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Efficient Application of the Schauder-Tychonoff Theorem to Functional Perturbations of $x^{(n)}=0$

By

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1. Introduction

We consider the scalar functional equation

$$y^{(n)} = F(t; y) \quad (n \geq 2) \tag{1.1}$$

as a perturbation of $x^{(n)}=0$. Our main theorem requires no specific assumptions on the form of the functional F, except that for each t in some interval $[t_0, \infty)$ and y in a suitably restricted family of $n-1$ times differentiable functions, $F(t; y)$ is determined by the values of $y, \ldots, y^{(n-1)}$ at one or more (possibly infinitely many) points in some interval $[a, \infty)$. Consequently, it is sufficiently general to be applicable to ordinary differential equations, equations with deviating arguments, and integro-differential equations.

Our objective is to use the Schauder-Tychonoff theorem to establish sufficient conditions for (1.1) to have a solution y_0 on some interval $[t_0, \infty)$ such that

$$y_0^{(r)}(t) = p^{(r)}(t) + o(t^{m-r}), \quad 0 \leq r \leq n-1, \tag{1.2}$$

where p is a given polynomial and m is an integer, with

$$0 \leq m \leq \deg p \leq n-1. \tag{1.3}$$

Neither this problem nor the application of the Schauder-Tychonoff theorem to it is new; nevertheless, we believe that our method of applying the theorem has some advantages over the standard approach, in which the integrability conditions imposed on F virtually always imply that there are constants a and M and a positive function W such that

$$\int^{\infty} t^{n-m-1} W(t)dt < \infty \tag{1.4}$$

and

$$|F(t; y)| \leq W(t) \tag{1.5}$$

whenever y is in the family

(1.6) $Y=\{y\in C^{(n-1)}[a,\infty)\,|\,|y^{(r)}(t)-p^{(r)}(t)|\le Mt^{m-r},\ 0\le r\le n-1,\ t\ge a\}$.

Assumptions like this preclude the possibility of exploiting oscillatory properties of the functions h in the family

$$H=\{h\,|\,h(t)=F(t;\,y),\ y\in Y\}.$$

To put it another way, (1.4) and (1.5) require that integrals of the form

$$\int_t^\infty s^{n-m-1}|F(s;\,y)|ds,\quad y\in Y, \tag{1.7}$$

converge to zero uniformly with respect to y as $t\to\infty$. Our integrability assumptions on F are stated in terms of ordinary (rather than absolute) convergence, and require only that the quantitites

$$\left|\int_t^\infty s^{n-m-1}F(s;\,y)ds\right|,\quad y\in S, \tag{1.8}$$

converge to zero uniformly (with respect to y) sufficiently rapidly as $t\to\infty$, where S is a suitably restricted subfamily of Y as defined in (1.6); thus, we *integrate first and then take absolute values*. This enables us to obtain results in situations where the integrals in (1.8) may converge only conditionally, and to make good use of their rate of convergence (which is in general faster than that of (1.7), even if the latter converge), so as to advantageously restrict the family of functions to which the fixed point theorem is to be applied, and to obtain better estimates of the errors $y_0^{(r)}-p^{(r)}$ $(0\le r\le n-1)$ as $t\to\infty$.

We have presented our main result (Theorem 1) in such a way that it can replace the Schauder-Tychonoff theorem in specific applications. To apply Theorem 1, it is only necessary to show that the functional F in (1.1) satisfies three assumptions, two of which are trivially verifiable in most situations. One can then infer from Theorem 1 that (1.1) has a solution with the required asymptotic properties, without explicitly converting (1.1) into a suitable integral equation and verifying that the associated transformation satisfies the hypotheses of the fixed point theorem. Since the forms of the functional perturbations of interest vary greatly and can be quite complicated (for a very incomplete list of examples, see [3, 4, 5, 7, 8, 10, 14]), this would appear to be a useful feature. It is to be hoped that an analog of Theorem 1 can be developed for perturbations of the general disconjugate equation

$$\frac{1}{p_n}\frac{d}{dt}\frac{1}{p_{n-1}}\cdots\frac{1}{p_1}\frac{d}{dt}\frac{x}{p_0}=0,$$

which has recently attracted considerable interest; see, e.g. [1, 2, 6, 9, 11, 12, 15, 16, 17, 18] (again, a very incomplete list).

2. Preliminary considerations

We use "O" and "o" in the standard way to indicate behavior as $t\to\infty$. Whenever we write an improper integral in stating hypotheses, we are assuming that it converges, and the convergence may be conditional, except where the integrand is obviously nonnegative.

The following standing assumption applies throughout the paper.

Assumption A. Let p be a given polynomial and m a given integer satisfying (1.3). Let $0<a\le t_0$, and denote

$$\hat{t} = \begin{cases} t_0 & \text{if } a \le t < t_0, \\ t & \text{if } t \ge t_0. \end{cases} \tag{2.1}$$

Suppose that ϕ is continuous, positive, and nonincreasing on $[0, \infty)$, and, if $m\ge 1$, define

$$\hat{\phi}(t) = \frac{1}{t}\int_0^t \phi(\tau)d\tau, \quad t\ge 0 \quad (\hat{\phi}(0)=\phi(0)). \tag{2.2}$$

We obtain our results by applying the Schauder-Tychonoff theorem to the transformation $\mathscr{T}_m$ defined by

$$(\mathscr{T}_m y)(t) = p(t) - L_m(t;\, F(\ ;\, y)), \tag{2.3}$$

where L_m is given by the following essentially standard definition, slightly complicated by the requirement that it be applicable to the case where the value of $F(t;\, y)$ may depend on values $y(\tau)$ with $\tau<t$.

