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AN ALGORITHM FOR THE INVERSION OF FINITE HANKEL MATRICES*

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1. Introduction. By a Hankel matrix of order $n + 1$, we mean a matrix of the form

$$H_n = \begin{pmatrix} C_0 & C_1 & \cdots & C_n \\ C_1 & C_2 & \cdots & C_{n+1} \\ & & \cdots & \\ C_n & C_{n+1} & \cdots & C_{2n} \end{pmatrix} = (C_{i+j})_{i,j=0}^{n}.$$

Hankel quadratic forms,

$$Q_n(x_0, x_1, \cdots, x_n) = \sum_{i,j=0}^{n} C_{i+j} x_i x_j,$$

are closely connected with the Hamburger moment problem [1, pp. 4–5]. Matrices of this form also arise as coefficient matrices of the normal equations in problems of least squares polynomial curve fitting. Most writers suggest that the normal equations be solved by Gauss's method of elimination or one of its variants. For example, see [2, pp. 147–163].

In this paper, we present an algorithm for the inversion of the matrix H_n which yields the exact inverse. When all of the matrices $H_0, H_1, \cdots, H_n$ are nonsingular, the number of multiplications required to invert H_n is proportional to $(n + 1)^2$, rather than to $(n + 1)^3$, as in the conventional methods for the inversion of an arbitrary symmetric matrix of order $n + 1$.

The author has previously [3] presented a similar algorithm for the inversion of finite Toeplitz matrices, which are of the form

$$T_n = (C_{i-j})_{i,j=0}^{n}.$$

2. Derivation of the algorithm. Throughout this section, we assume that $n \geqq 1$, and that H_{n-1}, H_n, H_{n+1} are nonsingular. Denote H_n^{-1} by

$$H_n^{-1} = B_n = (b_{rsn})_{r,s=0}^{n}.$$

B_n is symmetric, a fact which we will use without specifically citing it. By definition,

$$\sum_{j=0}^{n} C_{r+j} b_{jsn} = \delta_{rs}, \qquad 0 \leqq r, s \leqq n. \tag{1}$$

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We also define

$$u_{sn} = -\sum_{j=0}^{n} C_{n+j+1} b_{jsn}, \qquad 0 \leqq s \leqq n. \tag{2}$$

For notational convenience, we define

$$u_{n+1,n} = 1, \qquad u_{n+2,n} = u_{-1,n} = 0, \tag{3}$$

and

$$b_{-1,sn} = b_{n+1,sn} = 0, \qquad 0 \leqq s \leqq n+1, \tag{4}$$

for every n. We will apply (3) and (4) often in the following derivation, without specifically stating it each time. These definitions allow us to take liberties with indices and limits of summation in ways which prove to be quite convenient.

From (1) and (2),

$$\sum_{j=0}^{n+1} C_{r+j} b_{jsn} = \delta_{rs} - \delta_{r,n+1} u_{sn}, \qquad 0 \leqq r \leqq n+1, 0 \leqq s \leqq n.$$

For fixed s, $0 \leqq s \leqq n$, consider this as a system in the unknowns b_{rsn}, $0 \leqq r \leqq n+1$. It has the solution

$$b_{rsn} = b_{rs,n+1} - b_{r,n+1,n+1} u_{sn}, \qquad 0 \leqq r \leqq n+1, 0 \leqq s \leqq n. \tag{5}$$

With $r = n+1$, this implies that

$$b_{n+1,s,n+1} = b_{n+1,n+1,n+1} u_{sn}, \qquad 0 \leqq s \leqq n+1. \tag{6}$$

Multiplying both sides by C_{s+n+1} and summing from $s = 0$ to $s = n+1$ yields

$$b_{n+1,n+1,n+1} = \lambda_{n+1}^{-1}, \tag{7}$$

where

$$\lambda_{n+1} = \sum_{s=0}^{n+1} C_{n+s+1} u_{sn}. \tag{8}$$

From (6) and (7) we substitute into (5) to obtain

$$b_{rs,n+1} = b_{rsn} + \frac{u_{rn} u_{sn}}{\lambda_{n+1}}, \qquad 0 \leqq r, s \leqq n+1. \tag{9}$$

