EXISTENCE OF SOLUTIONS OF FUNCTIONAL DIFFERENTIAL
SYSTEMS WITH PRESCRIBED ASYMPTOTIC BEHAVIOR

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We consider the $n \times n$ ($n \geq 1$) system of functional differential equations

\[ X' = FX, \quad t > t_0. \]

For now we make no specific assumptions on the form of the functional $F$. For example, (1) may be a system of ordinary differential equations, an integro–differential system, a system with one or more deviating arguments, or a combination of these. To allow for the possibility that the values of $(FX)(t)$ for $t \geq t_0$ may depend on the values of $X(\tau)$ for some $\tau < t_0$ (as in the case of a delay equation, for example), we make the following definition.

**Definition 1.** If $-\infty < t_0 < \infty$, then $C_n(t_0)$ is the space of continuous $n$-vector functions $X = (x_1, \ldots, x_n)$ on $(-\infty, \infty)$ which are constant on $(-\infty, t_0]$, with the topology induced by the following definition of convergence:

\[ X_j \to X \quad \text{as} \quad j \to \infty \]

if

\[ \lim_{j \to \infty} \left[ \sup_{-\infty < t \leq T} \|X_j(t) - X(t)\| \right] = 0 \]

for every $T$ in $(-\infty, \infty)$. (Here $\| \cdot \|$ is any convenient vector norm.)
Notice that $C_n(t_1) \subset C_n(t_0)$ if $t_0 \leq t_1$. We will say $X$ is a solution of (1) on $[t_0, \infty)$ if $X \in C_n(a)$ for some $a \leq x_0$ and $X$ satisfies (1) for $t \geq t_0$ (derivative from the right at $t_0$). We are interested in giving conditions on the functional $F$ which imply that (1) has a solution $\hat{X}$ such that $\lim_{t \to \infty} \dot{X}(t) = C$, where $C$ is a given constant vector.

The Schauder–Tychonoff theorem has proved to be a powerful tool for establishing existence theorems of the kind that interest us here. More precisely, the following special case of this theorem, which is essentially the form stated by Coppel [1] has yielded many useful results.

**Lemma 1.** Let $S$ be a closed convex subset of $C_n(t_0)$, and suppose that $T$ is a transformation of $S$ such that (a) $T(S) \subset S$; (b) $T$ is continuous (i.e., if $\{X_j\} \subset S$ and $X_j \to X$, then $TX_j \to TX$); and (c) the family of functions $T(S)$ is uniformly bounded and equicontinuous on every compact subinterval of $[t_0, \infty)$. Then there is an $\hat{X}$ in $S$ such that $T \hat{X} = \hat{X}$.

The following theorem illustrates one way in which Lemma 1 can be applied to our problem. We omit the proof, since this theorem follows from Theorem 3, below.

**Theorem 1.** Suppose that there are constants $a$ and $M$ ($M > 0$) and a continuous function $w: [a, \infty) \to (0, \infty)$ such that $FX \in C_n[a, \infty)$ and $\|(FX)(t)\| \leq w(t)$ for $t \geq a$ whenever

\[
\text{if } X \in C_n(a) \text{ and } \|X(t)\| \leq M, \ t \geq a. 
\]

Suppose further that

\[
\int_a^\infty w(s) \, ds < \infty,
\]

and that $\lim_{j \to \infty}(FX_j)(t) = (FX)(t)$ (pointwise) if each $X_j$ satisfies (2) and $X_j \to X$. Let $C$ be a given constant, with $\|C\| < M$. Then the system (1) has a solution $\hat{X}$ on some interval $[t_0, \infty)$, such that $\lim_{t \to \infty} \hat{X}(t) = C$.

Although useful results can be obtained from this theorem, it is clear that the integrability condition on the functional $F$ is very strong, since it implies that the integrals

\[
\int_t^\infty \|(FX)(s)\| \, ds, \ X \in S,
\]
all converge, and even uniformly for all $X$ in $S$ (i.e. \(\int_t^\infty \|FX\| ds \leq \int_t^\infty w(s) ds\)). It is quite possible to obtain useful results without requiring that the integrals (3) converge at all, so long as the integrals \(\int_0^\infty (FX)(s) ds\) ($X \in S$) converge in the ordinary (i.e., perhaps conditional) sense, and satisfy a uniform estimate of the form

\[
\| \int_t^\infty (FX)(s) ds \| \leq \rho(t), \quad X \in S,
\]

for some function $\rho$ such that $\lim_{t \to \infty} \rho(t) = 0$. Moreover, it is important to exploit not just the assumption that the integrals in (4) converge, but also their rate of convergence. *Whenever possible, we should integrate before taking absolute values.* This point is often missed.

