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ASYMPTOTIC INTEGRATION OF A PERTURBED CONSTANT COEFFICIENT DIFFERENTIAL EQUATION UNDER MILD INTEGRAL SMALLNESS CONDITIONS*

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Abstract. The problem of asymptotic behavior of solutions of an nth order linear differential equation is reconsidered, and a result obtained by Hartman and Wintner under integral smallness conditions on the perturbing terms which require absolute integrability is shown to hold under weaker integrability conditions requiring only ordinary (perhaps conditional) convergence of some of the improper integrals that occur. The estimates of the asymptotic behavior of solutions of the perturbed equation are also sharper than in the classical result.

Key words. linear perturbations, asymptotic behavior, integral smallness conditions

AMS(MOS) subject classifications. 34C10, 34D10, 34E10

1. Introduction. We consider the scalar equation

$$x^{(n)}+a_1x^{(n-1)}+\cdots+a_kx^{(n-k)}+P_1(t)x^{(n-1)}+\cdots+P_n(t)x=f(t), \qquad t>0, \tag{1.1}$$

where $a_1, \cdots, a_k$ are complex constants, with

$$1\leqq k\leqq n-1, \qquad a_k\neq 0. \tag{1.2}$$

It is assumed through that $P_1, \cdots, P_n$, and f are complex-valued and continuous on $(0,\infty)$. We give conditions on them which imply that (1.1) has a solution which behaves asymptotically like a given polynomial of degree $<n-k$.

The following theorem of Hartman and Wintner [1, Thm. 17.3, p. 316] addresses this question. (We use "o" and "O" in the standard way to denote behavior as $t\to\infty$.)

THEOREM 1. *Suppose that the polynomial*

$$Q(\lambda)=\lambda^k+a_1\lambda^{k-1}+\cdots+a_k$$

has no purely imaginary zeros, and that

$$\int^\infty |P_j(t)|t^q\,dt<\infty, \qquad 1\leqq j\leqq k+1, \tag{1.3}$$

and

$$\int^\infty |P_j(t)|t^{j-k-1+q}\,dt<\infty, \qquad k+2\leqq j\leqq n, \tag{1.4}$$

for some $q\geqq 0$. Then, for each $l=0,1,\cdots,n-k-1$, the equation

$$x^{(n)}+a_1x^{(n-1)}+\cdots+a_kx^{(n-k)}+P_1(t)x^{(n-1)}+\cdots+P_n(t)x=0, \qquad t>0, \tag{1.5}$$

has a solution x_l such that

$$\left(x_l(t)-\frac{t^l}{l!}\right)^{(r)}=\begin{cases} o(t^{l-r-q}), & 0\leqq r\leqq n-k-1,\\ o(t^{-n+l+k+1-q}), & n-k\leqq r\leqq n-1.\end{cases}$$

Prevatt [3] has obtained the conclusion of Theorem 1 under weaker integrability conditions on $|P_1|, \cdots, |P_n|$, and Hartman [2] has recently extended Prevatt's results

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to the case where $Q(\lambda)$ may have purely imaginary zeros. Here we retain the assumption that $Q(\lambda)$ has no purely imaginary zeros, and we obtain results which imply the conclusion of Theorem 1 under integral smallness conditions on $P_1, \cdots, P_n$ and f which allow conditional convergence of some (in some cases all) of the improper integrals involved. We also give more precise estimates of the asymptotic behavior of the desired solutions.

The results obtained here are anaolgous to those obtained in [8] for the equation

$$x^{(n)}+P_1(t)x^{(n-1)}+\cdots+P_n(t)x=f(t) \tag{1.6}$$

(see also [7]); however, the condition (1.2) necessitates a distinctly nontrivial extension of the methods used in [7] and [8]. (For example, see Lemma 1 and its proof.)

In work related to the present paper in the sense that the integrability conditions on $P_1, \cdots, P_n$, and f are stated in terms of possibly conditional convergence, Šimša [4]-[6], has studied (1.1) with $k=n$, regarding it as a perturbation of the constant coefficient equation

$$x^{(n)}+a_1x^{(n-1)}+\cdots+a_nx=0.$$

The author [9] has also considered this problem.

2. The main theorem. It is to be understood below that improper integrals appearing in hypotheses are assumed to converge, and that the convergence may be conditional unless, of course, the integrand is necessarily nonnegative.

It is convenient to collect some technical definitions in the following standing assumption, which holds throughout the paper.

Assumption A. Let

$$Q(\lambda)=(\lambda-\lambda_1)^{d_1}\cdots(\lambda-\lambda_L)^{d_L} \tag{2.1}$$

where $\lambda_l=\mu_l+i\nu_l$ are distinct, $\mu_l\neq 0$ $(1\leqq l\leqq L)$, and $\mu_1\leqq\mu_2\leqq\cdots\leqq\mu_L$. Let N be the unique integer in $\{1, \cdots, L+1\}$ such that

$$\mu_l<0 \quad \text{if } 1\leqq l\leqq N-1 \tag{2.2}$$

and $\mu_l>0$ if $N\leqq l\leqq L$. Suppose that p is a given polynomial of degree $<n-k$, and define

$$g=f-\sum_{j=k+1}^{n} P_jp^{(n-j)}. \tag{2.3}$$

Let m be an integer in $\{0, \cdots, n-k-1\}$. Let ϕ be continuous, positive, and nonincreasing on $[a,\infty)$ for some $a>0$. If $m\neq 0$, suppose that $t^\gamma\phi(t)$ is nondecreasing for some $\gamma<1$. If $N\geqq 2$ (so that (2.2) is nonvacuous), let there be a number α such that $0<\alpha<-\mu_{N-1}$ and $e^{\alpha t}t^{-n+m+k+1}\phi(t)$ is nondecreasing on $[a,\infty)$.

The following is our main theorem.

