EXISTENCE OF SOLUTIONS OF FUNCTIONAL DIFFERENTIAL SYSTEMS WITH PRESCRIBED ASYMPTOTIC BEHAVIOR

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We consider the $n \times n$ $(n \geq 1)$ system of functional differential equations

$$X' = FX, \; t > t_0. \tag{1}$$

For now we make no specific assumptions on the form of the functional F. For example, (1) may be a system of ordinary differential equations, an integro–differential system, a system with one or more deviating arguments, or a combination of these. To allow for the possibility that the values of $(FX)(t)$ for $t \geq t_0$ may depend on the values of $X(\tau)$ for some $\tau < t_0$ (as in the case of a delay equation, for example), we make the following definition.

Definition 1. *If $-\infty < t_0 < \infty$, then $\mathcal{C}_n(t_0)$ is the space of continuous n-vector functions $X = (x_1, \dots, x_n)$ on $(-\infty, \infty)$ which are constant on $(-\infty, t_0]$, with the topology induced by the following definition of convergence:*

$$X_j \to X \quad as \quad j \to \infty$$

if

$$\lim_{j\to\infty} [\sup_{-\infty < t \leq T} \|X_j(t) - X(t)\|] = 0$$

for every T in $(-\infty, \infty)$. (Here $\| \cdot \|$ is any convenient vector norm.)

Notice that $\mathcal{C}_n(t_1) \subset \mathcal{C}_n(t_0)$ if $t_0 \leq t_1$. We will say X is a *solution of (1) on* $[t_0, \infty)$ if $X \in \mathcal{C}_n(a)$ for some $a \leq x_0$ and X satisfies (1) for $t \geq t_0$ (derivative from the right at t_0). We are interested in giving conditions on the functional F which imply that (1) has a solution $\hat{X}$ such that $\lim_{t\to\infty} \hat{X}(t) = C$, where C is a given constant vector.

The Schauder–Tychonoff theorem has proved to be a powerful tool for establishing existence theorems of the kind that interest us here. More precisely, the following special case of this theorem, which is essentially the form stated by Coppel [1] has yielded many useful results.

Lemma 1. *Let $\mathcal{S}$ be a closed convex subset of $C_n(t_0)$, and suppose that $\mathcal{T}$ is a transformation of $\mathcal{S}$ such that (a) $\mathcal{T}(S) \subset S$; (b) $\mathcal{T}$ is continuous (i.e., if $\{X_j\} \subset S$ and $X_j \to X$, then $\mathcal{T}X_j \to \mathcal{T}X$); and (c) the family of functions $\mathcal{T}(\mathcal{S})$ is uniformly bounded and equicontinuous on every compact subinterval of $[t_0, \infty)$. Then there is an $\hat{X}$ in $\mathcal{S}$ such that $\mathcal{T}\hat{X} = \hat{X}$.*

The following theorem illustrates one way in which Lemma 1 can be applied to our problem. We omit the proof, since this theorem follows from Theorem 3, below.

Theorem 1. *Suppose that there are constants a and M $(M > 0)$ and a continuous function $w\colon [a, \infty) \to (0, \infty)$ such that $FX \in C_n[a, \infty)$ and $\|(FX)(t)\| \leq w(t)$ for $t \geq a$ whenever*

$$X \in \mathcal{C}_n(a) \text{ and } \|X(t)\| \leq M,\ t \geq a. \tag{2}$$

Suppose further that

$$\int_a^\infty w(s)\, ds < \infty,$$

and that $\lim_{j\to\infty}(FX_j)(t) = (FX)(t)$ (pointwise) if each X_j satisfies (2) and $X_j \to X$. Let C be a given constant, with $\|C\| < M$. Then the system (1) has a solution $\hat{X}$ on some interval $[t_0, \infty)$, such that $\lim_{t\to\infty}\hat{X}(t) = C$.

Although useful results can be obtained from this thereom, it is clear that the integrability condition on the functional F is very strong, since it implies that the integrals

$$\int_t^\infty \|(FX)(s)\| ds,\ X \in \mathcal{S}, \tag{3}$$

all converge, and even uniformly for all X in $\mathcal{S}$ (i.e. $\int_t^\infty \|FX\|\,ds \le \int_t^\infty w(s)\,ds$). It is quite possible to obtain useful results without requiring that the integrals (3) converge at all, so long as the integrals $\int^\infty (FX)(s)\,ds$ ($X \in \mathcal{S}$) converge in the ordinary (i.e., perhaps conditional) sense, and satisfy a uniform estimate of the form

$$\left\| \int_t^\infty (FX)(s)\,ds \right\| \le \rho(t), \;\; X \in \mathcal{S}, \tag{4}$$

for some function ρ such that $\lim_{t\to\infty} \rho(t) = 0$. Moreover, it is important to exploit not just the assumption that the integrals in (4) converge, but also their rate of convergence. *Whenever possible, we should integrate before taking absolute values.* This point is often missed.