The author has pursued this theme in several papers (see, e.g., [2]–[5]). The results given here have been extended in [5].

Consider the following classical result for the linear system

\[
\begin{bmatrix}
x'_1 \\
x'_2 \\
\vdots \\
x'_n
\end{bmatrix} =
\begin{bmatrix}
a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\
a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t)
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{bmatrix},
\]

which follows easily from Theorem 1.

**Theorem 2.** Suppose that \(\{a_{ij}\}\) are continuous on \([a, \infty)\) and \(\int_a^\infty |a_{ij}(t)| dt < \infty\) for \(1 \leq i, j \leq n\). Let \(C = (c_1, c_2, \ldots, c_n)\) be a given constant vector. Then the system (5) has a solution \(\dot{X}\) such that \(\lim_{t \to \infty} \dot{X} = C\).

**Example 1.** Consider the system

\[
\begin{bmatrix}
x'_1 \\
x'_2
\end{bmatrix} = \frac{\sin t}{t^\alpha} \begin{bmatrix}
a_1 t^{-1} & b_1 \\
a_2 t^{-2} & b_2 t^{-1}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}, \quad t \geq a > 0,
\]

where $b_1, b_2 \neq 0$ and $\alpha > 0$. Since

\[
\int_a^\infty t^{-\alpha} |\sin t| dt \begin{cases} = \infty & \text{if } \alpha \leq 1, \\
< \infty & \text{if } \alpha > 1,
\end{cases}
\]
Theorem 2 does not apply to this system if $0 \leq \alpha \leq 1$; if $\alpha > 1$, then Theorem 2 implies that if $c_1$ and $c_2$ are given constants, then (6) has a solution $\hat{X} = (\hat{x}_1, \hat{x}_2)$ such that

$$\lim_{t \to \infty} x_i(t) = c_i, \ i = 1, 2.$$ 

Theorem 2 provides no estimate of the order of convergence here, but it is straightforward to show that if $\alpha > 1$, then

$$x_1(t) = c_1 + O(t^{-\alpha+1}) \text{ and } x_2(t) = c_2 + O(t^{-\alpha}).$$

However, a more efficient use of integrability conditions for problems like this will show later that the true situation is as follows:

Suppose that $\alpha > 0$. Then:

(i) If $c_1$ is arbitrary and $c_2 \neq 0$, then (6) has a solution $\hat{X}$ such that

$$x_1(t) = c_1 + O(t^{-\alpha}) \text{ and } x_2(t) = c_2 + O(t^{-\alpha-1}).$$

(ii) If $c_1$ is arbitrary and $c_2 = 0$, then (6) has a solution $\hat{X}$ such that

$$x_1(t) = c_1 + O(t^{-\alpha-1}) \text{ and } x_2(t) = O(t^{-\alpha-2}).$$

The following theorem makes more efficient use of the Schauder–Tychonoff theorem (Lemma 1). Here it is convenient to rewrite (1) in component form as

$$x_i' = f_i X, \ 1 \leq i \leq n, \ t > t_0.$$ 

**Theorem 3.** Let $C = (c_1, c_2, \ldots, c_n)$ be a given constant vector. Let $\gamma_1, \ldots, \gamma_n$ be continuous, positive and nonincreasing on $[t_0, \infty)$ and let $M_1, \ldots, M_n$ be positive constants. Let $S$ be the set of functions $X = (x_1, \ldots, x_n)$ in $C_n(t_0)$ such that

$$| x_i(t) - c_i | \leq M_i \gamma_i(t), \ t \geq t_0, \ 1 \leq i \leq n.$$
Suppose that \( F \) satisfies the following assumptions:

(i) \( FX \in C_n[t_0, \infty) \) if \( X \in S \).

(ii) The family of functions \( F = \{FX \mid X \in S\} \) is uniformly bounded on each subinterval of \([t_0, \infty)\).