Definition 1. Suppose that $u\in C[t_0, \infty)$ and the integral

$$\int^\infty s^{n-m-1}u(s)ds$$

converges. Let

$$L_m(t;\, u) = \begin{cases} \displaystyle\int_{\hat{t}}^\infty \frac{(t-s)^{n-1}}{(n-1)!}u(s)ds & \textit{if } m=0, \\ \displaystyle\int_a^t \frac{(t-\tau)^{m-1}}{(m-1)!}d\tau \int_{\hat{\tau}}^\infty \frac{(\tau-s)^{n-m-1}}{(n-m-1)!}u(s)ds & \textit{if } 1\le m\le n-1. \end{cases} \tag{2.4}$$

Dirichlet's test implies that the improper integral in (2.4) converges. It is easily verified that $L_m(\ ;\, u)\in C^{(n-1)}[a, \infty)$, and that

$$(2.5)\quad L_m^{(r)}(t;u)=\begin{cases}\displaystyle\int_a^t \frac{(t-\tau)^{m-r-1}}{(m-r-1)!}\,d\tau\int_{\hat\tau}^\infty \frac{(\tau-s)^{n-m-1}}{(n-m-1)!}\,u(s)ds, & 0\le r\le m-1,\\[2mm] \displaystyle\int_{\hat t}^\infty \frac{(t-s)^{n-r-1}}{(n-r-1)!}\,u(s)ds, & m\le r\le n-1.\end{cases}$$

Moreover,

$$(2.6)\qquad L_m^{(n)}(t;u)=\begin{cases}-u(t), & t\ge t_0,\\ 0\;, & a\le t\le t_0,\end{cases}$$

with the appropriate one-sided interpretations at $t=t_0$. From this and (2.3), we see that if $F(\;;y)$ is continuous on $[t_0,\infty)$ and

$$\int^\infty s^{n-m-1}F(s;y)ds$$

converges, then

$$(2.7)\qquad (\mathscr{T}_m y)^{(n)}(t)=\begin{cases}F(t;y), & t\ge t_0,\\ 0\;, & a\le t\le t_0,\end{cases}$$

so that a fixed point (function) y_0 of $\mathscr{T}_m$ is a solution of (1.1) on $[t_0,\infty)$. Although we do not emphasize it below, it is also clear that

$$(\mathscr{T}_m y)^{(r)}(a)=p^{(r)}(a),\quad 0\le r\le m-1,$$

and that $\mathscr{T}_m y$ is a polynomial of degree $\leqq n-1$ on $[a,t_0)$ if $a<t_0$.

The following elementary (but nontrivial) lemma is crucial to our approach. Special cases of this lemma have appeared in [20] and [21].

Lemma 1. *With u as in Definition 1, suppose that*

$$(2.8)\qquad \sup_{\tau\ge t}\left|\int_\tau^\infty s^{n-m-1}u(s)ds\right|\le\psi(t),\quad t\ge t_0,$$

where ψ is nonincreasing and positive on $[t_0,\infty)$. If $a<t_0$ in Assumption A, extend ψ over $[a,t_0)$ by defining

$$(2.9)\qquad \psi(t)=\psi(t_0),\quad a\le t\le t_0.$$

Let

$$k_{rmn}=\begin{cases}\dfrac{1}{(n-m-1)!(m-r-1)!}, & 0\le r\le m-1,\\[2mm] \dfrac{1}{(n-m-1)!}, & r=m,\\[2mm] \dfrac{2}{(n-r-1)!}, & m+1\le r\le n-1.\end{cases}$$

Then

$$|L_m^{(r)}(t; u)| \leq k_{rmn}\psi(t)t^{m-r}, \quad m \leq r \leq n-1, \quad t \geq a. \tag{2.10}$$

Now assume that

$$\psi(t) = O(\phi(t)), \tag{2.11}$$

with ϕ *as in Assumption A, and define*

$$\psi_1(t) = \sup_{\tau \geq t} \frac{\psi(\tau)}{\phi(\tau)}. \tag{2.12}$$

Then, if $m \geq 1$,

$$|L_m^{(r)}(t; u)| \leq k_{rmn}\psi_1(t_0)\hat{\phi}(t)t^{m-r}, \quad 0 \leq r \leq m-1, \quad t \geq a. \tag{2.13}$$

If

$$\psi(t) = o(\phi(t)), \tag{2.14}$$

then

$$L_m^{(r)}(t; u) = o(\phi(t)t^{m-r}), \quad m \leq r \leq n-1; \tag{2.15}$$

moreover, if $m \geq 1$ *and*

$$\int^\infty \phi(s)ds = \infty \tag{2.16}$$

in addition to (2.14), *then*

$$L_m^{(r)}(t; u) = o(\hat{\phi}(t)t^{m-r}), \quad 0 \leq r \leq m-1. \tag{2.17}$$

Before proving this lemma, we make some observations concerning ϕ and $\hat{\phi}$. First, it is easily verified that $\hat{\phi}(t) \geq \phi(t)$, $\hat{\phi}'(t) \leq 0$, and

$$\lim_{t\to\infty} \hat{\phi}(t) = 0 \quad \text{if} \quad \lim_{t\to\infty} \phi(t) = 0. \tag{2.18}$$

Second, the assumption (2.16), which is needed to prove (2.17) if $m \geq 1$, is quite natural, since it is easy to show by integrating by parts that if (2.8) and (2.11) hold for a nonincreasing function ϕ such that

$$\int^\infty \phi(s)ds < \infty, \tag{2.19}$$

then $\int^\infty s^{n-m}u(s)ds$ converges. In our application of this lemma, this would mean that we could work with the transformation $\mathscr{T}_{m-1}$ rather than $\mathscr{T}_m$. A fixed point of the former would be a better solution to our problem, since it would have the asymptotic behavior

$$y_0^{(r)}(t) = p^{(r)}(t) + o(t^{m-r-1}), \quad 0 \le r \le n-1.$$

(Compare this with (1.2).)