THEOREM 2. *Suppose that Assumption* A *holds, that*

$$\int_t^\infty s^{n-m-k-1}g(s)\,ds=O(\phi(t)), \tag{2.4}$$

and that

$$I_j(t)=\int_t^\infty |P_j(s)|\phi(s)\,ds=o(\phi(t)), \qquad 1\leqq j\leqq k+1. \tag{2.5}$$

Suppose further that

$$F_j(t)=\int_t^\infty P_j(s)\,ds=o(t^{-j+k+1}),\qquad k+2\leqq j\leqq n, \tag{2.6}$$

and that

$$I_j(t)=\int_t^\infty |F_j(s)|s^{j-k-2}\phi(s)\,ds=o(\phi(t)),\qquad k+2\leqq j\leqq n. \tag{2.7}$$

Then (1.1) *has a solution* $\hat{x}$ *such that*

$$\hat{x}^{(r)}(t)-p^{(r)}(t)=\begin{cases}O(\phi(t)t^{m-r}), & 0\leqq r\leqq n-k-1,\\ O(\phi(t)t^{-n+m+k+1}), & n-k\leqq r\leqq n-1.\end{cases} \tag{2.8}$$

Moreover, if (2.4) *can be replaced by*

$$\int_t^\infty s^{n-m-k-1}g(s)\,ds=o(\phi(t)), \tag{2.9}$$

then (2.8) *can be replaced by*

$$\hat{x}^{(r)}(t)-p^{(r)}(t)=\begin{cases}o(\phi(t)t^{m-r}), & 0\leqq r\leqq n-k-1,\\ o(\phi(t)t^{-n+m+k+1}), & n-k\leqq r\leqq n-1.\end{cases} \tag{2.10}$$

The proof uses the Banach contraction principle. It is convenient to introduce the new dependent variable $h=x-p$, which transforms (1.1) into

$$Q(D)h^{(n-k)}=g-Mh, \tag{2.11}$$

with g as in (2.3) and

$$Mh=\sum_{j=1}^{n} P_jh^{(n-j)}. \tag{2.12}$$

We will construct a transformation $\mathcal{T}$ which, for t_0 sufficiently large, is a contraction of the Banach space $B(t_0)$ consisting of functions h in $C^{(n-1)}[t_0,\infty)$ such that

$$h^{(r)}(t)=\begin{cases}O(\phi(t)t^{m-r}), & 0\leqq r\leqq n-k-1,\\ O(\phi(t)t^{-n+m+k+1}), & n-k\leqq r\leqq n-1,\end{cases} \tag{2.13}$$

with norm

$$\|h\|=\sup_{t\geqq t_0}(\phi(t))^{-1}\left[\sum_{r=0}^{n-k-1} t^{r-m}|h^{(r)}(t)|+t^{n-m-k-1}\sum_{r=n-k}^{n-1}|h^{(r)}(t)|\right]. \tag{2.14}$$

For reference below, we define

$$B_0(t_0)=\{h\in B(t_0)|\ (2.13)\text{ holds with “}O\text{” replaced by “}o\text{”}\}. \tag{2.15}$$

If $\hat{h}$ is a solution of (2.11) which is in $B(t_0)$, then the function

$$\hat{x}=p+\hat{h} \tag{2.16}$$

is a solution of (1.1) on $[t_0,\infty)$ (which can be continued over $(0,\infty)$), and $\hat{x}$ has the desired asymptotic behavior (2.8); moreover, if $\hat{h}\in B_0(t_0)$, then $\hat{x}$ satisfies (2.10). Therefore, we wish to construct $\mathcal{T}$ so that if $\mathcal{T}\hat{h}=\hat{h}$, then $\hat{h}$ satisfies (2.11).

To this end, let $A_1, \cdots, A_L$ be the uniquely defined polynomials such that $\deg A_l < d_l$ (see (2.1)) and

$$\sum_{l=1}^{L} [A_l(t)e^{\lambda_l t}]^{(r)}|_{t=0} = \delta_{r,k-1}, \qquad 0 \leqq r \leqq k-1. \tag{2.17}$$

If $v \in C[t_0, \infty)$, define $\mathscr{L}_1 v$ formally by

$$(\mathscr{L}_1 v)(t) = \sum_{l=1}^{N-1} \int_{t_0}^{t} A_l(t-\tau)e^{\lambda_l(t-\tau)}v(\tau)\,d\tau - \sum_{l=N}^{L} \int_{t}^{\infty} A_l(t-\tau)\,e^{\lambda_l(t-\tau)}v(\tau)\,d\tau.$$

Then formal differentiation and (2.17) imply that

$$\begin{aligned}(\mathscr{L}_1 v)^{(r)}(t) = &\sum_{l=1}^{N-1} \int_{t_0}^{t} A_{lr}(t-\tau)\,e^{\lambda_l(t-\tau)}v(\tau)\,d\tau \\ &- \sum_{l=N}^{L} \int_{t}^{\infty} A_{lr}(t-\tau)\,e^{\lambda_l(t-\tau)}v(\tau)\,d\tau, \qquad 0 \leqq r \leqq k-1,\end{aligned} \tag{2.18}$$

and that

$$Q(D)\mathscr{L}_1 v = v, \tag{2.19}$$

where A_{lr} is the polynomial defined by

$$A_{lr}(t) = e^{-\lambda_l t}[A_l(t)\,e^{\lambda_l t}]^{(r)}, \qquad 0 \leqq r \leqq k-1.$$

If $w \in C[t_0, \infty)$, define $\mathscr{L}_2 w$ formally by

$$(\mathscr{L}_2 w)(t) = \int_{t}^{\infty} \frac{(t-s)^{n-k-1}}{(n-k-1)!} w(s)\,ds \quad \text{if } m = 0 \tag{2.20}$$

or by

$$(\mathscr{L}_2 w)(t) = \int_{t_0}^{t} \frac{(t-\lambda)^{m-1}}{(m-1)!}\,d\lambda \int_{\lambda}^{\infty} \frac{(\lambda-s)^{n-k-m-1}}{(n-k-m-1)!} w(s)\,ds \quad \text{if } 1 \leqq m \leqq n-k-1. \tag{2.21}$$

In either case,

$$(\mathscr{L}_2 w)^{(n-k)} = -w. \tag{2.22}$$

Now define

$$G = \mathscr{L}_2(\mathscr{L}_1 g) \quad \text{(see (2.3))}, \tag{2.23}$$

$$\mathscr{L}h = \mathscr{L}_2(\mathscr{L}_1(Mh)) \quad \text{(see (2.12)}, \tag{2.24}$$

and

$$\mathscr{T}h = -G + \mathscr{L}h. \tag{2.25}$$

Formal manipulations using (2.19) and (2.22) show that

$$Q(D)(\mathscr{T}h)^{(n-k)} = g - Mh;$$

therefore, a fixed point (function) $\hat{h}$ of $\mathscr{T}$ satisfies (2.11). We will show that $\mathscr{T}$ is a contraction of $B(t_0)$, and therefore has a fixed point in $B(t_0)$, provided that t_0 is sufficiently large. It is convenient to present the lengthy proof of this assertion in a series of lemmas.