(iii) If \( \{X_j\} \subset S \) and \( X_j \to X \) (uniform convergence on every interval \(( -\infty, T ] \) ), then

\[
\lim_{j \to \infty} (FX_j)(t) = (FX)(t) \text{ (pointwise)}, \quad t \geq t_0.
\]

(iv) The integrals \( \int_0^\infty (FX)(s) \, ds \) \( X \in S \), converge, perhaps conditionally, and there are nonincreasing functions \( \rho_1, \rho_2, \ldots, \rho_n \) such that

\[
0 < \rho_i(t) \leq M_i \gamma_i(t), \quad 1 \leq i \leq n,
\]

\[
\lim_{t \to \infty} \rho_i(t) = 0, \quad 1 \leq i \leq n,
\]

and, if \( X \in S \) and \( t \geq t_0 \),

\[
| \int_t^\infty f_i X \, ds | \leq \rho_i(t), \quad 1 \leq i \leq n.
\]

Then (1) has a solution \( \hat{X} \) on \([t_0, \infty)\) such that

\[
| \hat{x}_i(t) - c_i | \leq \rho_i(t), \quad t \geq t_0, \quad 1 \leq i \leq n.
\]

**Proof.** We define the transformation \( Y = T X \) in terms of components as

\[
y_i(t) = \begin{cases} 
  c_i - \int_t^\infty (f_i X)(s) \, ds, & t \geq t_0, \\
  c_i - \int_{t_0}^\infty (f_i X)(s) \, ds, & t < t_0.
\end{cases} \quad 1 \leq i \leq n.
\]

Therefore, from (7), (8) and (9),

\[
|y_i(t) - c_i| \leq \rho_i(t) \leq M_i \gamma_i(t);
\]
hence, $T(S) \subset S$, and $T(S)$ is uniformly bounded on $[t_0, \infty)$, since $S$ is. Differentiating (9) shows that $y_i'(t) = (f_iX)(t)$ if $t \geq t_0$ and $y_i'(t) = 0$ if $t < t_0$; hence, the mean value theorem and assumption (iii) imply that the family $T(S)$ is equicontinuous on every interval $(-\infty, T]$. The proof that $T$ is continuous is somewhat more delicate than in Theorem 1, since the integrals in question may converge conditionally. Suppose that $\{X_j\} \subset S$ and $X_j \to X = (x_1, x_2, \ldots, x_n)$ as $j \to \infty$. Denote $X_j = (x_{1j}, x_{2j}, \ldots, x_{nj})$; then

$$y_{ij}(t) - y_i(t) = \begin{cases} \int_t^\infty (f_iX_j - f_iX) \, ds, & t \geq t_0, \\ \int_{t_0}^\infty (f_iX_j - f_iX) \, ds, & t < t_0. \end{cases}$$

Let

$$H_{ij} = \sup_{-\infty < t < \infty} |y_{ij}(t) - y_i(t)|, \quad 1 \leq i \leq n, \quad j = 1, 2, \ldots.$$

Then, if $t_1 \geq t_0$,

$$H_{ij} \leq \int_{t_0}^{t_1} |f_iX_j - f_iX| \, ds + \left| \int_{t_1}^\infty f_iX_j \, ds \right| + \left| \int_{t_1}^\infty f_iX \, ds \right|$$

$$\leq \int_{t_0}^{t_1} |f_iX_j - f_iX| \, ds + 2\rho_i(t_1),$$

from (8). Since the last integrand is uniformly bounded on $[t_0, t_1]$ for all $j$ and $\to 0$ pointwise as $t \to \infty$, the last integral $\to 0$ as $t \to \infty$, by the bounded convergence theorem. Hence,

$$\lim_{j \to \infty} H_{ij} \leq 2\rho_i(t_1)$$

for every $t_1$. Since $\lim_{t_1 \to \infty} \rho_i(t_1) = 0$, this implies that $\lim_{j \to \infty} H_{ij} = 0$ for $1 \leq i \leq n$; that is, $y_{ij}(t) \to y_i(t)$ uniformly on $(-\infty, \infty)$ as $j \to \infty$. Now Lemma 1 implies the conclusion.

**Theorem 4.** Let $S$, $\gamma_1, \gamma_2, \ldots, \gamma_n$, $M_1, M_2, \ldots, M_n$ and $C$ be as in Theorem 3, and suppose that $F$ satisfies assumptions (i) and (iii) on the set $S$ of functions $X = (x_1, \ldots, x_n)$ in $C_n(t_0)$ such that

$$|x_i(t) - c_i| \leq M_i \gamma_i(t), \quad t \geq t_0, \quad 1 \leq i \leq n.$$
Suppose further that \( \int_{\infty}^{\infty} FC \, dt \) converges (perhaps conditionally) and that
\[
\sup_{t \geq t_0} (\gamma_i(t))^{-1} \left| \int_{t}^{\infty} f_i C \, ds \right| = A_i < \infty, \quad 1 \leq i \leq n. 
\]

Suppose also that
\[
| (f_i X)(t) - (f_i C)(t) | \leq M_i w_i(t), \quad 1 \leq i \leq n, \quad t \geq t_0,
\]
for all \( X \) in \( S \), where
\[
\sup_{t \geq t_0} (\gamma_i(t))^{-1} \int_{t}^{\infty} w_i(s) \, ds = \theta_i < 1, \quad 1 \leq i \leq n.
\]

Finally, let
\[
M_i \geq \frac{A_i}{1 - \theta_i}.
\]

Then the conclusion of Theorem 3 holds.