Finally, we observe that it is easy to show that $\hat{\phi}(t) \le K\phi(t)$ $(t>0)$ for some constant K if $t^\gamma \phi(t)$ is eventually nondecreasing for some $\gamma<1$. On the other hand, $\lim_{t\to\infty} \hat{\phi}(t)/\phi(t) = \infty$ if $t^\gamma\phi(t)$ is eventually nonincreasing for some $\gamma \ge 1$. Although (2.16) precludes this for $\gamma>1$, it does not preclude the possibiity that $\phi(t) \sim K/t$ for some $K \ne 0$, in which case $\hat{\phi}(t) \sim K(\log t)/t$.

Proof of Lemma 1. The proof of (2.10) is similar to the proof of Lemma 1 of [23], but we include it here for the reader's convenience. Let

$$U(t) = \int_t^\infty s^{n-m-1} u(s)\,ds, \quad t \ge t_0,$$

and note that

$$|U(t)| \le \psi(t), \quad t \ge t_0, \tag{2.20}$$

because of (2.8). Writing

$$(t-s)^{n-r-1} u(s) = -\left(\frac{t}{s} - 1\right)^{n-r-1} s^{m-r} U'(s),$$

integrating by parts, and applying (2.20) yields the inequality

$$\left|\int_{\hat{t}}^\infty (t-s)^{n-r-1} u(s)\,ds\right| \le \psi(\hat{t})\left[\left(1 - \frac{t}{\hat{t}}\right)^{n-r-1} \hat{t}^{m-r} + \int_{\hat{t}}^\infty \left|\frac{d}{ds}\left(\frac{t}{s} - 1\right)^{n-r-1} s^{m-r}\right| ds\right], \quad m \le r \le n-1. \tag{2.21}$$

Since

$$\left|\frac{d}{ds}\left(\frac{t}{s} - 1\right)^{n-r-1} s^{m-r}\right| \le (r-m)\, s^{m-r-1} + \hat{t}^{m-r} \frac{d}{ds}\left(1 - \frac{t}{s}\right)^{n-r-1}$$

if $s \ge \hat{t}$ (cf. (2.1)) and $m \le r \le n-1$, (2.21) and our conventions (2.1) and (2.9) imply that

$$\left|\int_{\hat{t}}^\infty (t-s)^{n-r-1} u(s)\,ds\right| \le 2\psi(t) t^{m-r}, \quad m+1 \le r \le n-1, \quad t \ge a,$$

and that

$$\left|\int_{\hat{t}}^\infty (t-s)^{n-m-1} u(s)\,ds\right| \le \psi(t), \quad t \ge a. \tag{2.22}$$

These inequalities and (2.5) imply (2.10). Obviously, (2.10) and (2.14) imply (2.15), which completes the proof if $m=0$.

If $0 \le r \le m-1$, then (2.5) and (2.22) imply that

$$(2.23)\quad |L_m^{(r)}(t;u)| \le k_{rmn}\int_a^t (t-\tau)^{m-r-1}\psi(\tau)d\tau \le k_{rmn}t^{m-r-1}\int_a^t \psi(\tau)d\tau, \quad t \ge a,$$

since $a \ge 0$. It is straightforward to verify that this, (2.2), (2.9), and (2.12) imply (2.13). To see that (2.14) and (2.16) imply (2.17), we rewrite (2.23) as

$$|L_m^{(r)}(t;u)| \le k_{rmn}t^{m-r}\left[\frac{1}{t}\int_a^{t_1}\psi(\tau)d\tau + \frac{1}{t}\int_{t_1}^t \psi(\tau)d\tau\right],$$

where $t_0 \le t_1 \le t$. This, (2.2), and (2.12) imply that

$$(2.24)\quad |L_m^{(r)}(t;u)| \le k_{rmn}t^{m-r}\left[\frac{1}{t}\int_a^{t_1}\psi(\tau)d\tau + \psi_1(t_1)\hat{\phi}(t)\right].$$

From (2.2) and (2.16),

$$\lim_{t\to\infty} t\hat{\phi}(t) = \infty;$$

therefore, (2.24) implies that

$$(2.25)\quad \overline{\lim_{t\to\infty}}\,(\hat{\phi}(t)t^{m-r})^{-1}|L_m^{(r)}(t;u)| \le k_{rmn}\psi_1(t_1).$$

Since t_1 is arbitrary and (2.14) implies that $\lim_{t_1\to\infty}\psi_1(t_1)=0$, (2.25) implies (2.17). This completes the proof of Lemma 1.

