MIXED SUBLINEAR, SUPERLINEAR, AND SINGULAR
SYSTEMS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

WILLIAM F. TRENCH
Trinity University, San Antonio, TX USA

We consider systems of the form

\[ x'_i(t) = \sum_{j=1}^{n} a_{ij}(t)(x_j(g(t)))^{\gamma_j}, \quad t > t_0, \quad 1 \leq i \leq n, \quad (1) \]

where \( a_{ij} : [t_0, \infty) \rightarrow R \) and \( g : [t_0, \infty) \rightarrow R \) are continuous, and \( \gamma_1, \ldots, \gamma_n \) are nonzero rational numbers with odd denominators, so that the quantity \( x^{\gamma_j} \) is real-valued whenever \( x \) is real. However, this restriction is for notational convenience only; with trivial modifications our results are valid for the system

\[ x'_i(t) = \sum_{j=1}^{n} a_{ij}(t)|x_j(g(t)))|^{\gamma_j} \text{sgn}(x_j(g(t))), \quad t > t_0, \quad 1 \leq i \leq n. \]

The asymptotic behavior of systems of functional differential equations has recently begun to receive attention (see, e.g., [1]–[9]). Here we give condi-

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tions which imply that (1) has solutions on the half-line \([t_0, \infty)\) that approach a given constant vector \(C\) as \(t \to \infty\). Since there are no assumptions on the deviating argument \(g\) other than continuity, we must allow for the possibility that \(g(t) < t_0\) for some \(t > t_0\). For this reason we introduce the following definition.

**Definition.** If \(-\infty < t_0 < \infty\), then \(C_n(t_0)\) is the space of continuous \(n\)-vector functions on \((-\infty, \infty)\) which are constant on \((-\infty, t_0]\), with the topology induced by the following definition of convergence: \(X_j \to X\) as \(j \to \infty\) if \(\|X_j(t) - X(t)\| \to 0\) uniformly as \(j \to \infty\) on every half-line \((-\infty, b]\).

We say that a function \(X\) in \(C_n(t_0)\) is a solution of (1) if \(X\) is differentiable and satisfies (1) on \((t_0, \infty)\). We give conditions which guarantee the existence of a solution of (1) such that \(\lim_{t \to \infty} x_i(t) = c_i\), \(1 \leq i \leq n\), for given \(c_1, \ldots, c_n\). For convenience, we will abbreviate (1) as \(x'_i(t) = f_i(t; X)\), \(1 \leq i \leq n\), or in system form as \(X'(t) = F(t; X)\). We obtain our results by applying the Schauder–Tychonoff theorem to the transformation \(Y = TX\) defined by

\[
Y(t) = \begin{cases} 
C - \int_t^\infty F(s; X) \, ds, & t \geq t_0, \\
C - \int_{t_0}^\infty F(s; X) \, ds, & t < t_0.
\end{cases} \tag{2}
\]

The system (1) will be said to be linear, superlinear, sublinear, or singular with respect to \(x_i\) if, respectively, \(\gamma_i = 1\), \(\gamma_i > 1\), \(0 < \gamma_i < 1\), or \(\gamma_i < 0\). In the following \(A = \{i \mid 1 \leq i \leq n\}\) and \(B = \{i \mid 1 \leq i \leq n\}\), and \(\gamma_i > 0\). For a given constant vector \(C\), let \(N = \{i \mid 1 \leq i \leq n\}\) and \(Z = \{i \mid 1 \leq i \leq n\}\) and \(c_i = 0\). Any of the sets \(A, B, N, Z\) may be empty.
We impose the following integrability conditions on the coefficient functions \( \{a_{ij}\} \) in (1). It should be understood that this assumption applies throughout the remainder of the paper.

**Assumption A.** Let \( \gamma_i > 0 \) if \( i \in \mathbb{Z} \). Let \( \varphi_1 \ldots, \varphi_n \) be positive, non-increasing and continuous on \((-\infty, \infty)\), with \( \varphi_i(t) = 1, t \leq t_0 \). Suppose that the integrals \( \int_t^\infty a_{ij}(t)dt \) \((1 \leq i, j \leq n)\) converge (perhaps conditionally) and that for \( 1 \leq i \leq n \) and \( t \geq t_0 \),

\[
\alpha_{ij}(t) = | \int_t^\infty a_{ij}(s) \, ds | = O(\varphi_i(t)), \quad j \in \mathcal{N},
\]

\[
\beta_{ij}(t) = | \int_t^\infty |a_{ij}(s)| \varphi_j(g(s)) \, ds = O(\varphi_i(t)), \quad j \in \mathcal{N},
\]

and

\[
\sigma_{ij}(t) = \int_t^\infty |a_{ij}(s)| (\varphi_j(g(s)))^{\gamma_j} \, ds = O(\varphi_i(t)), \quad j \in \mathbb{Z}.
\]