LEMMA 1. *Suppose that v is complex-valued and continuous on $[t_0, \infty)$ with $t_0 \geqq a > 0$, and that $\int^\infty t^q v(t)\,dt$ converges for some nonnegative integer q. Let*

$$\psi(t) = \sup_{\tau \geqq t} \left| \int_\tau^\infty s^q v(s)\,ds \right|. \tag{2.26}$$

Let $\lambda = \mu + i\nu$ be a complex number and X be a polynomial. Then

(i) *If $\mu > 0$, the functions*

$$f_1(t) = \int_t^\infty X(t-\tau)\, e^{\lambda(t-\tau)} v(\tau)\,d\tau \tag{2.27}$$

and

$$f_2(t) = \int_t^\infty s^q\,ds \int_s^\infty X(s-\tau)\, e^{\lambda(s-\tau)} v(\tau)\,d\tau = \int_t^\infty s^q f_1(s)\,ds \tag{2.28}$$

are defined on $[t_0, \infty)$ and satisfy the inequalities

$$|f_1(t)| \leqq K_1 t^{-q} \psi(t), \qquad t \geqq t_0, \tag{2.29}$$

$$|f_2(t)| \leqq K_2 \psi(t), \qquad t \geqq t_0, \tag{2.30}$$

where K_1 and K_2 are constants which depend only on λ and X.

(ii) *If $\mu < 0$, suppose that $\psi(t) = 0(\phi(t))$, where ϕ is nonincreasing and continuous on $[a, \infty)$, and $e^{\alpha t} t^{-q} \phi(t)$ in nondecreasing on $[a, \infty)$ for some α such that*

$$0 < \alpha < -\mu. \tag{2.31}$$

Let

$$\psi_1(t) = \sup_{\tau \geqq t} \frac{\psi(t)}{\phi(t)}, \tag{2.32}$$

and define

$$f_3(t) = \int_{t_0}^t X(t-\tau)\, e^{\lambda(t-\tau)} v(\tau)\,d\tau. \tag{2.33}$$

Then

$$|f_3(t)| \leqq K_3 \psi_1(t_0) t^{-q} \phi(t), \qquad t \geqq t_0, \tag{2.34}$$

where K_3 is a constant depending only on X and λ, and the function

$$f_4(t) = \int_t^\infty s^q\,ds \int_{t_0}^s X(s-\tau)\, e^{\lambda(s-\tau)} v(\tau)\,d\tau = \int_t^\infty s^q f_3(s)\,ds \tag{2.35}$$

is defined on $[t_0, \infty)$, and it satisfies the inequality

$$|f_4(t)| \leqq K_4 \psi_1(t_0) \phi(t), \qquad t \geqq t_0, \tag{2.36}$$

where K_4 is a constant depending only on X and λ.

Finally, if

$$\psi(t) = o(\phi(t)), \tag{2.37}$$

then

$$f_3(t) = o(t^{-q} \phi(t)) \tag{2.38}$$

and

$$f_4(t) = o(\phi(t)). \tag{2.39}$$

The lengthy proof of this lemma is given in § 4.

(Note that because of (2.29) and (2.30), (2.37) implies that $f_1(t) = o(t^{-q}\phi(t))$ and $f_2(t) = o(\phi(t))$.)

LEMMA 2. *Suppose that ϕ and m are as in Assumption* A *and $w \in C[t_0, \infty)$ for some $t_0 \geqq a$. Suppose also that $\int^\infty t^{n-m-k-1}w(t)\,dt$ converges, and that*

$$\int_t^\infty s^{n-m-k-1}w(s)\,ds = O(\phi(t)). \tag{2.40}$$

Define

$$\rho(t) = \sup_{\tau \geqq t} (\phi(\tau))^{-1} \left| \int_\tau^\infty s^{n-k-m-1}w(s)\,ds \right|. \tag{2.41}$$

Then $\mathscr{L}_2 w \in C^{(n-k)}[t_0, \infty)$ (see (2.20) *and* (2.21)), *and there is a constant K which does not depend on w or t_0 such that*

$$|(\mathscr{L}_2 w)^{(r)}(t)| \leqq K\rho(t_0)\phi(t)t^{m-r}, \qquad t \geqq t_0, \quad 0 \leqq r \leqq n-k-1. \tag{2.42}$$

Moreover, if

$$\lim_{t\to\infty} \rho(t) = 0, \tag{2.43}$$

then

$$(\mathscr{L}_2 w)^{(r)}(t) = o(\phi(t)t^{m-r}), \qquad 0 \leqq r \leqq n-k-1. \tag{2.44}$$

This lemma follows immediately from Lemma 1 of [8]. Since its conclusion follows trivially from the assumption that

$$\int^\infty s^{n-k-m-1}|w(s)|\,ds < \infty,$$

it is important to emphasize here that the covergence in (2.40) may be conditional. (Note: The existence of the constant γ in Assumption A is required for this lemma.)