3. The main theorem

If $\{y_j\}$ is a sequence of functions in $C^{(n-1)}[a,\infty)$, we will say that $\{y_j\}$ converges to y, and write $y_j \to y$, if

$$\lim_{j\to\infty}\left[\sup_{a\le t\le T}\sum_{r=0}^{n-1}|y_j^{(r)}(t)-y^{(r)}(t)|\right] = 0$$

for every $T>a$. With this topology, $C^{(n-1)}[a,\infty)$ is a Fréchet (complete linear metric) space. If $M>0$, the subset S of $C^{(n-1)}[a,\infty)$ consisting of functions y such that

$$(3.1)\quad |y^{(r)}(t)-p^{(r)}(t)| \le \begin{cases} Mk_{rmn}\hat{\phi}(t)t^{m-r}, & 0 \le r \le m-1, \\ Mk_{rmn}\phi(t)t^{m-r}, & m \le r \le n-1, \end{cases} \quad t \ge a$$

(where p is the given polynomial in (1.2)), is a closed convex subset of $C^{(n-1)}[a,\infty)$.

In the following theorem we impose what we believe to be comparatively mild conditions on F which will imply that (a) $\mathcal{T}_m$ is defined on S and $\mathcal{T}_m(S) \subset S$; (b) $\mathcal{T}_m y_j \to \mathcal{T}_m y$ if $y_j \to y$ (i.e., $\mathcal{T}_m$ is continuous); and (c) $\mathcal{T}_m(S)$ has compact closure. The Schauder-Tychonoff theorem will then imply that $\mathcal{T}_m y_0 = y_0$ for some y_0 in S.

Theorem 1. *Suppose that Assumption A holds and there is a constant $M > 0$ such that $F(\ ;y) \in C[t_0, \infty)$ whenever y is in the subset S of $C^{(n-1)}[a, \infty)$ defined by* (3.1), *and that the following conditions are satisfied:*

(i) *The family $\{F(\ ;y) \mid y \in S\}$ is uniformly bounded on each finite subinterval of $[t_0, \infty)$.*

(ii) *If $\{y_j\}$ is a sequence in S such that $y_j \to y$, then*

$$\lim_{j\to\infty} F(t;\, y_j) = F(t;\, y) \quad (\textit{pointwise}), \quad t \geq t_0. \tag{3.2}$$

(iii) *The integral*

$$\int_{t_0}^{\infty} s^{n-m-1} F(s;\, y)\, ds \tag{3.3}$$

converges for every y in S, and there is a continuous, nonincreasing function ρ defined on $[a, \infty)$ such that

$$\lim_{t\to\infty} \rho(t) = 0, \tag{3.4}$$

$$\rho(t) = \rho(t_0), \quad a \leq t \leq t_0, \tag{3.5}$$

and

$$\left| \int_t^{\infty} s^{n-m-1} F(s;\, y)\, ds \right| \leq \rho(t) \leq M\phi(t), \quad t \geq a, \quad y \in S. \tag{3.6}$$

Then there is a function y_0 which is defined on $[a, \infty)$ and satisfies (1.1) *on $[t_0, \infty)$, such that*

$$|y_0^{(r)}(t) - p^{(r)}(t)| \leq \begin{cases} k_{rmn}\hat{\rho}(t)t^{m-r}, & 0 \leq r \leq m-1, \\ k_{rmn}\rho(t)t^{m-r}, & m \leq r \leq n-1, \end{cases} \quad t \geq a, \tag{3.7}$$

where

$$\hat{\rho}(t) = \frac{1}{t}\int_0^t \rho(s)\, ds.$$

Proof. By applying Lemma 1 with $u = F(\ ;y)$ and $\phi = \psi = \rho$, we infer from the first equality in (3.6) that

$$(3.8)\qquad |L_m^{(r)}(t;\, F(\ ;\, y))| \le \begin{cases} k_{rmn}\hat{\rho}(t)t^{m-r}, & 0 \le r \le m-1, \\ k_{rmn}\rho(t)t^{m-r}, & m \le r \le n-1, \end{cases} \quad t \ge a.$$

This, (2.3), and the second inequality in (3.6) imply that $\mathscr{T}_m y \in S$ (cf. (3.1)); therefore,

$$(3.9)\qquad \mathscr{T}_m(S) \subset S.$$

Now suppose that $\{y_j\}$ is a sequence in S such that $y_j \to y$. If $\varepsilon > 0$, choose $T > t_0$ such that $\rho(T) < \varepsilon/4$. (This is possible because of (3.4).) Then (3.6) implies that

$$(3.10)\qquad \left| \int_t^\infty s^{n-m-1}[F(s;\, y_j) - F(s;\, y)]ds \right| < \frac{\varepsilon}{2}, \quad t \ge T,$$

for all j. With T now fixed, choose j_0 so that

$$(3.11)\qquad \int_{t_0}^T s^{n-m-1}|F(s;\, y_j) - F(s;\, y)|ds < \frac{\varepsilon}{2}, \quad j \ge j_0,$$

which is possible because of assumptions (i) and (ii) and the bounded convergence theorem. Now (3.10) and (3.11) imply that

$$\left| \int_t^\infty s^{n-m-1}[F(s;\, y_j) - F(s;\, y)]ds \right| < \varepsilon, \quad t \ge t_0, \quad j \ge j_0,$$

and therefore Lemma 1 with $u = F(\ ;\, y_j) - F(\ ;\, y)$ and $\phi = \psi = \varepsilon$ implies that

$$|(\mathscr{T}_m y_j)^{(r)}(t) - (\mathscr{T}_m y)^{(r)}(t)| \le \varepsilon k_{rmn} t^{m-r}, \quad 0 \le r \le n-1, \quad t \ge a.$$

(See (2.3).) This implies that $\mathscr{T}_m y_j \to \mathscr{T}_m y$; hence, $\mathscr{T}_m$ is continuous.

