{U}_k \\ \tilde{V}_k \end{bmatrix} = \begin{bmatrix} 0_p \\ 0_q \end{bmatrix}.$$

This implies that

$$b_1 \tilde{W}_1 + \cdots + b_k \tilde{W}_k = \begin{bmatrix} 0_p \\ H \\ 0_q \end{bmatrix}, \tag{24}$$

with

$$H = b_1 H_1 + \cdots + b_k H_k$$

(cf. (19)). Because of (18), we can rewrite (24) as

$$b_1 W_1 + \cdots + b_k W_k = \Phi_m \begin{bmatrix} 0_p \\ H \\ 0_q \end{bmatrix}, \tag{25}$$

which makes it apparent that $H \neq 0_n$, since $\{W_1, \cdots, W_k\}$ is linearly independent. Now (16) and (25) imply that

$$C_m \Phi_m \begin{bmatrix} 0_p \\ H \\ 0_q \end{bmatrix} = \begin{bmatrix} U_0 \\ 0_n \\ V_0 \end{bmatrix}, \tag{26}$$

with

$$U_0 = \sum_{j=1}^{k} b_j U_j, \qquad V_0 = \sum_{j=1}^{k} b_j V_j .$$

However, (26) and Lemma 1 with $Y = 0_n$ and $X = H$ imply that $T_n H = 0_n$, and therefore T_n is noninvertible, since $H \neq 0_n$.

Henceforth we assume that Φ_m is invertible and that an efficient algorithm is available for solving systems with coefficient matrix Φ_m. Theorem 1 suggests a procedure for solving (2), as follows:

Step 1. Obtain a basis $W_1, \cdots, W_k$ for $\mathscr{W}$, and solve (18) for $\tilde{W}_1, \cdots, \tilde{W}_k$. If (2) is to be solved for more than one Y, then store $\tilde{W}_1, \cdots, \tilde{W}_k$ for repeated use.

Step 2. For the given Y, let F in (14) be a solution of (15), and solve (20) for $\tilde{F}$.

Step 3. Solve the $k \times k$ system (22) for $a_1, \cdots, a_k$. (If (22) has no solution, then (2) has no solution.)

Step 4. Compute X from (23), with $H_1, \cdots, H_k$ as defined in (19) and $\tilde{Y}$ as in (21).

The missing link in this procedure is a discussion of methods for obtaining F and $W_1, \cdots, W_k$. This is the subject of § 3.

3. Computation of F and $W_1, \cdots, W_k$. As mentioned earlier, there are many algorithms specifically designed to solve banded Toeplitz systems efficiently. If C_m is invertible, then we could obtain F by solving (17) with $U = 0_p$ and $V = 0_q$ by means of one of these algorithms. We could also obtain $W_1, \cdots, W_k$ by solving (16) in this way, with

$$\begin{bmatrix} U_1 & \cdots & U_k \\ V_1 & \cdots & V_k \end{bmatrix} = I_k .$$

However, all algorithms for solving banded Toeplitz systems require some kind of assumption on C_m; in fact, most require that C_m and all its principal submatrices be invertible. Therefore, we will suggest a recursive method for computing suitable vectors F and $W_1, \cdots, W_k$. This method requires no specific assumptions on C_m (even that it be invertible), and it addresses the question of stability; however, it does require information on the zeros of $C(z)$.

One solution (14) of (15) can be obtained from the recursion

$$f_i = \frac{1}{c_{-q}}\left[y_{i-q} - \sum_{l=-q+1}^{p} c_l f_{i-q-l}\right], \qquad q+1 \leqq i \leqq n+q, \tag{27}$$

with $f_i = 0$, if $-p+1 \leqq i \leqq q$. To exhibit a basis for $\mathscr{W}'$, we first consider the Maclaurin expansion

$$[z^q C(z)]^{-1} = \sum_{\nu=0}^{\infty} \alpha_\nu z^\nu.$$

The $\{\alpha_\nu\}$ can be computed recursively

$$\alpha_\nu = -\frac{1}{c_{-q}} \sum_{l=-q+1}^{p} c_l \alpha_{\nu-q-l}, \qquad \nu \geqq 1, \tag{28}$$