The author has pursued this theme in several papers (see, e.g., [2]–[5]). The results given here have been extended in [5].

Consider the following classical result for the linear system

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \tag{5}$$

which follows easily from Theorem 1.

Theorem 2. *Suppose that $\{a_{ij}\}$ are continuous on $[a,\infty)$ and $\int_a^\infty |a_{ij}(t)|\,dt < \infty$ for $1 \le i,j \le n$. Let $C = (c_1, c_2, \dots, c_n)$ be a given constant vector. Then the system (5) has a solution $\hat{X}$ such that* $\lim_{t\to\infty} \hat{X} = C$.

Example 1. Consider the system

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \frac{\sin t}{t^\alpha} \begin{bmatrix} a_1 t^{-1} & b_1 \\ a_2 t^{-2} & b_2 t^{-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; t \ge a > 0, \tag{6}$$

where $b_1, b_2 \ne 0$ and $\alpha > 0$. Since

$$\int^\infty t^{-\alpha} |\sin t|\,dt \begin{cases} = \infty \text{ if } & \alpha \le 1, \\ < \infty \text{ if } & \alpha > 1, \end{cases}$$

Theorem 2 does not apply to this system if $0 \leq \alpha \leq 1$; if $\alpha > 1$, then Theorem 2 implies that if c_1 and c_2 are given constants, then (6) has a solution $\hat{X} = (\hat{x}_1, \hat{x}_2)$ such that

$$\lim_{t\to\infty} x_i(t) = c_i, \;\; i = 1, 2.$$

Theorem 2 provides no estimate of the *order* of convergence here, but it is straightforward to show that if $\alpha > 1$, then

$$x_1(t) = c_1 + O(t^{-\alpha+1}) \text{ and } x_2(t) = c_2 + O(t^{-\alpha}).$$

However, a more efficient use of integrability conditions for problems like this will show later that the true situation is as follows:

Suppose that $\alpha > 0$. Then:

(i) If c_1 is arbitrary and $c_2 \neq 0$, then (6) has a solution $\hat{X}$ such that

$$x_1(t) = c_1 + O(t^{-\alpha}) \textit{ and } x_2(t) = c_2 + O(t^{-\alpha-1}).$$

(ii) If c_1 is arbitrary and $c_2 = 0$, then (6) has a solution $\hat{X}$ such that

$$x_1(t) = c_1 + O(t^{-\alpha-1}) \textit{ and } x_2(t) = O(t^{-\alpha-2}).$$

The following theorem makes more efficient use of the Schauder–Tychonoff theorem (Lemma 1). Here it is convenient to rewrite (1) in component form as

$$x_i' = f_i X, \;\; 1 \leq i \leq n, \;\; t > t_0.$$

Theorem 3. *Let $C = (c_1, c_2, \ldots, c_n)$ be a given constant vector. Let $\gamma_1, \ldots, \gamma_n$ be continuous, positive and nonincreasing on $[t_0, \infty)$ and let $M_1, \ldots, M_n$ be positive constants. Let $\mathcal{S}$ be the set of functions $X = (x_1, \ldots, x_n)$ in $C_n(t_0)$ such that*

$$|\, x_i(t) - c_i \,| \leq \; M_i \gamma_i(t), \;\; t \geq t_0, \;\; 1 \leq i \leq n.$$

Suppose that F satisfies the following assumptions:

(i) $FX \in C_n[t_0, \infty)$ if $X \in \mathcal{S}$.

(ii) The family of functions $\mathcal{F} = \{FX \mid X \in \mathcal{S}\}$ is uniformly bounded on each subinterval of $[t_0, \infty)$.