From (3.9) and the definition (3.1) of S, the family of vector-valued functions

$$(3.12)\qquad \{[(\mathscr{T}_m y), (\mathscr{T}_m y)', \ldots, (\mathscr{T}_m y)^{(n-1)}] \mid y \in S\}$$

is uniformly bounded on every finite subinterval of $[a, \infty)$, and the first $n-1$ components are also equicontinous on these intervals. Because of (2.7) and assumption (i), the nth components of the functions in (3.12) are also equicontinuous on finite subintervals of $[a, \infty)$. Hence, Arzela's theorem implies that $\mathscr{T}_m(S)$ has compact closure.

We have now verified that $\mathscr{T}_m$ satisfies the hypotheses of the Schauder-Tychonoff theorem on S, and therefore some y_0 in S is left fixed by $\mathscr{T}_m$; i.e.,

$$(3.13)\qquad y_0(t) = p(t) - L_m(t;\, F(\ ;\, y_0))$$

(cf. (2.3)). From this and (2.6) with $u = F(\ ;\, y_0)$, y_0 satisfies (1.1) on (t_0, ∞). Since (3.8) must hold with $y = y_0$, (3.13) implies (3.7).

Remark 1. Since (3.4) implies that $\hat{\rho}(t)=o(1)$, it is easy to see that (3.7) implies (1.2); however, (3.7) is in general sharper than (1.2).

To apply Theorem 1 to a specific problem we must have a way to choose the integer m and the function ϕ. Although we cannot specify a method for doing this which is guaranteed to be optimal in all situations, the following procedure seems reasonable, and has been successfully applied in [13, 19, 20, 21, 22, 23]:

(a) Let m be the smallest nonnegative integer such that the integral

$$E(t) = \int_t^\infty s^{n-m-1}F(s;\, p)ds \tag{3.14}$$

converges. (Notice that the function $h(t)=F(t;\, p)$ is the functional evaluated along the "target" polynomial.) Of course, if there is no such integer, then Theorem 1 does not apply.

(b) Let $\psi(t)=\sup_{\tau\geq t}|E(\tau)|$. It is clear that we must choose ϕ so that $\psi(t)=O(\phi(t))$. While it may seem reasonable to simply let $\psi=\phi$, this is usually inconvenient, since one usually has only an estimate for ψ. In the applications to ordinary differential equations in [19, 20, 21, 22, 23], the author has chosen ψ to be of the same order of magnitude as ϕ; i.e., $\lim_{t\to\infty}\psi(t)/\phi(t)=L\,(0<L<\infty)$. Clearly we cannot have $\phi(t)=o(\psi(t))$; however, in some applications to functional equations (e.g., see Theorem 2, below) or perhaps to more complicated nonlinear ordinary differential equations, it may be desirable or necessary that $\psi(t)=o(\phi(t))$. In any case, the choice $\phi=1$ (which is used in virtually all of the standard literature) is probably the least desirable, since it ignores information on the rate of convergence of (3.14), which is available in any specific problem. There are many examples in [13, 19, 20, 21, 22, 23] in which the hypotheses of Theorem 1 (particularly the integrability conditions in (iii)) do not hold with $\phi=1$, but do hold with some $\phi(t)=o(1)$. The reason for this is that ϕ enters into the proof of Theorem 1 in a nontrivial way, since, once M is chosen, ϕ determines the set S on which $\mathscr{T}_m$ must satisfy the hypotheses of the Schauder-Tychonoff theorem.

For a related discussion of this question, see Remark 2 of [24], which deals with functional perturbations of the second order equation

$$(r(t)x')' + q(t)x = 0.$$

4. Applications of the main theorem

The author has applied the idea which led to the formulation of Theorem 1 to ordinary differential equations; e.g., see [19, 21, 23] for linear perturbations of $x^{(n)}=0$, and [20, 22] for nonlinear perturbations. Kusano and Trench [13] have obtained a global existence theorem for a nonlinear equation by a similar approach. More specific restrictions were placed on ϕ in these papers, and, except in [13], the Banach contraction principle was used rather than the Schauder-Tychonoff theorem; nevertheless, the essential ideas in all these cases are the same as those presented here, and the results of those papers can be obtained from Theorem 1. This would seem to establish that Theorem 1 has nontrivial applications to ordinary differential equations; therefore, for considerations of length, we will confine our attention here to some simple applications of Theorem 1 to functional equations. We believe that these results are new.

Throughout this section all quantities are assumed to be real. For notational convenience, we define

$$\phi_{rm} = \begin{cases} \hat{\phi}, & 0 \le r \le m-1, \\ \phi, & m \le r \le n-1, \end{cases} \tag{4.1}$$

where ϕ and m are as in Assumption A and $\hat{\phi}$ is as in (2.2).