with $\alpha_\nu = 0$ if $\nu < 0$ and $\alpha_0 = 1/c_{-q}$. The vectors (13) with

$$w_i^{(j)} = \alpha_{i-j+p}, \quad -p+1 \leqq i \leqq n+q, \quad 1 \leqq j \leqq k, \tag{29}$$

form a basis for $\mathscr{W}'$. To see that they satisfy (12), observe that if (29) holds, then

$$\sum_{l=-q}^{p} c_l w_{i-l}^{(j)} = \sum_{l=-q}^{p} c_l \alpha_{i-j+p-l}. \tag{30}$$

However, from (28)

$$\sum_{l=-q}^{p} c_l \alpha_{\mu-l} = 0, \qquad \mu > -q.$$

Therefore, the right side of (30) vanishes if $i \geqq 1$ and $1 \leqq j \leqq k$, since then $i-j+p > -q$. To see that $W_1, \cdots, W_k$ are linearly independent, it suffices to observe that the first k rows of the $(n+k) \times k$ matrix

$$[W_1, \cdots, W_k] \tag{31}$$

form an upper triangular matrix with $1/c_{-q}$ in each diagonal position; hence, (31) has rank k.

This procedure provides a formal method for obtaining F and $W_1, \cdots, W_k$; however, it is computationally useless for large n if $C(z)$ has zeros in $|z| < 1$. To be specific, let $z_1, \cdots, z_L$ be the distinct zeros of $C(z)$, with respective multiplicities $m_1, \cdots, m_L$. Then

$$\alpha_i = \sum_{l=1}^{L} p_l(i) z_l^{-i},$$

where p_l is a polynomial of degree $m_l - 1$. This means that the sequence $\{\alpha_i\}$ grows very rapidly with increasing i if $|z_l| < 1$ for one or more values of l. Since the recursion (27) has the explicit solution

$$f_i = \sum_{\nu=1}^{i-q} \alpha_{i-\nu-q} y_\nu, \qquad q+1 \leqq i \leqq n+q,$$

with $f_i = 0$ if $-p+1 \leqq i \leqq q$, f_i also becomes large as i increases. Therefore, these recursions can lead to overflow for large n. Moreover, it is well known that the propagation of errors renders the recursion formula (27) useless if $|z_l| < 1$ for some l. (To a lesser extent, the presence of repeated roots on $|z| = 1$ is also a source of instability.)

If $C(z)$ has no zeros *outside* the unit circle, then it makes sense to replace the recursion (27) by

$$f_{n-i} = \frac{1}{c_p}\left[y_{n+p-i} - \sum_{l=-q}^{p-1} c_l f_{n+p-i-l}\right], \qquad p \leqq i \leqq n+p-1,$$

with $f_{n-i} = 0$ if $-q \leqq i \leqq p-1$. This also yields a solution (14) of (15). To obtain a basis for $\mathscr{W}'$ in this case, we consider the Laurent series

$$[z^{-p}C(z)]^{-1} = \sum_{\nu=0}^{\infty} \beta_\nu z^{-\nu},$$

convergent for large z. The $\{\beta_\nu\}$ can be computed recursively

$$\beta_\nu = -\frac{1}{c_p}\sum_{l=-q}^{p-1} c_l \beta_{\nu-p+l}, \qquad \nu > 0, \tag{32}$$

with $\beta_\nu = 0$ if $\nu < 0$ and $\beta_0 = 1/c_p$. The vectors (13) with

$$w_i^{(j)} = \beta_{n-p+j-i}, \quad -p+1 \leqq i \leqq n+q, \quad 1 \leqq j \leqq k, \tag{33}$$

form a basis for $\mathscr{W}'$. To see that they satisfy (12), observe that if (33) holds, then

$$\sum_{l=-q}^{p} c_l w_{i-l}^{(j)} = \sum_{l=-q}^{p} c_l \beta_{n-p+j-i+l}. \tag{34}$$

However, from (32),

$$\sum_{l=-q}^{p} c_l \beta_{\mu+l} = 0, \qquad \mu > -p;$$

therefore, the right side of (34) vanishes if $i \leqq n$ and $j \geqq 1$, since then $n - p + j - i > -p$. To see that $W_1, \cdots, W_k$ are linearly independent, observe that in this case the last k rows of (31) form an upper triangular matrix with $1/c_p$ in each diagnoal position.