This relation provides a convenient method for obtaining B_{n+1} from B_n. However, there is a recurrence formula of a different type which reduces the problem of inverting H_n to a still simpler form. By starting from (1) and suitably manipulating indices, it can be shown that

$$\sum_{j=0}^{n} C_{r+j} b_{j-1,s+1,n} = \delta_{rs} - \delta_{rn} u_{s+1,n} - C_{n+r+1} b_{n,s+1,n},$$
$$0 \leqq r \leqq n, 0 \leqq s \leqq n-1.$$

For fixed s, we solve this system to obtain

$$b_{r-1,s+1,n} = b_{rsn} - b_{rnn}u_{s+1,n} - b_{n,s+1,n}\sum_{j=0}^{n} b_{rjn}C_{n+j+1}, \quad 0 \leqq r \leqq n, 0 \leqq s \leqq n - 1.$$

From (2), (6), and (7), this can be rewritten

$$b_{rsn} = b_{r-1,s+1,n} + \lambda_n^{-1}(u_{r,n-1}u_{s+1,n} - u_{rn}u_{s+1,n-1}), \quad 0 \leqq r, s \leqq n + 1. \tag{10}$$

This relation reduces the problem of inverting H_n to that of obtaining $\{u_{rn}\}$ and $\{u_{r,n-1}\}$. These quantities in turn can be computed with a simple recursion formula. Substituting (10) into (2), we find that

$$u_{sn} = -\sum_{j=0}^{n+1} C_{n+j+1}b_{j-1,s+1,n} - \lambda_n^{-1}u_{s+1,n}\sum_{j=0}^{n+1} C_{n+j+1}u_{j,n-1} + \lambda_n^{-1}u_{s+1,n-1}\sum_{j=0}^{n+1} C_{n+j+1}u_{jn}, \quad 0 \leqq s \leqq n. \tag{11}$$

By shifting the index of summation and using (9), we can write

$$\sum_{j=0}^{n+1} C_{n+j+1}b_{j-1,s+1,n} = \sum_{j=0}^{n+1} C_{n+j+2}b_{j,s+1,n} = \sum_{j=0}^{n+1} C_{n+j+2}b_{j,s+1,n+1} - \lambda_{n+1}^{-1} u_{s+1,n}\sum_{j=0}^{n+1} C_{n+j+2}u_{jn}, \quad 0 \leqq s \leqq n. \tag{12}$$

From (2), we can rewrite (12) as

$$\sum_{j=0}^{n+1} C_{n+j+1}b_{j-1,s+1,n} = -u_{s+1,n+1} - \lambda_{n+1}^{-1}\gamma_{n+1}u_{s+1,n}, \quad 0 \leqq s \leqq n, \tag{13}$$

where

$$\gamma_{n+1} = \sum_{j=0}^{n+1} C_{n+j+2}u_{jn}. \tag{14}$$

Substituting (13) into (11), and applying (8) and (14) yields

$$u_{sn} = u_{s+1,n+1} + \lambda_{n+1}^{-1}\gamma_{n+1}u_{s+1,n} - \lambda_n^{-1}\gamma_n u_{s+1,n} + \lambda_n^{-1}\lambda_{n+1}u_{s+1,n-1}, \quad 0 \leqq s \leqq n.$$

By replacing s with $s - 1$, and solving for $u_{s,n+1}$, we obtain

$$u_{s,n+1} = (\lambda_n^{-1}\gamma_n - \lambda_{n+1}^{-1}\gamma_{n+1})u_{sn} + u_{s-1,n} - \lambda_n^{-1}\lambda_{n+1}u_{s,n-1}, \quad 0 \leqq s \leqq n + 1, \tag{15}$$

except that a separate verification is required for $s = 0$. This verification follows similar lines, and is omitted.

Equations (8), (9), (10), (14), and (15) provide the basis for a simple algorithm for the inversion of Hankel matrices. For the reader's convenience, the equations are arranged in a reasonable computational order in the next section.