LEMMA 3. *Suppose that v is complex-valued and continuous on $[t_0, \infty)$ with $t_0 \geqq a$, and that $\int^\infty t^{n-m-k-1}v(t)\,dt$ converges. Let*

$$\psi(t) = \sup_{\tau \geqq t} \left| \int_\tau^\infty s^{n-m-k-1}v(s)\,ds \right| = O(\phi(t)), \tag{2.45}$$

and define ψ_1 as in (2.32). *Then the function*

$$u = \mathscr{L}_2(\mathscr{L}_1 v) \tag{2.46}$$

is in $B(t_0)$, and

$$\|u\| \leqq W\psi_1(t_0), \tag{2.47}$$

where W is a constant independent of t_0 and v. Moreover, if

$$\psi(t) = o(\phi(t)), \tag{2.48}$$

then

$$u \in B_0(t_0) \quad (\textit{see } (2.15)). \tag{2.49}$$

Proof. From (2.18), $\mathscr{L}_1 v$ and its first $k-1$ derivatives are linear combinations of integrals of the forms (2.27) and (2.33) with $X=A_{lr}$: hence, Lemma 1 with $q=n-m-k-1$ (specifically, (2.29), (2.32) and (2.34)) implies that $\mathscr{L}_1 v\in C^{(k-1)}[t_0,\infty)$, and that

$$|(\mathscr{L}_1 v)^{(j)}(t)|\leqq \alpha_1\psi_1(t_0)\phi(t)t^{-n+m+k+1},\qquad 0\leqq j\leqq k-1, \tag{2.50}$$

where α_1 is a constant independent of t_0 and v. Lemma 1 also implies that $\int^\infty t^{n-m-k-1}(\mathscr{L}_1 v)(t)\,dt$ converges (since it is a linear combination of integrals of the forms (2.28) and (2.35), again with $q=n-m-k-1$ and $X=A_{lr}$), and that

$$\left|\int_t^\infty s^{n-m-k-1}(\mathscr{L}_1 v)(s)\,ds\right|\leqq \alpha_2\psi_1(t_0)\phi(t),\qquad t\geqq t_0, \tag{2.51}$$

where α_2 is a constant independent of t_0 and v. (See (2.30), (2.32), and (2.36).)

Now we apply Lemma 2 with $w=\mathscr{L}_1 v$. Then (2.51) implies (2.40) and (2.41), with $\rho(t)\leqq \alpha_2\psi_1(t_0)$. Recalling (2.46), we now see from (2.42) with $w=\mathscr{L}_1 v$ that

$$|u^{(r)}(t)|\leqq K\alpha_2\psi_1(t_0)\phi(t)t^{m-r},\qquad 0\leqq r\leqq n-k-1. \tag{2.52}$$

Moreover, since

$$u^{(n-k+j)}=-(\mathscr{L}_1 v)^{(j)},\qquad 0\leqq j\leqq k-1 \tag{2.53}$$

(from (2.22) with $w=\mathscr{L}_1 v$), (2.50) implies that

$$|u^{(r)}(t)|\leqq \alpha_1\psi_1(t_0)\phi(t)t^{-n+m+k+1},\qquad n-k\leqq r\leqq n-1. \tag{2.54}$$

Now (2.14), (2.52) and (2.54) imply (2.47), with $W=\max\{\alpha_1, K\alpha_2\}$.

It remains only to show that (2.48) implies (2.49). From the closing paragraph of Lemma 1, (2.48) implies that

$$(\mathscr{L}_1 v)^{(j)}(t)=o(\phi(t)t^{-n+m+k+1}),\qquad 0\leqq j\leqq k-1,$$

and therefore

$$u^{(r)}(t)=o(\phi(t)t^{-n+m+k+1}),\qquad n-k\leqq r\leqq n-1, \tag{2.55}$$

because of (2.53). The closing paragraph of Lemma 1 also implies that

$$\int_t^\infty s^{n-m-k-1}(\mathscr{L}_1 v)(s)\,ds=o(\phi(t)),$$

because of (2.48). Therefore, (2.43) holds if $w=\mathscr{L}_1 v$ in (2.41), and so

$$u^{(r)}(t)=o(\phi(t)t^{m-r}),\qquad 0\leqq r\leqq n-k-1, \tag{2.56}$$

from (2.44). Since (2.55) and (2.56) are equivalent to (2.49), this completes the proof of Lemma 3.

LEMMA 4. *Suppose that the assumptions of Theorem 2 hold, and let $\mathscr{L}$ be as defined in (2.24). Suppose also that $h\in B(t_0)$ for some $t_0\geqq a$. Then $\mathscr{L}h\in B_0(t_0)$ and*

$$\|\mathscr{L}h\|\leqq W\sigma(t_0)\|h\|, \tag{2.57}$$

where σ is defined on $[a,\infty)$,

$$\lim_{t\to\infty}\sigma(t)=0, \tag{2.58}$$

and W is as in (2.47).

Proof. We first consider the integral

$$J(t;h)=\int_t^\infty s^{n-m-k-1}(Mh)(s)\,ds = \sum_{j=1}^{n}\int_t^\infty s^{n-m-k-1}P_j(s)h^{(n-j)}(s)\,ds. \tag{2.59}$$

We will show that the integrals in this sum converge, and estimate them. We remind the reader that $\|h\|$ is defined in (2.14).

From (2.5),

$$\left|\int_t^\infty s^{n-m-k-1}P_j(s)h^{(n-j)}(s)\,ds\right| \leqq \|h\| I_j(t), \qquad 1\leqq j\leqq k+1. \tag{2.60}$$

If $k+2\leqq j\leqq n$, then integration by parts yields

$$\int_t^\infty s^{n-m-k-1}P_j(s)h^{(n-j)}(s)\,ds = t^{n-m-k-1}h^{(n-j)}(t)F_j(t)+\int_t^\infty F_j(s)[s^{n-m-k-1}h^{(n-j)}(s)]'\,ds \quad \text{(see (2.6))}. \tag{2.61}$$

To justify this we first observe that

$$\lim_{T\to\infty} T^{n-m-k-1}h^{(n-j)}(T)F_j(T)=0, \qquad k+2\leqq j\leqq n,$$

because of (2.6) and (2.13). Moreover, since (2.13) and (2.14) imply that

$$|[s^{n-m-k-1}h^{(n-j)}(s)]'|\leqq (n-m-k)\|h\|\phi(s)s^{j-k-2}, \qquad k+2\leqq j\leqq n, \tag{2.62}$$

(2.7) implies that the integral on the right side of (2.61) converges (absolutely). We now conclude that $J(t;h)$ exists on $[t_0,\infty)$ if $h\in B(t_0)$; moreover, (2.59), (2.60), (2.61) and (2.62) imply that $|J(t;h)|\leqq \|h\|\Gamma(t)$, where

$$\Gamma(t)=\sum_{j=1}^{k+1} I_j(t)+(n-m-k)\sum_{j=k+1}^{n} I_j(t)+\phi(t)\sum_{j=k+2}^{n} t^{j-k-1}|F_j(t)|.$$

(See (2.6) and (2.7).) Now define $\sigma(t)=\sup_{\tau\geqq t}\Gamma(\tau)/\phi(\tau)$, and note that σ satisfies (2.58), because of (2.5), (2.6) and (2.7).