We first consider the linear functional equation

$$y^{(n)}(t) = P(t)y(g(t)). \tag{4.2}$$

Theorem 2. *Suppose that P and g′ are continuous and* $g' \geq 0$ *on* $[a_0, \infty)$ *for some* $a_0 \geq 0$, *with*

$$\lim_{t\to\infty} g(t) = g(\infty) > 0. \tag{4.3}$$

Suppose also that the integral

$$\int^{\infty} s^{n-m-1}P(s)(g(s))^{\ell}ds$$

converges (perhaps conditionally), where ℓ *and m are integers such that*

$$0 \le m \le \ell \le n-1, \tag{4.4}$$

and define

$$E(t) = \int_t^{\infty} s^{n-m-1}P(s)(g(s))^{\ell}ds, \tag{4.5}$$

$$\psi(t) = \sup_{\tau\geq t} |E(\tau)|,$$

and

$$Q(t) = \int_t^{\infty} s^{n-m-1}P(s)ds. \tag{4.6}$$

Let ϕ *be as in Assumption A, and suppose that either* (i) $m<\ell$; *or* (ii) $m=\ell$ *and*

(4.7) $\overline{\lim_{t\to\infty}}(\phi(t))^{-1}\int_t^\infty |Q(s)|\phi_{1\ell}(g(s))(g(s))^{\ell-1}g'(s)ds = \alpha < 1/k_{lmn}.$

Then (4.2) *has a solution* y_0 *with the asymptotic behavior*

$$(y_0(t)-t^\ell)^{(r)} = O(\phi_{rm}(t)t^{m-r}), \quad 0 \le r \le n-1, \tag{4.8}$$

provided that either

$$g(\infty) = \infty \quad \textit{and} \quad \psi(t) = O(\phi(t)) \tag{4.9}$$

or

$$\psi(t) = o(\phi(t)) \tag{4.10}$$

(*even if* $g(\infty) < \infty$).

Proof. We first observe that $Q=E$ if $\ell=0$; if $\ell>0$, integration by parts as in the proof of Abel's convergence theorem shows that Q is defined and satisfies the inequality

$$|Q(t)| \le 2\psi(t)(g(t))^{-\ell}. \tag{4.11}$$

Now suppose that $g(t_0)>0$ (recall (4.3)), and choose a so that

$$0 < a < \min[t_0, g(t_0)].$$

Here the set S of Theorem 1 consists of those function y in $C^{(n-1)}[a, \infty)$ such that

$$|(y(t)-t^\ell)^{(r)}| \le Mk_{rmn}\phi_{rm}(t)t^{m-r}, \quad 0 \le r \le n-1, \quad t \ge a \tag{4.12}$$

(cf. (3.1) and (4.1)), and

$$F(t; y) = P(t)y(g(t)); \tag{4.13}$$

therefore, assumptions (i) and (ii) are obviously satisfied for any choice of M. We will now show that it is possible to satisfy (3.6) with a suitably chosen ρ by choosing M sufficiently large, provided that the function

(4.14)
$$\sigma(t) = 2k_{0mn}\psi(t)\phi_{0m}(g(t))(g(t))^{m-\ell} + k_{1mn}\int_t^\infty |Q(s)|\phi_{1m}(g(s))(g(s))^{m-1}g'(s)ds$$

is defined and satisfies the inequality

$$\sigma(t) \le \theta\phi(t), \quad t \ge t_0, \tag{4.15}$$

for some $\theta<1$. We will then show that our hypotheses imply the latter for t_0 sufficiently large, and the remainder of the proof will be easy.

For convenience, denote

$$z(t) = y(t) - t^{\ell}. \tag{4.16}$$

Then, from (4.5) and (4.13),

$$\int_t^{\infty} s^{n-m-1}F(s;\, y) = E(t) + \lim_{T\to\infty} J_T(t;\, y), \tag{4.17}$$

where

$$J_T(t;\, y) = \int_t^T s^{n-m-1}P(s)z(g(s))ds = -\int_t^T Q'(s)z(g(s))ds \quad \text{(cf. (4.6))}$$

$$= -\,Q(s)z(g(s))\,\Big|_t^T + \int_t^T Q(s)z'(g(s))g'(s)ds.$$

Now routine estimates based on (4.11), (4.12), and (4.16) show that the integral on the left of (4.17) converges and satisfies the first inequality in (3.6) if $y \in S$, with

$$\rho(t) = \psi(t) + M\sigma(t), \quad t \geq t_0, \tag{4.18}$$

and $\rho(t)$ as defined in (3.5) for $a \leq t \leq t_0$. Since $\psi(t) = O(\phi(t))$, there is a constant K such that $\psi(t) < K\phi(t)$ for $t \geq a$; thus, (4.18) implies that

$$\rho(t) \leq K\phi(t) + M\sigma(t), \quad t \geq t_0. \tag{4.19}$$

Now let us consider σ. We first observe that the first term on the right of (4.14) is $o(\phi(t))$. This is obvious if (4.10) holds, which is in turn obvious if $\lim_{t\to\infty} \phi(t) > 0$. On the other hand, if $\lim_{t\to\infty} \phi(t) = 0$, then $\lim_{t\to\infty} \phi_{0m}(t) = 0$ (recall (2.18) and (4.1)), and then (4.9) implies this statement. From (4.11), the integral on the right of (4.14) is $o(\phi(t))$ if $m < \ell$. Therefore, we can conclude in any of the cases considered that there is a t_0 satisfying (4.15), where θ is any number in (0, 1) if (i) applies, or in $(\alpha k_{1mn}, 1)$ if (ii) applies (cf. (4.7)). With t_0 chosen in this way, we see from (4.19) that we need only choose $M > K/(1-\theta)$ to satisfy the second inequality in (3.6).

