GENERAL FUNCTIONAL DIFFERENTIAL SYSTEMS WITH ASYMPTOTICALLY CONSTANT SOLUTIONS

William F. Trench

Trinity University, San Antonio, TX USA

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Recently the asymptotic properties of solutions of systems of functional differential equations have begun to be studied. (See, e.g., [1]–[10].) Here we give sufficient conditions for rather general functional differential systems to have solutions that approach given constant vectors as $t \to \infty$. This question has been thoroughly investigated for ordinary differential equations, and it seems clear that the methods applied to them can be adapted to functional equations. For example, in [10] the author and T. Kusano obtained sufficient conditions for a functional differential system of the form

$$x_i'(t) = f_i(t, x_1(g_{i1}(t)), \dots, x_n(g_{in}(t))), \quad 1 \le i \le n, \tag{1}$$

to have solutions which approach constant vectors as $t \to \infty$, given that $f_i : [a, \infty) \times \mathbf{R}^n \to \mathbf{R}$ and $g_{ij} : [a, \infty) \to \mathbf{R}$, $1 \le i, \ j \le n$, are continuous and that

$$|f_i(t, \xi_1, \dots, \xi_n)| \le w_i(t, |\xi_1|, \dots, |\xi_n|), \quad 1 \le i \le n, \tag{2}$$

where $w_1, \dots, w_n$ satisfy certain monotonicity and integrability conditions.

Assumptions of this kind are standard in connection with systems

$$x_i'(t) = f_i(t, x_1(t), \dots, x_n(t)), \quad 1 \le i \le n,$$

of ordinary differential equations. It is interesting that they lead to results for the system (1), in which no assumptions other than continuity are imposed on the deviating arguments $\{g_{ij}\}$. Nevertheless, although (1) is a considerably more general system than (2), it is not very general in the context of functional differential equations, which may take a great variety of forms. For example, one may wish to consider a functional system

$$X'(t) = F(t; X), \tag{3}$$

or, in component form,

$$x_i'(t) = f_i(t; X), \quad 1 \leq i \leq n$$

(here $X = [x_1, \ldots, x_n]$), in which each $f_i(t; X)$ depends on X evaluated at several (perhaps infinitely many) values of the independent variable t. Therefore, it would seem to be useful to replace assumptions like those stated above by conditions which are easy to check for specific systems, but are not strictly limited in their applicability to systems of a given special form. This is our objective here.

We begin with the following definition from [10].

Definition 1. *If $-\infty < t_0 < \infty$, then $\mathcal{C}_n(t_0)$ is the space of continuous n-vector functions $X = (x_1, \ldots, x_n)$ on $(-\infty, \infty)$ which are constant on $(-\infty, t_0]$, with the topology induced by the following definition of convergence: $X_j \to X$ as $j \to \infty$ if $\lim_{j\to\infty} [\sup_{-\infty < t \leq T} \|X_j(t) - X(t)\|] = 0$ for every T in $(-\infty, \infty)$.*

Here $\|\cdot\|$ is any convenient vector norm. Notice that $\mathcal{C}_n(t_0)$ is a Fréchet (= complete linear metric) space and that $\mathcal{C}_n(t_0) \subset \mathcal{C}_n(a)$ if $t_0 \geq a$.

We now state our main assumption on the functional F in (3). It is to be understood that this assumption holds throughout the paper.

Assumption A. *Suppose that* $a \in (-\infty, \infty)$ *and* r *is a given integer,* $1 \leq r \leq n$. *Let* $\mathcal{C}_{nr}(a)$ *be the set of functions* X *in* $\mathcal{C}_n(a)$ *such that* $|x_i(t)| > 0,\ -\infty < t < \infty,\ r+1 \leq i \leq n$. *Suppose further that:*

(i) For each X *in* $\mathcal{C}_{nr}(a)$, $F(\cdot; X)$ *is continuous on* $[a, \infty)$.

(ii) If $\{X_j\} \subset \mathcal{C}_{nr}(a)$ *and* $X_j \to X \in \mathcal{C}_{nr}(a)$, *then* $\lim_{j\to\infty} F(t; X_j) = F(t; X)$ *(pointwise),* $t \geq a$.