For convenience below we define

\[
\overline{\alpha}_{ij} = \sup_{t \geq t_0} \alpha_{ij}(t)/\varphi_i(t), \quad j \in \mathcal{N},
\]

\[
\overline{\beta}_{ij} = \sup_{t \geq t_0} \beta_{ij}(t)/\varphi_i(t), \quad j \in \mathcal{N},
\]

\[
\overline{\sigma}_{ij} = \sup_{t \geq t_0} \sigma_{ij}(t)/\varphi_i(t), \quad j \in \mathbb{Z},
\]

and

\[
M_{ij} = \overline{\pi}_{ij} + \theta (1 \pm \theta)^{\gamma_j}^{-1} |\gamma_j| \overline{\beta}_{ij},
\]

3
where \( \theta \) is a given number in \((0, 1)\) and the “±” is “+” if \( \gamma_j \geq 1 \) or “−” if \( \gamma_j < 1 \). It is also convenient here to define the functions \( \lambda_i(t), 1 \leq i \leq n \), by

\[
\lambda_i(t) = \sum_{j \in \mathbb{Z}} r_j \sigma_{ij}(t) + \sum_{j \in \mathbb{N}} |c_j| \gamma_j [\alpha_{ij}(t) + \theta |\gamma_j|(1 \pm \theta)^{-1} \beta_{ij}(t)] \tag{10}
\]

if \( t \geq t_0 \) and \( \lambda_i(t) = \lambda_i(t_0) \) if \( t < t_0 \).

**Theorem 1.** If \( r_i \ (i \in \mathbb{Z}) \) and \( c_i \ (i \in \mathbb{N}) \) are constants such that

\[
\sum_{j \in \mathbb{Z}} \sigma_{ij} \gamma_j^j + \sum_{j \in \mathbb{N}} M_{ij} |c_j| \gamma_j^j \leq \begin{cases} \theta |c_i|, & i \in \mathbb{N}, \\ r_i, & i \in \mathbb{Z}, \end{cases} \tag{11}
\]

then (1) has a solution \( \hat{X} \) such that

\[
|\hat{x}_i(t) - c_i| \leq \lambda_i(t) \leq \theta |c_i| \varphi_i(t) \ (i \in \mathbb{N}), \quad -\infty < t < \infty, \tag{12}
\]

and

\[
|\hat{x}_i(t)| \leq \lambda_i(t) \leq r_i \varphi_i(t) \ (i \in \mathbb{Z}), \quad -\infty < t < \infty. \tag{13}
\]

**Proof.** We apply the Schauder–Tychonoff theorem to show that \( \hat{X} = T\hat{X} \ (\text{cf.} (2)) \) for some \( \hat{X} \) in the closed convex subset \( \mathcal{S} \) consisting of functions \( X \) in \( C_n(t_0) \) such that

\[
|x_i(t) - c_i| \leq \theta |c_i| \varphi_i(t) \ (i \in \mathbb{N}), \quad -\infty < t < \infty, \tag{14}
\]

and

\[
|x_i(t)| \leq r_i \varphi_i(t) \ (i \in \mathbb{Z}), \quad -\infty < t < \infty. \tag{15}
\]

Since

\[
0 < (1 - \theta)|c_i| \leq |x_i(\tau)| \leq (1 + \theta)|c_i| \ (i \in \mathbb{N}), \quad -\infty < \tau < \infty, \tag{16}
\]
the continuity of the \( \{a_{ij}\} \) implies that the functions

\[
f_i(t; X) = \sum_{i=1}^{n} a_{ij}(t)(x_j(g(t)))^{\gamma_j}, \quad 1 \leq i \leq n, \quad X \in S,
\]

are continuous on \([t_0, \infty)\). Moreover,

\[
| \int_t^\infty f_i(s; X) \, ds | \leq | \int_t^\infty f_i(s; C) \, ds | + \int_t^\infty |f_i(s; X) - f_i(s; C)| \, ds \tag{17}
\]

if the integrals on the right converge, which we will now verify. From (3),

\[
| \int_t^\infty f_i(s; C) \, ds | \leq \sum_{j \in \mathcal{N}} |c_j|^{\gamma_j} \alpha_{ij}(t). \tag{18}
\]