Since

$$\beta_i = \sum_{l=1}^{L} q_l(i) z_l^i,$$

where q_l is a polynomial of degree $m_l - 1$, it can be shown that this method of computation is stable if $|z_l| \leqq 1$ and $m_l = 1$ if $|z_l| = 1$ $(1 \leqq l \leqq L)$.

If $C(z)$ has zeros in both the interior and exterior of the unit disc, then the recursive procedures that we have considered so far are both unstable. We will now propose a procedure applicable to this situation. It requires that we know a factorization

$$C(z) = z^{s-q} A(z) B(1/z), \tag{35}$$

where

$$A(z) = \sum_{\mu=0}^{r} a_\mu z^\mu \qquad (a_0 a_r \neq 0),$$

and

$$B(z) = \sum_{\nu=0}^{s} b_\nu z^\nu \qquad (b_0 b_s \neq 0),$$

with $r > 0$, $s > 0$, and $r + s = p + q = k$, such that $A(z)$ has no zeros in $|z| < 1$, $z^s B(1/z)$ has no zeros in $|z| > 1$, and $A(z)$ and $z^s B(1/z)$ have no zeros in common. (This last assumption is clearly superfluous if $C(z) \neq 0$ for $|z| = 1$; however, if $C(z)$ has zeros on $|z| = 1$, it may be convenient to allocate them between $A(z)$ and $z^s B(1/z)$ subject to this restriction. This would be so, for example, if $C(z) = C(1/z)$, so that C_m is symmetric. In this case an appropriate factorization would be $C(z) = A(z)A(1/z)$, where the zeros of $A(z)$ are in $|z| \leqq 1$.)

Since $A(z)$ and $z^s B(1/z)$ are relatively prime by assumption, there are unique polynomials $g(z)$ and $h(z)$ such that deg $g < r$, deg $h < s$, and

$$z^s g(z) B(1/z) + h(z) A(z) = 1; \tag{36}$$

moreover, the coefficients of $g(z)$ and $h(z)$ can be found by solving a $k \times k$ linear system. Now define

$$Y(z) = \sum_{l=1}^{n} y_l z^l$$

and

$$\tilde{Y}(z) = \sum_{l=1}^{n} y_{n-l+1} z^{l-1},$$

and notice that

$$Y(z) = z^n \tilde{Y}(1/z). \tag{37}$$

Consider the expansions

$$\frac{Y(z)}{A(z)} = \sum_{i=0}^{\infty} \xi_i z^{i+1} \tag{38}$$

and

$$\frac{\tilde{Y}(1/z)}{B(1/z)} = \sum_{i=0}^{\infty} \eta_i z^{-i}. \tag{39}$$

Notice that $\{\xi_i\}$ and $\{\eta_i\}$ can be computed recursively, as follows:

$$\xi_i = \frac{1}{a_0}\left[y_{i+1} - \sum_{l=1}^{r} a_l \xi_{i-l}\right], \qquad i \geqq 0, \tag{40}$$

and

$$\eta_i = \frac{1}{b_0}\left[y_{n-i+1} - \sum_{l=1}^{s} b_l \eta_{i-l}\right], \qquad i \geqq 0, \tag{41}$$

where, for convenience, we define $y_i = 0$ if $i \leqq 0$ or $i \geqq n+1$, and $\xi_i = \eta_i = 0$ if $i < 0$.