We now apply Lemma 3 with $v=Mh$; then the function ψ defined in (2.45) satisfies the inequality

$$\psi(t)\leqq \|h\|\Gamma(t)=o(\phi(t)).$$

Since $u=\mathscr{L}h$ in (2.46) when $v=Mh$ (see (2.24)), we conclude that $\mathscr{L}h\in B_0(t_0)$, and (2.47) with $\psi_1=\sigma\|h\|$ implies (2.57). This proves Lemma 4.

We can now complete the proof of Theorem 2. From (2.4) and Lemma 3 (with $v=g$), G as defined in (2.23) is also in $B(t_0)$. Now (2.25) and Lemma 4 imply that $\mathscr{T}(B(t_0))\subset B(t_0)$; moreover, $\mathscr{T}$ is obviously a contraction if $\mathscr{L}$ is, and the latter is so if

$$\sigma(t_0)<1/W \quad \text{(see (2.57))}. \tag{2.63}$$

From (2.58), we can choose t_0 to satisfy (2.63). By the contraction mapping principle, there is an $\hat{h}$ in $B(t_0)$ such that $\mathscr{T}\hat{h}=\hat{h}$; hence, $\hat{x}$ as defined in (2.16) satisfies (1.1) and (2.8).

Now suppose that (2.9) holds. Then Lemma 3 implies that $G \in B_0(t_0)$. Since $\mathscr{L}\hat{h} \in B_0(t_0)$ (by Lemma 4) and

$$\hat{h} = -G + \mathscr{L}\hat{h} \quad (\text{see } (2.25)),$$

it now follows that $\hat{h} \in B_0(t_0)$. This and (2.16) imply (2.10), which completes the proof of Theorem 2.

3. Corollaries. If $P \in C[a, \infty)$ and $\int^\infty |P(t)|\,dt < \infty$, then obviously

$$\int_t^\infty |P(s)|\phi(s)\,ds = o(\phi(t))$$

if ϕ is nonincreasing. Also, if $\int^\infty t^\alpha |P(t)|\,dt < \infty$ for some $\alpha > 0$, then

$$\int_t^\infty P(s)\,ds = o(t^{-\alpha}) \quad \text{and} \quad \int^\infty t^{\alpha-1}\left|\int_t^\infty P(\tau)\,d\tau\right| dt < \infty$$

(see Corollary 3 of [8]), which in turn implies that

$$\int_t^\infty s^{\alpha-1}\left|\int_s^\infty P(\tau)\,d\tau\right| \phi(s)\,ds = o(\phi(t))$$

if ϕ is nonincreasing. Since the converses of these statements are false, the integrability conditions (1.3) and (1.4)—even with $q=0$—are stronger than (2.5), (2.6) and (2.7).

We remind the reader that Assumption A is still in force.

COROLLARY 1. *Let l be an integer in $\{0, 1, \cdots, n-k-1\}$, and suppose that*

$$\int_t^\infty s^{j-k-1}P_j(s)\,ds = O(t^{-q}), \qquad n-l \leqq j \leqq n, \tag{3.1}$$

for some $q \geqq 0$ such that

$$q \neq 0, 1, \cdots, l. \tag{3.2}$$

Let

$$\int_t^\infty P_j(s)\,ds = o(t^{-j+k+1}), \qquad k+2 \leqq j \leqq n-l-1, \tag{3.3}$$

and define

$$\beta = \begin{cases} q-[q] & \text{if } l \geqq [q] \ (= \text{integer part of } q), \\ q-l & \text{if } l < [q]. \end{cases} \tag{3.4}$$

Finally, suppose that

$$\int_t^\infty |P_j(s)| s^{-\beta}\,ds = o(t^{-\beta}), \qquad 1 \leqq j \leqq k+1. \tag{3.5}$$

Then (1.5) *has a solution x_l such that*

$$\left(x_l(t) - \frac{t^l}{l!}\right)^{(r)} = \begin{cases} O(t^{l-r-q}), & 0 \leqq r \leqq n-k-1, \\ O(t^{l-n+k+1-q}), & n-k \leqq r \leqq n-1. \end{cases} \tag{3.6}$$

Moreover, if (3.1) *can be replaced by*

$$\int_t^\infty P_j(s)\,ds = o(t^{-j+k+1-q}), \qquad n-l \leqq j \leqq n, \tag{3.7}$$

then (3.6) *can be replaced by*

$$\left(x_l(t)-\frac{t^l}{l!}\right)^{(r)}=\begin{cases} o(t^{l-r-q}), & 0\leqq r\leqq n-k-1,\\ o(t^{l-n+k+1-q}), & n-k\leqq r\leqq n-1.\end{cases} \tag{3.8}$$

Proof. We start by observing that if $0<a<b$ and

$$\int_t^\infty P(s)\,ds=O(t^{-b}), \tag{3.9}$$

then

$$\int_t^\infty s^a P(s)\,ds=O(t^{a-b}). \tag{3.10}$$

If $f=0$ (as in (1.5)) and $p(t)=t^l/l!$, then

$$g(t)=-\sum_{j=n-l}^{n} P_j(t)\frac{t^{l-n+j}}{(l-n+j)!} \tag{3.11}$$

(see (2.3)); hence, (3.1) implies that

$$\int_t^\infty s^{n-m-k-1}g(s)\,ds=O(t^{l-m-q})$$

if $l-m<q$. We now apply Theorem 2 with

$$m=\max\{0,\,l-[q]\} \tag{3.12}$$

and

$$\phi(t)=O(t^{l-m-q})=O(t^{-\beta}), \tag{3.13}$$

with β as in (3.4). (Note that if $m>0$, then $\beta=q-[q]<1$, and ϕ as in (3.13) satisfies Assumption A with $\gamma=\beta$.)