(iii) The component functionals $f_1(\cdot; X), \ldots, f_n(\cdot; X)$ *satisfy the inequalities*

$$|f_i(t; X)| \leq w_i(t, \rho_1, \ldots, \rho_n),\ t \geq a,\ 1 \leq i \leq n, \tag{4}$$

whenever $X \in \mathcal{C}_{nr}(a)$ *and*

$$|x_i(t)| \leq \rho_i,\ -\infty < t < \infty,\ 1 \leq i \leq r, \tag{5}$$

and

$$|x_i(t)| \geq \rho_i,\ -\infty < t < \infty,\ r+1 \leq i \leq n, \tag{6}$$

where, for each $i = 1, \ldots, n$, $w_i : [a, \infty) \times (0, \infty)^r \times (0, \infty)^{n-r} \to [0, \infty)$ *is continuous, nondecreasing in* $\rho_1, \ldots, \rho_r$, *and nonincreasing in* $\rho_{r+1}, \ldots, \rho_n$. *Suppose also that*

$$\int_a^\infty w_i(t, \rho_1, \ldots, \rho_n)\ dt \leq \infty,\ 1 \leq i \leq n, \tag{7}$$

for all $\rho_1, \ldots, \rho_n > 0$.

Some of these conditions are obviously vacuous if $r = 0$ or $r = n$. This is also true of other assumptions and of some of the conclusions below. The proofs simplify in obvious ways in these special cases.

We first prove a lemma which will facilitate the derivation of our main results.

Lemma 1. *Suppose that* $t_0 \geq a$ *and* $\rho_1, \dots, \rho_n$, $\theta_1, \dots, \theta_n$ *are positive numbers such that* $0 < \theta_i < 1$ *if* $r+1 \leq i \leq n$,

$$\int_{t_0}^{\infty} w_i(t, \rho_1, \dots, \rho_n)\, dt \leq \frac{\theta_i}{1+\theta_i}\rho_i, \quad 1 \leq i \leq r, \tag{8}$$

and

$$\int_{t_0}^{\infty} w_i(t, \rho_1, \dots, \rho_n)\, dt \leq \frac{\theta_i}{1-\theta_i}\rho_i, \quad r+1 \leq i \leq n. \tag{9}$$

Let $c_1, \dots, c_n$ *be given constants such that*

$$|c_i| \leq \frac{\rho_i}{1+\theta_i}, \; 1 \leq i \leq r, \tag{10}$$

and

$$|c_i| \geq \frac{\rho_i}{1-\theta_i}, \; r+1 \leq i \leq n. \tag{11}$$

Then there is a function $\hat{X}$ *in* $\mathcal{C}_{nr}(t_0)$ *such that*

$$\hat{X}'(t) = F(t; \hat{X}), \; t \geq t_0, \tag{12}$$

$$|\hat{x}_i(t) - c_i| \leq \frac{\theta_i}{1+\theta_i}\rho_i, \; -\infty < t < \infty, \; 1 \leq i \leq r, \tag{13}$$

$$|\hat{x}_i(t) - c_i| \le \frac{\theta_i}{1-\theta_i}\rho_i,\ -\infty < t < \infty,\ r+1 \le i \le n, \tag{14}$$

and

$$lim_{t\to\infty}\hat{x}_i(t) = c_i,\ 1 \le i \le n. \tag{15}$$

Proof. We obtain $\hat{X}$ as a fixed point (function) of the transformation $Y = TX$ defined by

$$y_i(t) = \begin{cases} c_i - \int_t^\infty f_i(s;X)\ ds, & t \ge t_0, \\ \\ c_i - \int_{t_0}^\infty f_i(s;X)\ ds, & t < t_0, \end{cases} \qquad 1 \le i \le n. \tag{16}$$

which is to act on the set $\mathcal{S}$ of functions X in $\mathcal{C}_n(t_0)$ such that

$$|x_i(t) - c_i| \le \frac{\theta_i}{1+\theta_i}\rho_i,\ -\infty < t < \infty,\ 1 \le i \le r, \tag{17}$$

and

$$|x_i(t) - c_i| \le \frac{\theta_i}{1-\theta_i}\rho_i,\ -\infty < t < \infty,\ r+1 \le i \le n. \tag{18}$$

Since $\mathcal{S}$ is a closed convex subset of $\mathcal{C}_n(t_0)$, the Schauder–Tychonoff theorem asserts that $\hat{X} = T\hat{X}$ for some $\hat{X}$ in $\mathcal{S}$ provided that

(a) T is defined on $\mathcal{S}$;

(b) $T(\mathcal{S}) \subset \mathcal{S}$;

(c) $TX_j \to TX$ if $\{X_j\} \subset \mathcal{S}$ and $X_j \to X$; and

(d) $T(\mathcal{S})$ has compact closure.

For the rest of the proof we assume that $X \in \mathcal{S}$. Then (10) and (17) imply (5), while (11) and (18) imply (6), so that $\mathcal{S} \subset \mathcal{C}_{nr}(a)$. Now (5) and (6) imply (4); hence, (8) and (9) imply that $Y = TX$ is defined, and that (17) and (18) remain valid with x_i replaced by y_i (cf.(16)). This establishes hypotheses (a) and (b) of the Schauder–Tychonoff theorem.