(iii) If $\{X_j\} \subset \mathcal{S}$ and $X_j \to X$ (uniform convergence on every interval $(-\infty, T]$), then

$$\lim_{j\to\infty}(FX_j)(t) \;=\; (FX)(t) \text{ (pointwise)},\; t \geq t_0.$$

(iv) The integrals $\int^\infty (FX)(s)\,ds$ ($X \in \mathcal{S}$), converge, perhaps conditionally, and there are nonincreasing functions $\rho_1, \rho_2, \ldots, \rho_n$ such that

$$0 < \rho_i(t) \leq M_i\gamma_i(t),\; 1 \leq i \leq n, \tag{7}$$

$$\lim_{t\to\infty}\rho_i(t) = 0,\; 1 \leq i \leq n,$$

and, if $X \in \mathcal{S}$ and $t \geq t_0$,

$$\left|\int_t^\infty f_iX\; ds\right| \leq \rho_i(t),\; 1 \leq i \leq n. \tag{8}$$

Then (1) has a solution $\hat{X}$ on $[t_0, \infty)$ such that

$$|\,\hat{x}_i(t) - c_i\,| \leq \rho_i(t),\; t \geq t_0,\; 1 \leq i \leq n.$$

Proof. We define the transformation $Y = \mathcal{T}X$ in terms of components as

$$y_i(t) = \begin{cases} c_i - \int_t^\infty (f_iX)(s)\,ds, & t \geq t_0, \\ \\ c_i - \int_{t_0}^\infty (f_iX)(s)\,ds, & t < t_0. \end{cases} \qquad 1 \leq i \leq n. \tag{9}$$

Therefore, from (7),(8) and (9),

$$|y_i(t) - c_i| \leq \rho_i(t) \leq M_i\gamma_i(t);$$

hence, $\mathcal{T}(\mathcal{S}) \subset \mathcal{S}$, and $\mathcal{T}(\mathcal{S})$ is uniformly bounded on $[t_0, \infty)$, since $\mathcal{S}$ is. Differentiating (9) shows that $y_i'(t) = (f_iX)(t)$ if $t \geq t_0$ and $y_i'(t) = 0$ if $t < t_0$; hence, the mean value theorem and assumption (iii) imply that the family $\mathcal{T}(\mathcal{S})$ is equicontinuous on every interval $(-\infty, T]$. The proof that $\mathcal{T}$ is continuous is somewhat more delicate than in Theorem 1, since the integrals in question may converge conditionally. Suppose that $\{X_j\} \subset \mathcal{S}$ and $X_j \to X = (x_1, x_2, \dots, x_n)$ as $j \to \infty$. Denote $X_j = (x_{1j}, x_{2j}, \dots, x_{nj})$; then

$$y_{ij}(t) - y_i(t) = \begin{cases} \int_t^\infty (f_iX_j - f_iX)\,ds, & t \geq t_0, \\ \int_{t_0}^\infty (f_iX_j - f_iX)\,ds, & t < t_0. \end{cases}$$

Let

$$H_{ij} = \sup_{-\infty < t < \infty} |y_{ij}(t) - y_i(t)|, \ 1 \leq i \leq n, \ j = 1, 2, \cdots.$$

Then, if $t_1 \geq t_0$,

$$H_{ij} \leq \int_{t_0}^{t_1} |f_iX_j - f_iX|\,ds + \Big|\int_{t_1}^\infty f_iX_j\,ds\Big| + \Big|\int_{t_1}^\infty f_iX\,ds\Big|$$

$$\leq \int_{t_0}^{t_1} |f_iX_j - f_iX|\,ds + 2\rho_i(t_1),$$

from (8). Since the last integrand is uniformly bounded on $[t_0, t_1]$ for all j and $\to 0$ pointwise as $t \to \infty$, the last integral $\to 0$ as $t \to \infty$, by the bounded convergence theorem. Hence,

$$\overline{\lim}_{j\to\infty} H_{ij} \leq 2\rho_i(t_1)$$

for every t_1. Since $\lim_{t_1\to\infty} \rho_i(t_1) = 0$, this implies that $\lim_{j\to\infty} H_{ij} = 0$ for $1 \leq i \leq n$; that is, $y_{ij}(t) \to y_i(t)$ uniformly on $(-\infty, \infty)$ as $j \to \infty$. Now Lemma 1 implies the conclusion.

Theorem 4. *Let $\mathcal{S}$, $\gamma_1, \gamma_2, \dots, \gamma_n$, $M_1, M_2, \dots M_n$ and C be as in Theorem 3, and suppose that F satisfies assumptions (i) and (iii) on the set $\mathcal{S}$ of functions $X = (x_1, \dots, x_n)$ in $C_n(t_0)$ such that*

$$|\, x_i(t) - c_i \,| \leq \ M_i\gamma_i(t), \ t \geq t_0, \ 1 \leq i \leq n.$$

Suppose further that $\int^\infty FC\,dt$ *converges (perhaps conditionally) and that*

$$\sup_{t\geq t_0}(\gamma_i(t))^{-1}\Big|\int_t^\infty f_iC\,ds\Big| = A_i < \infty,\ 1 \leq\ i \leq n.$$

Suppose also that

$$|(f_iX)(t) - (f_iC)(t)| \leq M_i w_i(t),\ 1 \leq i \leq n,\ t \geq t_0,$$

for all X *in* $\mathcal{S}$*, where*

$$\sup_{t\geq t_0}(\gamma_i(t))^{-1}\int_t^\infty w_i\,ds = \theta_i < 1,\ 1 \leq i \leq n.$$

Finally, let

$$M_i \geq \frac{A_i}{1-\theta_i}.$$

Then the conclusion of Theorem 3 holds.