Because of the assumptions on the zeros of $A(z)$ and $B(1/z)$, the recursions (40) and (41) are stable, or at worst, mildly unstable if $C(z)$ has repeated zeros on $|z| = 1$.

Now define the formal Laurent series

$$F(z) = z^{q+1}g(z)\sum_{i=0}^{\infty}\xi_i z^i + z^{n+q-s}h(z)\sum_{i=0}^{\infty}\eta_i z^{-i} \tag{42}$$
$$= \sum_{l=-\infty}^{\infty} f_l z^l.$$

Then (35), (36), (37), (38), and (39) imply that $C(z)F(z) = Y(z)$. Therefore, (14) with $f_{-p+1}, \cdots, f_{n+q}$ as in (42) satisfies (15).

To obtain a basis $W_1, \cdots, W_k$ for $\mathscr{W}$, we first define

$$\Gamma(z) = z^s g(z)\sum_{\mu=0}^{\infty}\alpha_\mu z^\mu + h(z)\sum_{\mu=0}^{\infty}\beta_\mu z^{-\mu} \tag{43}$$
$$= \sum_{l=-\infty}^{\infty}\gamma_l z^l,$$

where

$$[A(z)]^{-1} = \sum_{\mu=0}^{\infty}\alpha_\mu z^\mu \tag{44}$$

and

$$[B(1/z)]^{-1} = \sum_{\mu=0}^{\infty}\beta_\mu z^{-\mu}. \tag{45}$$

The coefficients $\{\alpha_\mu\}$ and $\{\beta_\mu\}$ can be computed recursively, with $\alpha_\mu = \beta_\mu = 0$ if $\mu < 0$, $\alpha_0 = 1/a_0$, $\beta_0 = 1/b_0$,

$$\alpha_\mu = -\frac{1}{a_0}\sum_{l=1}^{r} a_l\alpha_{\mu-l}, \qquad \mu \geqq 1,$$

and

$$\beta_\mu = -\frac{1}{b_0}\sum_{l=1}^{s} b_l\beta_{\mu-l}, \qquad \mu \geqq 1.$$

It is shown in [7] (see also [6]) that the Toeplitz matrix

$$\Gamma_{n+k} = (\gamma_{i-j})_{i,j=1}^{n+k}$$

is invertible. We will now show that the first r and last s columns of Γ_{n+k} form a basis for $\mathscr{W}$. (This follows from the main result of [7]; however, we include its brief verification here for the reader's convenience.) To see this, let W be the νth column of Γ_{n+k}, i.e.,

$$W = [w_{-p+1}, \cdots, w_{n+q}]^t = [\gamma_{i-\nu}, \cdots, \gamma_{n+k-\nu}]^t,$$

so that $w_i = \gamma_{i+p-\nu}$, $-p+1 \leqq i \leqq n+q$. Then

$$\sum_{l=-q}^{p} c_l w_{i-l} = \sum_{l=-q}^{p} c_l\gamma_{i-l+p-\nu}, \tag{46}$$

which is the coefficient of $z^{i+p-\nu}$ in the formal Laurent expansion of $C(z)\Gamma(z)$. However, (35), (36), (43), (44), and (45) imply that $C(z)\Gamma(z) = z^{s-q}$; therefore, the right side of (46) vanishes for $1 \leqq i \leqq n$ provided that $i + p - \nu \neq s - q$ for $1 \leqq i \leqq n$. This condition holds if $1 \leqq \nu \leqq r$ or $n + r + 1 \leqq \nu \leqq n + k$, which proves our assertion.

4. Toeplitz systems with matrices generated by rational functions. If $\Phi(z) = 1$, then T_n in (3) is the $n \times n$ band matrix

$$T_n = (c_{i-j})_{i,j=1}^n,$$

and $\Phi_m = I_m$. Now Steps 1–4 of § 2 simplify to yield a procedure for solving (2) in which the only simultaneous system to be dealt with is of order k.

Step 1. Obtain $W_1, \cdots, W_k$ recursively, as described in § 3. If (2) is to be solved for more than one Y, store these vectors.