3. Statement of the algorithm. Suppose that it is required to invert the matrix H_{m+1}, that $k \leqq m$, and $H_{k-1}, H_k, \cdots, H_{m+1}$ are all nonsingular. Obtain $u_{0,k-1}, u_{1,k-1}, \cdots, u_{k-1,k-1}$, and $u_{0k}, u_{1k}, \cdots, u_{kk}$ by solving the system

$$\sum_{j=0}^{n} C_{r+j} u_{jn} = -C_{n+r+1}, \qquad 0 \leqq r \leqq n,$$

for $n = k - 1$ and $n = k$. Also, compute λ_k and γ_k from (8) and (14), with $n = k - 1$. Then compute recursively as follows, for $k \leqq n \leqq m - 1$:

$$\lambda_{n+1} = \sum_{j=0}^{n+1} C_{n+j+1} u_{jn}, \tag{16}$$
$$\gamma_{n+1} = \sum_{j=0}^{n+1} C_{n+j+2} u_{jn},$$

$$u_{s,n+1} = (\lambda_n^{-1}\gamma_n - \lambda_{n+1}^{-1}\gamma_{n+1})u_{sn} + u_{s-1,n} - \lambda_n^{-1}\lambda_{n+1}u_{s,n-1}, \qquad 0 \leqq s \leqq n + 1, \tag{17}$$

where $u_{-1,n} = u_{n+1,n-1} = 0$, $u_{n+1,n} = 1$,

$$b_{rsm} = b_{r-1,s+1,m} + \lambda_m^{-1}(u_{r,m-1}u_{s+1,m} - u_{rm}u_{s+1,m-1}), \quad 0 \leqq r \leqq s \leqq m, \tag{18}$$

where $b_{-1,s+1,m} = b_{r-1,m+1,m} = 0$. Then, compute λ_{m+1} and H_{m+1}^{-1}:

$$\lambda_{m+1} = \sum_{j=0}^{m+1} C_{m+j+1} u_{jm}, \tag{19}$$

$$b_{rs,m+1} = b_{rsm} + \lambda_{m+1}^{-1} u_{rm} u_{sm}, \quad 0 \leqq r \leqq s \leqq m + 1, \tag{20}$$
$$b_{rs,m+1} = b_{sr,m+1}, \qquad 0 \leqq s < r \leqq m + 1.$$

At first glance, it may seem more reasonable to employ (17) up to $n = m$, rather than to $m - 1$, and to compute $b_{rs,m+1}$ from (18), with m replaced by $m + 1$, thus dispensing with (19) and (20). However, a critical examination shows that this would require the use of C_{2m+3}, (in (16), with $n = m$), which does not appear in H_{m+1}. The algorithm as stated does not suffer from this defect.

If all the matrices $H_0, H_1, \cdots, H_{m+1}$ are nonsingular, then one can take

$k = 1$ in the above algorithm, and use starting conditions

$$u_{00} = C_0^{-1}C_1,$$
$$u_{01} = (C_0C_2 - C_1^2)^{-1}(C_1C_3 - C_2^2),$$
$$u_{11} = (C_0C_2 - C_1^2)^{-1}(C_1C_2 - C_0C_3),$$

and

$$\lambda_0 = C_0, \qquad \gamma_0 = C_1.$$

4. Connection with the theory of orthogonal polynomials. If the matrix H_n is positive definite, it is known [1, p. 5], that there exists a nondecreasing function $F(x)$ such that

$$C_r = \int_{-\infty}^{\infty} x_r \, dF(x), \qquad 0 \leqq r \leqq n.$$

There is a sequence [4, pp. 24–25], $P_0(x)$, $P_1(x)$, $\cdots$, $P_n(x)$, of polynomials

$$P_m(x) = \sum_{r=0}^{m} p_{rm} x^r$$

such that $p_{mm} > 0$ and

$$\int_{-\infty}^{\infty} P_m(x)P_k(x) \, dF(x) = \delta_{mk}.$$

The kernel function

$$K_n(x, y) = \sum_{m=0}^{n} P_m(x)P_m(y)$$

has been extensively studied [4, pp. 37–43]. It can be shown that

$$K_n(x, y) = \sum_{r,s=0}^{n} b_{rsn} x^r y^s;$$

that is, $K_n(x, y)$ is the generating function for the inverse of H_n. This fact appears to have been ignored, or at least not applied in the problem of computing H_n^{-1}. Of course, the solution of least squares curve fitting problems by means of orthogonal polynomials is an old technique [2, pp. 163–173], but the extension to matrix inversion has not been carried out, to the author's knowledge. The algorithm given here can be derived, for the case where H_n is positive definite, from the properties of the orthogonal polynomials and the kernel function. However, the derivation given here is free of the assumption of positive definiteness, is self-contained, and is elementary in nature.

A similar connection exists between the author's previous work on Toeplitz matrices and the theory of polynomials orthogonal on the unit circle [4, pp. 280–288]. However, the author had not perceived this connection when that work was in progress.

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