**Proof.** See [5].

We now apply Theorem 4 to the linear system (5).

**Theorem 5.** Suppose that \( \{a_{ij}\} \) are continuous on \([a, \infty)\) and \( \int_{a}^{\infty} a_{ij}(t) \, dt \) converges (perhaps conditionally) for \( 1 \leq i, j \leq n \). Let \( C = (c_1, c_2, \ldots, c_n) \) be a given constant vector, and suppose that \( \gamma_1, \gamma_2, \ldots, \gamma_n \) are nonincreasing positive functions on \([a, \infty)\) such that
\[
\int_{t}^{\infty} f_i C \, ds = O(\gamma_i(t)), \quad 1 \leq i \leq n,
\]
and define
\[
w_i(t) = \sum_{j=1}^{n} |a_{ij}(t)| \gamma_j(t).
\]

Suppose further that
\[
\lim (\gamma_i(t))^{-1} \int_{t}^{\infty} w_i(s) \, ds = \theta_i < 1, \quad 1 \leq i \leq n.
\]

Then the system \( X' = AX \) has a solution \( \hat{X} \) such that
\[
\hat{x}_i(t) = c_i + O(\gamma_i(t)),
\]
for $1 \leq i \leq n$; moreover, if $\theta_i = 0$, then this can be replaced by more precise estimate

$$\hat{x}_i(t) = c_i + \int_t^\infty f_i C \, ds + O\left(\int_t^\infty w_i \, ds\right).$$

**Proof.** See [5].

**Example 1 (Continuation).** As mentioned above, a linear system $X' = A(t)X$ (with $A$ continuous on $[0, \infty)$) has a solution $\dot{X}$ satisfying an arbitrary final condition $\lim_{t \to \infty} \hat{x}(t) = C$ if $\int^\infty \| A(t) \| \, dt < \infty$. This result can be obtained from Theorem 5 by simply taking $\gamma_i = 1$, $1 \leq i \leq n$. The system (6) with $b_1 \neq 0$ does not satisfy this integrability condition if $\alpha \leq 1$; moreover, even if $\alpha > 1$, the standard theorem merely implies that if $c_1$ and $c_2$ are given constants, then (6) has a solution $(\hat{x}_1, \hat{x}_2)$ such that $\lim_{t \to \infty} \hat{x}_i(t) = c_i$ ($i = 1, 2$). However, Theorem 5 implies that if $\alpha > 0$ and $(c_1, c_2)$ is arbitrary, then (6) has a solution $(\hat{x}_1, \hat{x}_2)$ such that

$$\hat{x}_1(t) = c_1 (1 - a_1 S_{\alpha+1}(t)) - b_1 c_2 S_{\alpha}(t) + 0(t^{-2\alpha})$$

and

$$\hat{x}_2(t) = -a_2 c_1 S_{\alpha+2}(t) + c_2 (1 - b_2 S_{\alpha+1}(t)) + 0(t^{-2\alpha-1}),$$

where

$$S_\beta(t) = \int_t^\infty s^{-\beta} \sin s \, ds = 0(t^{-\beta}), \quad \beta > 0.$$  

This conclusion is obtained by letting $\gamma_1(t) = t^{-\alpha}$ and $\gamma_2(t) = t^{-\alpha-1}$. A sharper result is available if $c_2 = 0$; i.e., for every constant $c_1$, (6) has a solution $(\hat{x}_1, \hat{x}_2)$ such that

$$\hat{x}_1(t) = c_1 (1 - a_1 S_{\alpha+1}(t)) + 0(t^{-2\alpha-1}),$$

and

$$\hat{x}_2(t) = -a_2 c_1 S_{\alpha+2}(t) + 0(t^{-2\alpha-2}).$$

This is obtained by letting $\gamma_1(t) = t^{-\alpha-1}$ and $\gamma_2(t) = t^{-\alpha-2}$.

We will now obtain a global result for the nonlinear integral equation

(10)  

$$x'(t) = g(t)(x(t))^\alpha + \int_0^t P(t, \tau)(x(\tau))^{\beta} \, d\tau, \quad t > 0,$$
where \( g \in C[0, \infty) \) and \( P \) is continuous on \([0, \infty) \times [0, \infty)\).