Now (3.1), (3.2) and (3.3) imply (2.6), and (3.5) is the same as (2.5) with $\phi(t)=t^{-\beta}$. Since (3.2) and (3.4) imply that $\beta>0$, (2.6) automatically implies (2.7) with ϕ as in (3.13), without any absolute integrability assumptions on $P_{k+2},\cdots,P_n$. Now Theorem 2 implies that (1.5) has a solution x_l such that

$$\left(x_l(t)-\frac{t^l}{l!}\right)^{(r)}=\begin{cases} O(t^{m-r-\beta}), & 0\leqq r\leqq n-k-1,\\ O(t^{m-n+k+1-\beta}), & n-k\leqq r\leqq n-1,\end{cases}$$

which, in view of (3.4) and (3.12), is equivalent to (3.6).

If (3.9) holds with "O" replaced by "o", then so does (3.10). From Theorem 2, this also implies the assertion concerning (3.7) and (3.8).

We now consider the exceptional cases where (3.2) is not satisfied.

COROLLARY 2. *Let q and l be integers, with $0\leqq q\leqq l\leqq n-k-1$. Suppose that the integrals*

$$\int^\infty t^{j-k-1+q}P_j(t)\,dt,\qquad n-l\leqq j\leqq n, \tag{3.14}$$

converge, that (3.3) *holds, that*

$$\int^\infty |P_j(t)|\,dt<\infty,\qquad 1\leqq j\leqq k+1, \tag{3.15}$$

and that

$$\int^\infty t^{j-k-2}|F_j(t)|\,dt<\infty \tag{3.16}$$

for $k+2 \leqq j \leqq n-l-1$. *Then* (1.5) *has a solution* x_l *which satisfies* (3.8) *if* $q>0$. *If* $q=0$, *the same conclusion is valid if the above assumptions hold and, in addition,* (3.16) *also holds for* $n-l \leqq j \leqq n$.

Proof. Now (3.14) implies that $\int^\infty t^{n-m-k-1} g(t)\,dt$ converges (see (3.11)) with $m=l-q$, and that (3.7) holds. We apply Theorem 2 with $\phi=1$, in which case (2.5) is equivalent to (3.15). Also, (2.7) is then equivalent to (3.16) for $k+2 \leqq j \leqq n$. Since the convergence of (3.14) implies (3.7), which automatically implies (3.16) for $n-l \leqq j \leqq n$, the proof is complete.

Corollaries 1 and 2 imply the following corollary, which in turn implies Theorem 1.

COROLLARY 3. *Suppose that the integrals*

$$\int^\infty t^{j-k-1+q} P_j(t)\,dt, \qquad k+1 \leqq j \leqq n, \tag{3.17}$$

converge for some $q \geqq 0$. *Suppose also that*

$$\int_t^\infty |P_j(s)| s^{[q]-q}\,ds = o(t^{[q]-q}), \qquad 1 \leqq j \leqq k+1. \tag{3.18}$$

Then the conclusions of Theorem 1 *hold if either* (i) $q>0$; *or* (ii) $q=0$ *and*

$$\int^\infty s^{j-k-2} \left| \int_s^\infty P_j(\tau)\,d\tau \right| ds < \infty, \qquad k+2 \leqq j \leqq n. \tag{3.19}$$

It is important to note that (3.17) implies (3.19) if $q>0$; therefore, it is not necessary to assume (3.19) in this case.

Corollary 1 implies that if

$$\int_t^\infty s^{j-k-1} P_j(s)\,ds = O(\phi(t)), \qquad k+1 \leqq j \leqq n, \tag{3.20}$$

where $\phi(t) \to 0$ as $t \to \infty$ like some positive power of $1/t$, then (1.5) has solutions $x_l(t) \sim t^l/l!$ $(0 \leqq l \leqq n-k-1)$, without any further integrability assumptions on $P_{k+1}, \cdots, P_n$, provided that $P_1, \cdots, P_k$ satisfy (2.5). The following corollary shows that this conclusion remains valid even if ϕ decays more slowly than this.

COROLLARY 4. *Suppose that* ϕ *is as in Assumption* A, *and also that* $t^\gamma \phi(t)$ *is eventually nondecreasing for* $\gamma<1$, *and*

$$\int_t^\infty \frac{\phi^2(s)}{s}\,ds = o(\phi(t)). \tag{3.21}$$

Suppose also that (2.5) *and* (3.20) *hold. Then* (1.5) *has solutions* $x_0, \cdots, x_{n-k-1}$ *such that*

$$\left(x_l(t) - \frac{t^l}{l!}\right)^{(r)} = \begin{cases} O(\phi(t) t^{l-r}), & 0 \leqq r \leqq n-k-1, \\ O(\phi(t) t^{-n+l+k+1}), & n-k \leqq r \leqq n-1. \end{cases} \tag{3.22}$$

Moreover, if (3.20) *holds with "O" replaced by "o," then so does* (3.22).

Proof. For each $l=0,1,\cdots,n-k-1$, we apply Theorem 2 with $p(t)=t^l/l!$ and $m=l$. The assumption (3.20) implies (2.4) and also that

$$F_j(t) = \int_t^\infty P_j(s)\,ds = O(\phi(t) t^{-j+k+1}), \qquad k+2 \leqq j \leqq n.$$

This implies (2.6) and (2.7) (the latter because of (3.21)), which completes the proof.

A similar argument yields the following related corollary.

COROLLARY 5. *Let ϕ be as in Corollary 4, except that (3.21) is replaced by*

$$\int_t^\infty \frac{\phi^2(s)}{s}\,ds = O(\phi(t)).$$

Suppose also that (2.5) holds, and that

$$\int_t^\infty s^{j-k-1}P_j(s)\,ds = o(\phi(t)), \qquad k+1 \leqq j \leqq n.$$

Then (1.5) has solutions $x_0, \cdots, x_{n-k-1}$ which satisfy (3.22) with "O" replaced by "o."