Proof. See [5].

We now apply Theorem 4 to the linear system (5).

Theorem 5. *Suppose that* $\{a_{ij}\}$ *are continuous on* $[a,\infty)$ *and* $\int_a^\infty a_{ij}(t)\,dt$ *converges (perhaps conditionally) for* $1 \leq i,j \leq n$*. Let* $C = (c_1, c_2, \ldots, c_n)$ *be a given constant vector, and suppose that* $\gamma_1, \gamma_2, \ldots, \gamma_n$ *are nonincreasing positive functions on* $[a,\infty)$ *such that*

$$\int_t^\infty f_iC\,ds = O(\gamma_i(t)),\ 1 \leq i \leq n,$$

and define

$$w_i(t) = \sum_{j=1}^n |a_{ij}(t)|\gamma_j(t).$$

Suppose further that

$$\overline{\lim}(\gamma_i(t))^{-1}\int_t^\infty w_i(s)\,ds = \theta_i < 1,\ 1 \leq i \leq n.$$

Then the system $X' = AX$ *has a solution* $\hat{X}$ *such that*

$$\hat{x}_i(t) = c_i + O(\gamma_i(t)),$$

for $1 \leq i \leq n$*; moreover, if* $\theta_i = 0$*, then this can be replaced by more precise estimate*

$$\hat{x}_i(t) = c_i + \int_t^\infty f_i C\, ds + O\Big(\int_t^\infty w_i\, ds\Big).$$

<u>Proof</u>. See [5].

<u>Example 1 (Continuation)</u>. As mentioned above, a linear system $X' = A(t)X$ (with A continuous on $[0, \infty)$) has a solution $\hat{X}$ satisfying an arbitrary final condition $\lim_{t\to\infty} \hat{x}(t) = C$ if $\int^\infty \parallel A(t) \parallel dt < \infty$. This result can be obtained from Theorem 5 by simply taking $\gamma_i = 1,\ 1 \leq i \leq n$. The system (6) with $b_1 \neq 0$ does not satisfy this integrability condition if $\alpha \leq 1$; moreover, even if $\alpha > 1$, the standard theorem merely implies that if c_1 and c_2 are given constants, then (6) has a solution $(\hat{x}_1, \hat{x}_2)$ such that $\lim_{t\to\infty} \hat{x}_i(t) = c_i$ $(i = 1, 2)$. However, Theorem 5 implies that if $\alpha > 0$ and (c_1, c_2) is arbitrary, then (6) has a solution $(\hat{x}_1, \hat{x}_2)$ such that

$$\hat{x}_1(t) = c_1(1 - a_1 S_{\alpha+1}(t)) - b_1 c_2 S_\alpha(t) + 0(t^{-2\alpha})$$

and

$$\hat{x}_2(t) = -a_2 c_1 S_{\alpha+2}(t) + c_2(1 - b_2 S_{\alpha+1}(t)) + 0(t^{-2\alpha-1}),$$

where

$$S_\beta(t) = \int_t^\infty s^{-\beta} \sin s\ ds = 0(t^{-\beta}),\ \ \beta > 0.$$

This conclusion is obtained by letting $\gamma_1(t) = t^{-\alpha}$ and $\gamma_2(t) = t^{-\alpha-1}$. A sharper result is available if $c_2 = 0$; i.e., for every constant c_1, (6) has a solution $(\hat{x}_1, \hat{x}_2)$ such that

$$\hat{x}_1(t) = c_1(1 - a_1 S_{\alpha+1}(t)) + 0(t^{-2\alpha-1}),$$

and

$$\hat{x}_2(t) = -a_2 c_1 S_{\alpha+2}(t) + 0(t^{-2\alpha-2}).$$

This is obtained by letting $\gamma_1(t) = t^{-\alpha-1}$ and $\gamma_2(t) = t^{-\alpha-2}$.

We will now obtain a global result for the nonlinear integral equation

$$x'(t) = g(t)(x(t))^\alpha + \int_0^t P(t,\tau)(x(\tau))^\beta\, d\tau,\ \ t > 0, \tag{10}$$

where $g \in C[0,\infty)$ and P is continuous on $[0,\infty) \times [0,\infty)$.