**Theorem 6.** Suppose that

\[\int_{t}^{\infty} g(s) \, ds = O(\gamma(t)),\]

\[\int_{t}^{\infty} |g(s)| \gamma(s) \, ds = O(\gamma(t)),\]

\[\int_{t}^{\infty} \int_{0}^{s} P(s, \tau) \, d\tau \, ds = O(\gamma(t)),\]

and

\[\int_{t}^{\infty} \int_{0}^{s} |P(s, \tau)| \gamma(\tau) \, d\tau \, ds = O(\gamma(t)),\]

where \( \gamma \) is positive and nonincreasing on \([0, \infty)\), and \( \lim_{t \to \infty} \gamma(t) = 0 \). Suppose also that \( 0 < \theta < 1 \).

Then there is a constant \( c_0 > 0 \) such that (10) has a solution \( \hat{x} \) on \([0, \infty)\) which satisfies the following conditions:

\[|\hat{x}(t) - c| \leq \theta c \quad (t \geq 0), \quad \hat{x}(t) = c + O(\gamma(t)).\]

provided that either

(a) \( \alpha, \beta > 1 \) and \( 0 < c < c_0 \); or

(b) \( \alpha, \beta < 1 \) and \( c > c_0 \).

(Notice that (11) and (12) do not imply that \( \int_{0}^{\infty} |g(s)| \, ds < \infty \), nor do (13) and (14) imply that \( \int_{0}^{\infty} \int_{0}^{s} |P(s, \tau)| \, d\tau \, ds < \infty \).)

**Proof.** For convenience, normalize \( \gamma \) so that \( \gamma(0) = 1 \). Here

\[ (Fx)(t) = g(t)(x(t))^\alpha + \int_{0}^{t} P(t, \tau)(x(\tau))^\beta \, d\tau. \]
and

\[(Fc)(t) = c^\alpha \int_t^\infty g(s) \, ds + c^\beta \int_0^t P(t, \tau) \, d\tau.\]

If \(c > 0\), let

\[S = \{x \in C[0, \infty) \mid |x(t) - c| \leq \theta c \gamma(t), \ t \geq 0\}.\]

Then, if \(x \in S\), \(|x(t) - c| \leq \theta c\) \((t \geq 0)\), since \(\gamma\) is nonincreasing. Obviously, \(F\) satisfies assumptions (i), (ii), and (iii) of Theorem 3. Now,

\[
\int_t^\infty Fx \, ds = \int_t^\infty Fc \, ds + \int_t^\infty (Fx - Fc) \, ds
\]

\[
= c^\alpha \int_t^\infty g(s) \, ds + c^\beta \int_t^\infty \int_0^s P(s, \tau) \, d\tau \, ds
\]

\[
+ \int_t^\infty g(s)[(x(s))^{\alpha} - c^\alpha] \, ds \int_t^\infty \int_0^s P(s, \tau)[(x(s))^{\beta} - c^\beta] \, d\tau \, ds.
\]

By the mean value theorem,

\[|x^\alpha - c^\alpha| \leq K(\alpha) \equiv |\alpha| [(1 \pm \theta) c]^{\alpha - 1} |x - c|\]

if \(|x - c| \leq \theta c\) (with “+” if \(\alpha > 1\), “−” if \(\alpha < 1\)). Since \(|x(t) - c| \leq \theta c \gamma(t)\) if \(x \in S\), this means that

\[
\left| \int_t^\infty Fx \, ds \right| \leq c^\alpha \left| \int_t^\infty g(s) \, ds \right| + c^\beta \left| \int_t^\infty \int_0^s P(s, \tau) \, d\tau \, ds \right|
\]

\[
K c^\alpha \int_t^\infty |g(s)| \gamma(s) \, ds + K c^\beta \int_t^\infty \int_0^s |P(s, \tau)| \gamma(\tau) \, d\tau \, ds,
\]
where \( K \) is a constant which does not depend on \( c \). Since all four integrals on the right are \( O(\gamma(t)) \), this means that

\[
(\gamma(t))^{-1} \left| \int_{t}^{\infty} Fx\,ds \right| \leq Ac^\alpha + Bc^\beta, \quad x \in S, \quad t \geq 0,
\]

where \( A \) and \( B \) are constants which do not depend on \( c \). Since our requirement is that

\[
(\gamma(t))^{-1} \left| \int_{t}^{\infty} Fx\,ds \right| \leq \theta c, \quad x \in S, \quad t \geq 0,
\]

we have only to choose \( c \) so that

\[
Ac^{\alpha-1} + Bc^{\beta-1} \leq \theta.
\]

This is true for \( c \) sufficiently large if \( \alpha, \beta < 1 \), or for \( c \) sufficiently small if \( \alpha, \beta > 1 \).


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