We have now verified that the hypotheses of Theorem 1 hold if t_0 and M are suitably chosen; hence, (4.2) has a solution which satisfies (3.7) with $p(t) = t^{\ell}$. Since $\rho(t) = O(\phi(t))$, this implies (4.8) and completes the proof.

Remark 2. If (4.10) holds, then it is easily seen from (4.14) that $\sigma(t) = o(\phi(t))$ if $m < \ell$, and the same conclusion follows if $m = \ell$, provided that $\alpha = 0$ in (4.7). Therefore, $\rho(t) = o(\phi(t))$ in these cases (cf. (4.18) and the closing paragraph of Lemma 1 implies that "O" can be replaced by "o" in (4.8) for $m \leq r \leq n-1$, and also for $0 \leq r \leq m-1$ if $m \geq 1$ and (2.16) holds.

Remark 3. In Theorem 2 we exploited the inequalities (4.12) for $r = 0$ and $r = 1$ only. If $n > 2$, we could exploit the rest $(2 \leq r \leq n-1)$ by repeated integration by parts; however, the deviating argument makes this cumbersome. If $g(t)=t$, then repeated integration by parts leads to better results than we have obtained here for linear equations (e.g., [21]), and for certain nonlinear equations (e.g., [20]).

Now we consider the integro-differential equation

$$y^{(n)}(t) = f(t) + \int_{t-\tau}^{t} w(t, u, y(u), \ldots, y^{(n-1)}(u))du. \tag{4.20}$$

Theorem 3. *Suppose that $\tau > 0$ and w is continuous and satisfies the Lipschitz condition*

$$|w(t, u, y_0, \ldots, y_{n-1}) - w(t, u, \tilde{y}_0, \ldots, \tilde{y}_{n-1})| \leq \sum_{r=0}^{n-1} Q_r(t, u)|y_r - \tilde{y}_r| \tag{4.21}$$

on $[\tau, \infty) \times [0, \infty) \times \boldsymbol{R}^n$, where $Q_0, \ldots, Q_{n-1}$ are positive and continuous on $[\tau, \infty) \times [0, \infty)$. Let f be continuous on $[\tau, \infty)$ and, with p as in (1.3), *suppose that the integral*

$$E(t) = \int_t^\infty s^{n-m-1} \left[f(s) + \int_{s-\tau}^{s} w(s, u, p(u), \ldots, p^{(n-1)}(u))\, du \right] ds \tag{4.22}$$

converges, and that

$$E(t) = O(\phi(t)), \tag{4.23}$$

and that the function

$$\sigma(t) = \sum_{r=0}^{n-1} k_{rmn} \int_t^\infty s^{n-m-1}\, ds \int_{s-\tau}^{s} Q_r(s, u)\phi_{rm}(u)u^{m-r}du \tag{4.24}$$

is defined and satisfies the inequality

$$\overline{\lim_{t\to\infty}}\, (\phi(t))^{-1}\sigma(t) = \theta < 1. \tag{4.25}$$

Then (4.20) *has a solution y_0 which is defined for t sufficiently large and has the asymptotic behavior*

$$y_0^{(r)}(t) - p^{(r)}(t) = O(\phi_{rm}(t)t^{m-r}), \quad 0 \leq r \leq n-1.$$

Proof. Suppose that $t_0 > \tau$ and $M > 0$, and let $a = t_0 - \tau$. Let S be as defined by (3.1) (and recall (4.1)). For convenience, define

$$W(t, u; y) = w(t, u, y(u), \ldots, y^{(n-1)}(u))$$

and

$$\lambda(t) = \sum_{r=0}^{n-1} k_{rmn} \int_{t-\tau}^{t} Q_r(t, u)\phi_{rm}(u)u^{m-r}du, \tag{4.26}$$

and let $F(t; y)$ denote the right side of (4.20). Then

$$F(t; y) = F(t; p) + \int_{t-\tau}^{t} [W(t, u; y) - W(t, u; p)]du, \tag{4.27}$$

and therefore (3.1), (4.21), and (4.26) imply that

$$|F(t; y)| \leq |F(t; p)| + M\lambda(t), \quad t \geq t_0, \quad y \in S.$$

This implies Assumption (i) of Theorem 1.

Now suppose that $\{y_j\}$ is a sequence in S such that $y_j \to y$. From (4.27),

$$F(t; y_j) - F(t; y) = \int_{t-\tau}^{t} [W(t, u; y_j) - W(t, u; y)]du.$$

For fixed t, the integrand on the right approaches zero on $[t-\tau, t]$ as $j \to \infty$, and

$$|W(t, u; y_j) - W(t, u; y)| \leq 2M \sum_{r=0}^{n-1} k_{rmn} Q_r(t, u)\phi_{rm}(u)u^{m-r}.$$

Therefore, the bounded convergence theorem implies (3.2), which verifies assumption (ii) of Theorem 1.

From (4.22) and (4.27),

$$\int_t^\infty s^{n-m-1}F(s; y)ds = E(t) + \int_t^\infty s^{n-m-1}ds \int_{s-\tau}^{s} [W(s, u; y) - W(s, u; p)]du,$$

which, because of (3.1), (4.21), and the assumed existence of σ in (4.24), implies that the integrals (3.3) converge, and that

$$\left| \int_t^\infty s^{n-m-1}F(s; y)ds \right| \leq |E(t)| + M\sigma(t), \quad t \geq t_0, \quad y \in S.$$

We now invoke (4.25); choose θ_1 and $t_0 \geq \tau$ such that $\theta < \theta_1 < 1$ and

$$\sigma(t) \leq \theta_1 \phi(t), \quad t \geq t_0.$$

Let $A = \sup_{t \geq t_0} |E(t)|/\phi(t)$; then $A < \infty$, from (4.23). Now choose $M > A/(1-\theta_1)$. Then (3.6) holds, with $\rho(t) = |E(t)| + M\sigma(t)$ $(t \geqq t_0)$. Hence, Theorem 1 implies the stated conclusion, and the proof is complete.