Theorem 6. *Suppose that*

$$\int_t^\infty g(s)\,ds = O(\gamma(t)), \tag{11}$$

$$\int_t^\infty |g(s)|\gamma(s)\,ds = O(\gamma(t)), \tag{12}$$

$$\int_t^\infty \int_0^s P(s,\tau)\,d\tau\,ds = O(\gamma(t)), \tag{13}$$

and

$$\int_t^\infty \int_0^s |P(s,\tau)|\gamma(\tau)\,d\tau\,ds = O(\gamma(t)), \tag{14}$$

where γ is positive and nonincreasing on $[0,\infty)$, and $\lim_{t\to\infty}\gamma(t) = 0$. Suppose also that $0 < \theta < 1$. Then there is a constant $c_0 > 0$ such that (10) has a solution $\hat{x}$ on $[0,\infty)$ which satisfies the following conditions:

$$|\hat{x}(t) - c| \le \theta c \ (t \ge 0)\ , \hat{x}(t) = c + O(\gamma(t)).$$

provided that either

(a) $\alpha,\beta > 1$ and $0 < c < c_0$; or

(b) $\alpha,\beta < 1$ and $c > c_0$.

(Notice that (11) and (12) do not imply that $\int^\infty |g(s)|\,ds < \infty$, nor do (13) and (14) imply that $\int^\infty \int_0^s |P(s,\tau)|\,d\tau\,ds < \infty$.)

Proof. For convenience, normalize γ so that $\gamma(0) = 1$. Here

$$(Fx)(t) = g(t)(x(t))^\alpha + \int_0^t P(t,\tau)(x(\tau))^\beta\,d\tau.$$

and

$$(Fc)(t) = c^\alpha \int_t^\infty g(s)\,ds + c^\beta \int_0^t P(t,\tau)\,d\tau.$$

If $c > 0$, let

$$\mathcal{S} = \{x \in C[0,\infty) \;\Big|\; |x(t) - c| \le \theta c\gamma(t),\ t \ge 0\}.$$

Then, if $x \in \mathcal{S}$, $|x(t) - c| \le \theta c$ $(t \ge 0)$, since γ is nonincreasing. Obviously, F satisfies assumptions (i), (ii), and (iii) of Theorem 3. Now,

$$\int_t^\infty Fx\,ds = \int_t^\infty Fc\,ds + \int_t^\infty (Fx - Fc)\,ds$$

$$= c^\alpha \int_t^\infty g(s)\,ds + c^\beta \int_t^\infty \int_0^s P(s,\tau)\,d\tau\,ds$$

$$+ \int_t^\infty g(s)[(x(s))^\alpha - c^\alpha]\,ds \int_t^\infty \int_0^s P(s,\tau)[(x(s))^\beta - c^\beta]\,ds.$$

By the mean value theorem,

$$|x^\alpha - c^\alpha| \le K(\alpha) =_{df} |\alpha|[(1 \pm \theta)c]^{\alpha-1}|x - c|$$

if $|x - c| \le \theta c$ (with "+" if $\alpha > 1$, "−" if $\alpha < 1$. Since $|x(t) - c| \le \theta c\gamma(t)$ if $x \in \mathcal{S}$, this means that

$$\left|\int_t^\infty Fx\,ds\right| \le c^\alpha \left|\int_t^\infty g(s)\,ds\right| + c^\beta \left|\int_t^\infty \int_0^s P(s,\tau)\,d\tau\,ds\right|$$

$$Kc^\alpha \int_t^\infty |g(s)|\gamma(s)\,ds + Kc^\beta \int_t^\infty \int_0^s |P(s,\tau)|\gamma(\tau)\,d\tau\,ds,$$

where K is a constant which does not depend on c. Since all four integrals on the right are $O(\gamma(t))$, this means that

$$(\gamma(t))^{-1}\left|\int_t^\infty Fx\,ds\right| \leq Ac^\alpha + Bc^\beta,\ x\in\mathcal{S},\ t\geq 0,$$

where A and B are constants which do not depend on c. Since our requirement is that

$$(\gamma(t))^{-1}\left|\int_t^\infty Fx\,ds\right| \leq \theta c,\ x\in,\mathcal{S},\ t\geq 0,$$

we have only to choose c so that

$$Ac^{\alpha-1} + Bc^{\beta-1} \leq \theta.$$

This is true for c sufficiently large if $\alpha,\beta<1$, or for c sufficiently small if $\alpha,\beta>1$.

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