Remark 1. Although we have assumed (1.2) throughout, our results are also valid for (1.6), which corresponds to the case where $k=0$, provided that obviously vacuous conditions (i.e., those involving $0 \leqq j \leqq k-1$ and $n-k \leqq r \leqq n-1$) are ignored. To see this, one has only to modify (in fact, simplify) the arguments as follows:

(a) Omit the now vacuous assumptions on the zeros of $Q(\lambda)$.

(b) Let $\mathscr{L}_1$ be the identity operator; i.e., $\mathscr{L}_1 v = v$.

(c) Omit Lemma 1.

Viewed in this way, the present results improve on those in [7] and [8], since we assumed in those papers that $\int^\infty |P_1(t)|\,dt < \infty$, which is stronger that (2.5), (3.5) and (3.18) with $k=0$ and $j=1$.

Remark 2. While preparing this paper the author discovered errors in [7] and [8], caused by his overlooking the need for special treatment of the exceptional cases where (3.2) does not hold. Theorem 2 of [7] requires the additional assumption that

$$\int^\infty \left| \int_t^\infty p_l(\tau)\,d\tau \right| t^{l-2}\,dt < \infty, \qquad 2 \leqq l \leqq n-r-1$$

(the notation here is that of [7]). Our present Corollary 2 (with $k=0$) extends this corrected result. Example 1 of [8] requires the additional assumption that $\alpha \neq 1, \cdots, n-1$. (The notation here is that of [8].) Corollary 1 (with $k=0$) extends this corrected result, and Corollary 2 deals with the excluded cases where $\alpha = 1, \cdots, n-1$.

4. Appendix. Proof of Lemma 1. We assume throughout that $t \geqq t_0 \geqq a$. If

$$V_r(t) = \int_t^\infty s^r v(s)\,ds, \qquad 0 \leqq r \leqq q, \tag{4.1}$$

then (2.26) implies that

$$|V_q(t)| \leqq \psi(t), \tag{4.2}$$

and by writing $s^r v(s) = s^{r-q}(s^q v(s))$, integrating by parts, and again invoking (2.26), we find that

$$|V_r(t)| \leqq 2t^{r-q}\psi(t), \qquad 0 \leqq r \leqq q-1. \tag{4.3}$$

Proof of (i). Now suppose that $\mu > 0$. If $t_0 \leqq s \leqq T$, then integration by parts shows that

$$\int_T^\infty X(s-\tau)\,e^{\lambda(s-\tau)}v(\tau)\,d\tau = X(s-T)\,e^{\lambda(s-T)}V_0(T) + \int_T^\infty V_0(\tau)[X(s-\tau)\,e^{\lambda(s-\tau)}]'\,d\tau.$$

Hence, (4.3) implies that

$$\left| \int_T^\infty X(s-\tau)\,e^{\lambda(s-\tau)}v(\tau)\,d\tau \right| \leqq T^{-q}\psi(T)\hat{X}(T-s)\,e^{\mu(s-T)}, \qquad t_0 \leqq s \leqq T, \tag{4.4}$$

where $\hat{X}$ is a polynomial with nonnegative coefficients determined by X and λ. Setting $s=T=t$ in (4.4) yields (2.29), with $K_1=\hat{X}(0)$.

To prove (2.30), we consider the integral

$$I(t,T)=\int_t^T s^q f_1(s)\,ds=I_1(t,T)+I_2(t,T), \tag{4.5}$$

where

$$I_1(t,T)=\int_t^T s^q\,ds\int_s^T X(s-\tau)\,e^{\lambda(s-\tau)}v(\tau)\,d\tau \tag{4.6}$$

and

$$I_2(t,T)=\int_t^T s^q\,ds\int_T^\infty X(s-\tau)\,e^{\lambda(s-\tau)}v(\tau)\,d\tau. \tag{4.7}$$

From (4.4) and (4.7),

$$\begin{aligned}|I_2(t,T)|&\leqq \psi(T)\int_t^T \hat{X}(T-s)\,e^{\mu(s-T)}\,ds\\ &<\psi(T)\int_0^\infty \hat{X}(\eta)\,e^{-\mu\eta}\,d\eta;\end{aligned}$$

hence,

$$\lim_{T\to\infty} I_2(t,T)=0. \tag{4.8}$$

Changing the order of integration in (4.6) yields

$$I_1(t,T)=\int_t^T \Gamma(t,\tau)v(\tau)\,d\tau, \tag{4.9}$$

where

$$\Gamma(t,\tau)=\int_t^\tau s^q X(s-\tau)\,e^{\lambda(s-\tau)}\,ds. \tag{4.10}$$

Repeated integration by parts shows that

$$\Gamma(t,\tau)=\sum_{r=0}^q\,[X_r(0)\tau^r-t^rX_r(t-\tau)\,e^{\lambda(t-\tau)}], \tag{4.11}$$

where $X_0,\cdots,X_q$ are polynomials determined by X and λ. Substituting (4.11) into (4.9), we obtain $I(t,T)$ in terms of integrals which converge as $T\to\infty$; therefore, (4.5), (4.8) and (4.11) imply that the integral $f_2(t)=I(t,\infty)$ (see (2.28)) converges, and that

$$f_2(t)=\sum_{r=0}^q X_r(0)\,V_r(t)-\sum_{r=0}^q t^r\int_t^\infty X_r(t-\tau)\,e^{\lambda(t-\tau)}v(\tau)\,d\tau \tag{4.12}$$

(see (4.1)). Replacing X by X_r in (4.4) and letting $s=T=t$ shows that

$$\left|\int_t^\infty X_r(t-\tau)\,e^{\lambda(t-\tau)}v(\tau)\,d\tau\right|\leqq K_{1r}t^{-q}\psi(t),$$

where K_{1r} is a constant determined by λ and X_r (and therefore by X). This, (4.2), (4.3) and (4.12) imply (2.30) for suitable K_2, a constant determined by X and λ. This proves (i).