Step 2. Obtain F recursively, as described in § 3.

Step 3. Solve the $k \times k$ system

$$\sum_{j=1}^{k} a_j w_i^{(j)} = f_i, \quad -p+1 \leqq i \leqq 0, \quad n+1 \leqq i \leqq n+q$$

for $a_1, \cdots, a_k$. (If this is impossible, then (2) has no solution.)

Step 4. Compute

$$x_i = f_i - \sum_{j=1}^{k} a_j w_i^{(j)}, \qquad 1 \leqq i \leqq n.$$

The number of operations required for this procedure is $O(kn)$ as n (as compared to $O(k^2 n)$ for methods for solving general $n \times n$ banded systems that do not have the Toeplitz structure). Although there are many "fast" methods for solving banded Toeplitz systems, most of them require recursion with respect to n and are based on the assumption that the principal submatrices of T_n are all invertible. Moreover, to the author's knowledge, the stability of these methods has not been studied, except insofar as Bunch's results [2] on stability of algorithms for general Toeplitz systems apply to them.

In the situation that we have just discussed, the matrices $\{T_n\}$ can be described as being generated by the Laurent polynomial $C(z)$. Now we consider the case where they are generated by the rational functions

$$T(z) = \frac{C(z)}{P(z)Q(1/z)},$$

where $C(z)$ is as in (6),

$$P(z) = \sum_{l=0}^{\mu} p_l z^l$$

and

$$Q(z) = \sum_{l=0}^{\nu} q_l z^l.$$

We assume here that $(\mu + \nu) p_0 q_0 p_\mu q_\nu \neq 0$, and that no two of the polynomials $P(z)$, $Q(1/z)$ and $C(z)$ have a common zero. Here we let $\Phi(z)$ be the formal Laurent expansion of

$$R(z) = [P(z)Q(1/z)]^{-1}$$

obtained as follows:

(a) If $Q = 1$, then

$$R(z) = [P(z)]^{-1} = \sum_{l=0}^{\infty} \phi_l z^l,$$

so that the matrices (5) are lower triangular.

(b) If $P = 1$, then

$$R(z) = [Q(1/z)]^{-1} = \sum_{l=-\infty}^{0} \phi_l z^l,$$

so that the matrices (5) are upper triangular.

(c) If $\mu > 0$ and $\nu > 0$, then $\Phi(z)$ is obtained from $P(z)$ and $Q(z)$ in the same way that $\Gamma(z)$ (cf. (42)) was obtained from $A(z)$ and $B(1/z)$ in § 3. (There is no need to assume here that the zeros of $A(z)$ are confined to $|z| \leqq 1$ while those of $B(1/z)$ are in $|z| \geqq 1$; however, if these conditions hold with strict inequalities, then $\Phi(z)$ is the unique Laurent series which converges to $T(z)$ in an annulus containing $|z| = 1$.)

In this situation, the inverses $\{\Phi_m^{-1}\}$ are banded matrices that are "quasi-Toeplitz" in a sense made explicit in [7], and systems of the form (4) can be solved explicitly with a number of operations that are $O((\mu + \nu)m)$ for large m; moreover, there is no possibility of instability here, since the computation does not involve recursion. Since the formula for Φ_m^{-1} is given in [7], we will not include further detail here. Combining this formula with the recursive methods of § 3 yields the solution of (2) in $O(n)$ (as $n \to \infty$) operations.

In [15] we gave explicit formulas for the solution of (2) when T_n is rationally generated in this way, in terms of Y and determinants involving the values of $P(z)$ and $Q(1/z)$ at the zeros of $C(z)$. Although some discussion of numerical implementation was included in [15], the principal interest there was in the formulas. To the author's knowledge, the only previously published $O(n)$ algorithm specifically designed to solve $n \times n$ systems with rationally generated Toeplitz matrices is due to Dickinson [5]. However, Dickinson's method requires that $T_1, \cdots, T_n$ all be invertible, and he did not consider stability.

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