Now consider

\[
f_i(t; X) - f_i(t; C) = \sum_{j \notin \mathcal{Z}} a_{ij}(t)(x_j(g(t)))^{\gamma_j} + \sum_{j \in \mathcal{N}} a_{ij}(t)[(x_j(g(t)))^{\gamma_j} - c_j^{\gamma_j}] \tag{19}
\]

From the mean value theorem, \(|x^\gamma - c^\gamma| \leq |\gamma|\|\hat{x}\|^{\gamma_j-1}|x-c|\) with \(\hat{x}\) between \(x\) and \(c\), provided that \(x\) and \(c\) \((\neq 0)\) have the same sign. Therefore, from (16) with \(\tau\) replaced by \(g(t)\),

\[
|(x_j(g(t)))^{\gamma_j} - c_j^{\gamma_j}| \leq |\gamma_j|\|\hat{x}_j\|^{\gamma_j-1}|(x_j(g(t)))^{\gamma_j} - c_j^{\gamma_j}|, \quad j \in \mathcal{N}, \tag{20}
\]
where
\[(1 - \theta)|c_j| < |\hat{x}_j| < (1 + \theta)|c_j|, \quad j \in \mathcal{N}. \tag{21}\]

Now (14), (20), and (21) imply that

\[
|(x_j(g(t)))^\gamma_j - c_j^\gamma_j| \leq \theta|\gamma_j|(1 \pm \theta)^{\gamma_j-1}|c_j|^\gamma_j \varphi_j(g(t)) \quad (j \in \mathcal{N}), \quad X \in \mathcal{S},
\]

where the “±” is “+” if \(\gamma_j \geq 1\), “−” if \(\gamma_j < 1\). Hence, (4), (5), (15), and (19) imply that

\[
\int_t^\infty |f_i(s; X) - f_i(s; C)|ds \leq \sum_{j \in \mathcal{Z}} r_j^\gamma_j \sigma_{ij}(t) + \theta \sum_{j \in \mathcal{N}} |\gamma_j|(1 \pm \theta)^{\gamma_j-1}|c_j|^\gamma_j \beta_{ij}(t),
\]

which, together with (17) and (18) yields the inequalities

\[
\left| \int_t^\infty f_i(s; X)ds \right| \leq \lambda_i(t), \quad 1 \leq i \leq n,
\]

with \(\lambda_i\) as defined in (10). Therefore, (6), (7), (8), (9), and (11) imply that if \(Y = TX\), then

\[
|y_i(t) - c_i| \leq \lambda_i(t) \leq \theta|c_i|\varphi_i(t), \quad (i \in \mathcal{N}) \quad \text{and} \quad |y_i(t)| \leq \lambda_i(t) \leq r_i\varphi_i(t), \quad (i \in \mathcal{Z}),
\]

for all \(t\). Hence, \(T(S) \subset \mathcal{S}\). Since it is routine to verify that \(T\) is continuous and \(T(S)\) has compact closure, the Schauder–Tychonoff theorem now implies
that $T\hat{X} = \hat{X}$ for some $\hat{X}$ in $\mathcal{S}$ with components which satisfy (12) and (13). This completes the proof.

Now let $A_0 = \{ j \in \mathcal{N} \mid \gamma_j > 0 \}$ and recall that $B = \{ j \mid 1 \leq j \leq n \text{ and } \gamma_j < 0 \} \subset \mathcal{N}$.

**Corollary 1.** Suppose that $r_i$ ($i \in \mathcal{Z}$) and $c_i$ ($i \in A_0$) are such that

$$\sum_{j \in \mathcal{Z}} \bar{s}_{ij} r_j^\gamma_j + \sum_{j \in A_0} M_{ij} |c_j|^\gamma_j < \begin{cases} \theta |c_i|, & i \in A_0 \\ r_i, & i \in \mathcal{Z}. \end{cases}$$

Then the conclusions of Theorem 1 hold if $|c_i|$ is sufficiently large for $i \in B$.

**Proof.** Clearly (22) implies (11) if $|c_i|$ ($i \in B$) are sufficiently large.

**Corollary 2.** The conclusions of Theorem 1 hold if either:

(i) $\gamma_i > 1$ for all $i$ in $A$ (i.e., the nonsingular part of (1) is purely superlinear), provided that $r_i$ is sufficiently small for $i$ in $\mathcal{Z}$, $|c_i|$ is sufficiently small for $i$ in $A_0$, and $|c_i|$ is sufficiently large for $i$ in $B$.

(ii) $\gamma_i < 1$ for all $i$ in $A$ (i.e., the nonsingular part of (1) is purely sublinear), and the constants $|c_i|$ and $r_i$ are all sufficiently large.

**References**


(to appear).