Proof of (ii). Now suppose that $\mu<0$. Integrating (2.33) by parts yields

$$f_3(t)=V_0(t_0)X(t-t_0)\,e^{\lambda(t-t_0)}-X(0)\,V_0(t)+\int_{t_0}^{t}V_0(\tau)[X(t-\tau)\,e^{\lambda(t-\tau)}]'\,d\tau.$$

Therefore, from (4.3) with $r=0$,

$$\text{(4.13)}\qquad \begin{aligned}|f_3(t)|&\leqq 2t_0^{-q}\psi(t_0)|X(t-t_0)|\,e^{\mu(t-t_0)}+2|X(0)|\psi(t)t^{-q}\\&\quad+2\int_{t_0}^{t}\psi(\tau)\tau^{-q}\tilde{X}(t-\tau)\,e^{\mu(t-\tau)}\,d\tau\end{aligned}$$

where $\tilde{X}$ is a polynomial with positive coefficients determined by X and λ. Now (2.32) and our assumptions on α and ϕ imply that

$$\psi(\tau)\tau^{-q}\leqq\psi_1(t_0)\,e^{\alpha(t-\tau)}\phi(t)t^{-q},\qquad t\geqq\tau\geqq t_0.$$

This and (4.13) imply (2.34), with

$$K_3=2\left[|X(0)|+\sup_{\xi\geqq 0}|X(\xi)|\,e^{(\alpha+\mu)\xi}+\int_0^{\infty}\tilde{X}(\eta)\,e^{(\alpha+\mu)\eta}\,d\eta\right],$$

which is finite, because of (2.31).

To see that (2.37) implies (2.38), observe from (2.33) that if $t_0\leqq t_1\leqq t$, then

$$\text{(4.14)}\qquad f_3(t)=\int_{t_0}^{t_1}X(t-\tau)\,e^{\lambda(t-\tau)}v(\tau)\,d\tau+\int_{t_1}^{t}X(t-\tau)\,e^{\lambda(t-\tau)}v(\tau)\,d\tau.$$

For fixed t_1,

$$\text{(4.15)}\qquad \left|\int_{t_0}^{t_1}X(t-\tau)\,e^{\lambda(t-\tau)}v(\tau)\,d\tau\right|\leqq M(t_0,t_1)\,Y(t)\,e^{\mu t},$$

where

$$M(t_0,t_1)=\max\{|v(\tau)|\,|\,t_0\leqq\tau\leqq t_1\}$$

and Y is a polynomial, while

$$\text{(4.16)}\qquad \left|\int_{t_1}^{t}X(t-\tau)\,e^{\lambda(t-\tau)}v(\tau)\,d\tau\right|\leqq K_3\psi_1(t_1)t^{-q}\phi(t),\qquad t\geqq t_1.$$

(Since this integral has the same form as $f_3(t)$ in (2.33), with t_0 replaced by t_1, (4.16) is obtained as was (2.34).)

Our assumptions on ϕ and α imply that

$$Y(t)\,e^{\mu t}=o(t^{-q}\phi(t)).$$

Therefore, (4.14), (4.15) and (4.16) imply that

$$\overline{\lim_{t\to\infty}}\,(t^{-q}\phi(t))^{-1}|f_3(t)|\leqq K_3\psi_1(t_1),\qquad t_1\geqq t_0.$$

Since (2.32) and (2.37) imply that $\lim_{t_1\to\infty}\psi_1(t_1)=0$, we now see that (2.37) implies (2.38).

For reference below, we observe here that

$$\text{(4.17)}\qquad \lim_{t\to\infty}t^q f_3(t)=0$$

in any case. This is obvious from (2.34) if $\phi(t)=o(1)$. If $\lim_{t\to\infty}\phi(t)>0$, then (2.26) implies (2.37), which implies (2.38) and, therefore, (4.17).

To prove (2.36), consider the integral

$$J(t, T)=\int_t^T s^q f_3(s)\,ds=\int_t^T s^q\,ds\int_{t_0}^s X(s-\tau)\,e^{\lambda(s-\tau)}v(\tau)\,d\tau.$$

Reversing the order of integration yields

$$J(t, T)=\int_{t_0}^t v(\tau)\,d\tau\int_t^T s^qX(s-\tau)\,e^{\lambda(s-\tau)}\,ds+\int_t^T v(\tau)\,d\tau\int_\tau^T s^qX(s-\tau)\,e^{\lambda(s-\tau)}\,ds.$$

Manipulating the limits of integration here shows that

$$J(t, T)=H(t)-H(T).$$

where

$$H(t)=\int_{t_0}^t \Gamma(t,\tau)v(\tau)\,d\tau \quad (\text{see } (4.10)).$$

Therefore, from (4.11),

$$H(t)=\sum_{r=0}^q X_r(0)\int_{t_0}^t \tau^r v(\tau)\,d\tau-\sum_{r=0}^q t^r\int_{t_0}^t X_r(t-\tau)\,e^{\lambda(t-\tau)}v(\tau)\,d\tau. \tag{4.18}$$

The integrals on the right converge as $t\to\infty$. Moreover, since the integrals in the second sum are of the same form as $f_3(t)$ (see (2.33)), (4.17) implies that this sum approaches zero as $t\to\infty$; hence,

$$H(\infty)=\lim_{T\to\infty}H(T)=\sum_{r=0}^q X_r(0)\int_{t_0}^\infty \tau^r v(\tau)\,d\tau. \tag{4.19}$$

Since $f_4(t)=H(\infty)-H(t)$, we now see from (4.1), (4.18) and (4.19) that

$$f_4(t)=\sum_{r=0}^q X_r(0)\,V_r(t)+\sum_{r=0}^q t^r\int_{t_0}^t X_r(t-\tau)\,e^{\lambda(t-\tau)}v(\tau)\,d\tau. \tag{4.20}$$

Recalling (2.32), (4.2), (4.3) and, again, that the integrals in the second sum here are of the same form as $f_3(t)$, and therefore satisfy an inequality like (2.34), we see from (4.20) that (2.36) holds for a suitable constant K_4 which is ultimately determined by X and λ.

Finally, suppose that (2.37) holds. Then obviously $V_r(t)=o(\phi(t))$, from (4.2) and (4.3). Moreover,

$$\int_{t_0}^t X_r(t-\tau)\,e^{\lambda(t-\tau)}v(\tau)\,d\tau=o(t^{-q}\phi(t)),$$

as can be seen by replacing X with X_r in (2.33) and applying the argument that led to (2.38). Now (4.20) implies (2.39), which completes the proof of Lemma 1.

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