Suppose that $\{X_j\} \subset \mathcal{S}$ and $X_j \to X$. Let $Y_j = TX_j = (y_{1j}, \dots, y_{nj})$ and $Y = TX = (y_1, \dots, y_n)$. From (16),

$$|y_{ij}(t) - y_i(t)| \leq \int_{t_0}^{\infty} |f_i(s; X_j) - f_i(s; X)|\, ds, \ -\infty < t < \infty, \ 1 \leq i \leq n. \quad (19)$$

Since part (ii) of Assumption A implies that the integrand here converges pointwise to zero and (iii) implies that

$$|f_i(t; X_j) - f_i(t; X)| \leq 2w_i(t, \rho_1, \dots, \rho_n), \ t \geq t_0, \ 1 \leq i \leq n,$$

(7) and Lebesgue's dominated convergence theorem imply that the integral in (19) converges to zero as $j \to \infty$; hence, $\{Y_j\}$ converges to Y uniformly on $(-\infty, \infty)$. This establishes hypothesis (c) of the Schauder-Tychonoff theorem.

To see that $T(\mathcal{S})$ has compact closure, we first observe that it is uniformly bounded on $(-\infty, \infty)$, since $T(\mathcal{S}) \subset \mathcal{S}$. Differentiating (16) and applying (4) yields the inequality

$$|y_i'(t)| \leq |f_i(t; X)| \leq w_i(t, \rho_1, \dots, \rho_n), \ t \geq t_0, \ 1 \leq \ i \leq n$$

(with the appropriate one–sided interpretation when $t = t_0$.) This implies that $T(\mathcal{S})$ is equicontinuous on compact subintervals of $[t_0, \infty)$, and now the Arzela–Ascoli theorem and Definition 1 imply that $T(\mathcal{S})$ has compact closure. Therefore, the Schauder-Tychonoff theorem implies that $T\hat{X} = \hat{X}$ for some X in $\mathcal{S}$. It is trivial to verify that $\hat{X}$ satisfies (12), (13), (14), and (15). This completes the proof of Lemma 1.

Theorem 1. *Let $c_1, \dots, c_n$ be given constants, with $c_i \neq 0$ if $r+1 \leq i \leq n$. Then, if t_0 is sufficiently large, there is an $\hat{X}$ in $\mathcal{C}_{nr}(t_0)$ which satisfies (3) for $t > t_0$ and has the asymptotic behavior (15).*

PROOF. Choose positive numbers $\theta_1, \dots, \theta_n$, with $0 < \theta_i < 1$ for $r+1 \leq i \leq n$. Then choose $\rho_1, \dots, \rho_n$ to satisfy (10) and (11). Finally, choose t_0 so that (8) and (9) hold, and apply Lemma 1.

Theorem 2. *Suppose that $\rho_1, \dots, \rho_n$ and t_0 $(\geq a)$ are such that*

$$\int_{t_0}^{\infty} w_i(t, \rho_1, \dots, \rho_n)\, dt < \rho_i, \; 1 \leq i \leq r. \tag{20}$$

Then there is an $\hat{X}$ in $\mathcal{C}_{nr}(t_0)$ which satisfies (3) for $t > t_0$ and has the asymptotic behavior (15), provided that $|c_i|$ is sufficiently small for $1 \leq i \leq r$ and $|c_i|$ is sufficiently large for $r+1 \leq i \leq n$.

PROOF. Choose $\theta_1, \dots, \theta_r$ sufficiently large to imply (8). (This is possible because of (20).) Next choose $\theta_{r+1}, \dots, \theta_n$ in $(0, 1)$ to satisfy (9). Now Lemma 1 implies the conclusion, provided that $c_1, \dots, c_n$ satisfy (10) and (11).

Theorem 3. *Suppose that $r = 0$; i.e., that $w_i(t, \rho_1, \ldots, \rho_n)$ is non-increasing with respect to $\rho_1, \ldots, \rho_n$ for $1 \le i \le n$. Then there is an $\hat{X}$ in $\mathcal{C}_{nr}(a)$ which satisfies (3) on the entire interval $[a, \infty)$ and has the asymptotic behavior (15), provided that $|c_1|, \ldots, |c_n|$ are sufficiently large.*

Proof. Here (8) and (10) are vacuous. Let $\rho_1, \ldots, \rho_n$ be arbitrary positive numbers, and choose $\theta_1, \ldots, \theta_n$ in $(0, 1)$ so that (9) holds with $t_0 = a$ and $r = 0$. Then Lemma 1 implies the conclusion, provided that (11) holds with $r = 0$.