We close with a global existence theorem for the integro-differential equation

$$y^{(n)}(t) = \int_1^t K(t, u)(y(u))^\gamma du, \quad t \geq 1. \tag{4.28}$$

Theorem 4. *Let γ be an arbitrary real number, except that $\gamma \neq 1$. Suppose that $K = K(t, u)$ is continuous for $1 \leq u \leq t < \infty$ and*

(4.29) $$H(t) = \int_t^\infty s^{n-m-1}ds \int_1^s K(s, u)u^{\ell\gamma}du = O(\phi(t)),$$

where m and ℓ are integers as in (4.4) *and* $\lim_{t\to\infty} \phi(t)=0$. *Suppose further that*

(4.30) $$\sigma(t) = \int_t^\infty s^{n-m-1}ds \int_1^s |K(s, u)|\phi_{0m}(u)u^{m+(\gamma-1)\ell}du = O(\phi(t)).$$

Finally, let c be a positive constant. Then (4.28) *has a solution* y_0 *which is defined on* $[1, \infty)$ *and has the asymptotic behavior*

$$(y_0(t)-ct^\ell)^{(r)} = O(\phi_{rm}(t)t^{m-r}), \quad 0 \le r \le n-1,$$

provided that c is sufficiently small if $\gamma>1$, *or sufficiently large if* $\gamma<1$.

Proof. We apply Theorem 1 with $a=t_0=1$, $p(t)=ct^\ell$, and

(4.31) $$M = \alpha c,$$

where α is a constant such that

(4.32) $$\alpha k_{0mn}\phi_{0m}(1) = \theta < 1.$$

This implies that if y is in the subset S of Theorem 1, then

(4.33) $$|y(t)-ct^\ell| \le c\alpha k_{0mn}\phi_{0m}(t)t^m \le c\theta t^\ell, \quad t \ge 1,$$

where the second inequality holds because of (4.4), (4.32), and the monotonicity of ϕ_{0m}. However, if ξ is a number such that

$$|\xi - ct^\ell| \le \theta ct^\ell,$$

with $0<\theta<1$, then

$$0 < \xi^{\gamma-1} \le (1\pm\theta)^{\gamma-1}(ct^\ell)^{\gamma-1},$$

where the "+" applies if $\gamma>1$, the "−" if $\gamma<1$. Therefore, (4.33) and the mean value theorem imply that

(4.34) $$|(y(t))^\gamma-(ct^\ell)^\gamma| \le Nc^\gamma\phi_{0m}(t)t^{m+(\gamma-1)\ell}, \quad t \ge 1, \quad y \in S.$$

for a suitable constant N (independent of y and c).

Now let $F(t; y)$ denote the right hand side of (4.28). Then

(4.35) $$F(t; y) = c^\gamma \int_1^t K(t, u)u^{\ell\gamma}du + \int_1^t K(t, u)\,[(y(u))^\gamma-(cu^\ell)^\gamma]du;$$

hence, from (4.34),

$$|F(t; y)| \le c^\gamma \left|\int_1^t K(t, u)u^{\gamma\ell}\,du\right| + c^\gamma\lambda(t),$$

where

(4.36) $$\lambda(t) = N\int_1^t |K(t, u)|\phi_{0m}(u)u^{m+(\gamma-1)\ell}du.$$

This implies Assumption (i) of Theorem 1.

Now suppose that $\{y_j\}$ is a sequence in S and $y_j \to y$; then

$$|F(t; y_j)-F(t; y)| \le \int_1^t |K(t, u)|\,|(y_j(u))^\gamma-(y(u))^\gamma|du,$$

and the bounded convergence theorem implies (3.2), which verifies assumption (ii) of Theorem 1.

Finally, multiplying (4.35) by s^{n-m-1}, integrating, and applying routine estimates based on (4.29), (4.30), and (4.34) yields the inequality

$$\left|\int_t^\infty s^{n-m-1}F(s; y)ds\right| \le c^\gamma[|H(t)|+N\sigma(t)], \quad t\ge 1, \quad y\in S.$$

From this and the second equalities in (4.29) and (4.30),

$$\left|\int_t^\infty s^{n-m-1}F(s; y)ds\right| \le Ac^\gamma\phi(t), \quad t\ge 1, \quad y\in S,$$

for some constant A. Therefore, because of (4.31), (3.6) holds with $\rho=Ac^\gamma\phi$ if $Ac^{\gamma-1}<\alpha$. The remainder of the proof is trivial.

It can be shown that the conclusion of Theorem 4 holds for $\gamma=1$ (with c arbitrary, of course), provided that (4.30) (with $\gamma=1$) is strengthened to

$$(\phi(t))^{-1}\int_t^\infty s^{n-m-1}ds\int_1^s |K(s, u)|\phi_{0m}(u)u^m du \le \mu/k_{0mn}, \quad t\ge 1$$

where $\mu<1$.

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