Theorem 4. *In addition to Assumption A, suppose that $1 \le r \le n$ and that the following conditions hold for $1 \le i \le r$:*

$$w_i(t, \rho_1, \ldots, \rho_n) = u_i(t, \rho_1, \ldots, \rho_r) + v_i(t, \rho_{r+1}, \ldots, \rho_n), \tag{21}$$

where

$$\lim_{\lambda \to \infty} v_i(t, \lambda, \ldots, \lambda) = 0 \tag{22}$$

if $r < n$ or $v_i = 0$ if $r = n$; $\rho^{-1} u_i(t, \rho, \ldots, \rho)$ is monotonic in ρ for each t, and, for some $t_0 \ge a$,

$$\lim_{\rho \to \alpha} \rho^{-1} u_i(t, \rho, \ldots, \rho) = A_i(t), \; t \ge t_0, \; 1 \le i \le r, \tag{23}$$

where

$$\int_{t_0}^{\infty} A_i(t)\; dt = \phi_i < 1, \; 1 \le i \le r, \tag{24}$$

and either $\alpha = 0^+$ or $\alpha = \infty$. Then there is an $\hat{X}$ in $\mathcal{C}_{nr}(t_0)$ which satisfies (3) for $t \ge t_0$ and has the asymptotic behavior (15), provided that $|c_i|$ is *sufficiently large for $r+1 \leq i \leq n$ and either (i) $\alpha = 0^+$ and $|c_1|, \dots, |c_r|$ are sufficiently small or (ii) $\alpha = \infty$.*

PROOF. Choose θ sufficiently large so that $\phi_i < \theta/(1+\theta),\ 1 \leq i \leq r$. Then, from (23), (24), and Levi's monotone convergence theorem,

$$\int_{t_0}^{\infty} u_i(t, \rho, \dots, \rho)\ dt < \frac{\theta}{1+\theta}\rho,\ 1 \leq i \leq r,$$

if ρ is sufficiently near α. Moreover, (22) and Lebesgue's dominated convergence theorem imply that

$$\lim_{\lambda \to \infty} \int_{t_0}^{\infty} v_i(t, \lambda, \dots, \lambda)\ dt = 0,\ 1 \leq i \leq r.$$

Therefore, we can choose ρ_0 sufficiently near α and λ_0 sufficiently large so that

$$\int_{t_0}^{\infty} u_i(t, \rho_0, \dots, \rho_0)\ dt + \int_{t_0}^{\infty} v_i(t, \lambda_0, \dots, \lambda_0)\ dt \leq \frac{\theta}{1+\theta}\rho_0,\ 1 \leq i \leq r,$$

which, from (21), is equivalent to

$$\int_{t_0}^{\infty} w_i(t, \rho_0, \dots, \rho_0, \lambda_0, \dots, \lambda_0)\ dt \leq \frac{\theta}{1+\theta}\rho_0,\ 1 \leq i \leq r. \tag{25}$$

If $r < n$, now choose $\hat{\theta}$ in $(0,1)$ so that

$$\int_{t_0}^{\infty} w_i(t, \rho_0, \dots, \rho_0, \lambda_0, \dots, \lambda_0)\ dt \leq \frac{\hat{\theta}}{1-\hat{\theta}}\rho_0,\ r+1 \leq\ i \leq n.$$

Now Lemma 1 implies that there is an $\hat{X}$ in $\mathcal{C}_{nr}(t_0)$ which satisfies (3) for $t > t_0$ and has the asymptotic behavior (15), provided that

$$|c_i| \leq \frac{\rho_0}{1+\theta},\ 1 \leq\ i \leq r, \tag{26}$$

and

$$|c_i| \geq \frac{\rho_0}{1-\hat{\theta}},\ r+1 \leq\ i \leq n.$$

Hence, the conclusion of the present theorem is now immediate under assumption (i) above. To see that it also holds under assumption (ii), we simply observe that if $\alpha = \infty$ and $|c_1|, \ldots, |c_r|$ are arbitrary, we can pick ρ_0 so large that (25) and (26) both hold.

Theorem 4 generalizes results obtained in [10] with $r = n$ and $A_i(t) \equiv 0,\ 1 \leq i \leq n$. Notice that if $A_i(t) \equiv 0$ for $1 \leq i \leq r$, then Theorem 4 is *global* result, since then we can conclude that there is an $\hat{X}$ in $\mathcal{C}_{nr}(a)$ which satisfies (3) on the entire interval $[a, \infty)$ and has the asymptotic behavior (15). Of course, Theorem 3 is also of this nature.

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