Asymptotic Behavior of Solutions of Functionally Perturbed Nonoscillatory Second Order Differential Equations

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ABSTRACT

Conditions are given for a functional perturbation of a nonoscillatory linear second order differential equation to have solutions which behave asymptotically like given solutions of the unperturbed equation. The general results require no specific assumptions on the form of the functional perturbation, and the integrability conditions imposed on the functional permit conditional convergence. An application to a nonlinear integro–differential equation is included, along with an example.

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1. Introduction.

We consider the functional differential equation

$$(p(t)y')' + q(t)y = F(t;y) \tag{1}$$

as a perturbation of the linear differential equation

$$(p(t)x')' + q(t)x = 0. \tag{2}$$

We give conditions on the functional F which imply that (1) has a solution $\overline{y}$ on an interval $[a,\infty)$ that behaves in some sense (made precise below) as $t\to\infty$ like a given solution $\overline{x}$ of (2). In Section 2 we obtain general results without imposing specific assumptions on the form of F; thus, (1) may be an ordinary differential equation, it may involve one or more deviating arguments, or it may be an integro–differential equation, to name a few possibilities. In Section 3 we apply the general results to a specific kind of nonlinear integro–differential equation. We give an example in Section 4.

It is known [2, p. 355] that if (2) is nonoscillatory at ∞ then it has solutions x_1 and x_2 such that if a is sufficiently large then

$$x_1(t), x_2(t) > 0\,(t \geq a) \text{ and } \lim_{t\to\infty} \frac{x_2(t)}{x_1(t)} = \infty. \tag{3}$$

For convenience we assume that

$$r(x_1x_2' - x_1'x_2) = 1. \tag{4}$$

We will seek conditions on F which imply that (1) has a solution such that

$$\frac{\overline{y}(t) - \overline{x}(t)}{x_i(t)} = o(1)$$

with $i = 1$ or 2. (We use "O" and "o" in the usual way to indicate asymptotic behavior as $t \to \infty$.) In this case our conditions will also imply that

$$\left(\frac{\overline{y}(t) - \overline{x}(t)}{x_i(t)}\right)' = o\left(\frac{\rho'(t)}{\rho(t)}\right),$$

where

$$\rho = x_2/x_1, t \geq a. \tag{5}$$

From (3) and (4), $\rho' = 1/rx_1^2 > 0$ and $\lim_{t\to\infty} \rho(t) = \infty$.

We demonstrate the existence of the desired solution as a fixed point of the transformation

$$(\mathcal{T}_1 y)(t) = \overline{x}(t) + \int_t^\infty [x_1(t)x_2(s) - x_1(s)x_2(t)]\, F(s; y)\, ds \tag{6}$$

if $i = 1$, or of

$$(\mathcal{T}_2 y)(t) = \overline{x}(t) - x_1(t) \int_a^t \rho'(\tau) \int_\tau^\infty x_1(s)F(s; y)\, ds\, d\tau \tag{7}$$

if $i = 2$. If the appropriate improper integral converges then

$$(p(t)(\mathcal{T}_i y)')' + q(t)\mathcal{T}_i y = F(t; y); \tag{8}$$

hence, $\overline{y}$ satisfies (1) if $\mathcal{T}_i\overline{y} = \overline{y}$.

2. General Results

To obtain our results we impose integrability conditions on F and apply the following adaptation of the Schauder–Tychonoff theorem. The proof of this lemma is like that of a similar lemma of Coppel [1, p. 9].

Lemma 1. *If $\{y_j\}$ is a sequence of functions in $C^1[a,\infty)$, we will say that $y_j \to y$ if*

$$\lim_{j\to\infty}\left[\sup_{a\le t\le T}|y_j(t)-y(t)|+|y_j'(t)-y'(t)|\right]=0$$

whenever $a \le T < \infty$. With this definition of convergence, let $\mathcal{S}$ be a closed convex subset of $C^1[a,\infty)$ and suppose that $\mathcal{T} : \mathcal{S} \to \mathcal{S}$ is continuous and the families of functions $\mathcal{T}(\mathcal{S})$ and $\{(\mathcal{T}y)' : y \in \mathcal{S}\}$ are uniformly bounded and equicontinuous on $[a, T]$ for all $T > a$. Then $\mathcal{T}\overline{y} = \overline{y}$ for some $\overline{y} \in \mathcal{S}$.

The problems that we study and the use of the Schauder–Tychonoff theorem in this way are not new; however, our results are new in that we consider a larger class of functional perturbations F and the integrability conditions that we impose on them allow conditional convergence. For related results on other kinds of functional equations, see [3-5] and their references.

Throughout the rest of the paper it is to be understood that all equations and inequalities involving functions of t are valid for $t \ge a$. We make the following standing assumption.

Assumption A. *Let p and q be real–valued and continuous and $p > 0$ on $[a,\infty)$. Suppose that (2) is nonoscillatory at ∞, and that x_1 and x_2 are solutions of (2) which satisfy (3) and (4).*

Theorem 1. *Let $\overline{x}$ be a given solution of (2), ϕ be positive, continuous, and nonincreasing on $[a,\infty)$, and $i = 1$ and $j = 2$ or $i = 2$ and $j = 1$. Suppose that Assumption A holds and let $\mathcal{S}$ be the set of functions y in $C^1[a,\infty)$ such that*

$$\left|\frac{y(t)-\overline{x}(t)}{x_i(t)}\right| \le \phi(t) \text{ and } \left|\left(\frac{y(t)-\overline{x}(t)}{x_i(t)}\right)'\right| \le 2\phi(t)\frac{\rho'(t)}{\rho(t)}. \tag{9}$$

Suppose that

(i) $F(\cdot; y) \in C[a,\infty)$ *if* $y \in \mathcal{S}$*;*

(ii) the family $\{F(\cdot; y) \,|\, y \in \mathcal{S}\}$ *is uniformly bounded on each finite subinterval of* $[a,\infty)$*;*

(iii) If $\{y_j\}$ *is a sequence in* $\mathcal{S}$ *such that* $y_j \to y$*, then*

$$\lim_{j\to\infty} F(t; y_j) = F(t; y) \text{ (pointwise)},\ t \geq a.$$

Suppose also that $\int_a^\infty x_j(s)F(s; y)\,ds$ *converges for every* y *in* $\mathcal{S}$*, and there is a continuous, nonincreasing function* σ *defined on* $[a,\infty)$ *such that*

$$\left|\int_t^\infty x_j(s)F(s; y)\,ds\right| \leq \sigma(t),\ y \in \mathcal{S}, \tag{10}$$

$$\lim_{t\to\infty} \sigma(t) = 0, \tag{11}$$

and

$$\sigma(t) \leq \phi(t),\ \text{ if } i = 1, \tag{12}$$

or

$$\max\left(\sigma(t), \frac{1}{\rho(t)} \int_a^t \rho'(\tau)\sigma(\tau)\,d\tau\right) \leq \phi(t) \text{ if } i = 2. \tag{13}$$

Then (1) a has solution $\overline{y}$ *on* $[a,\infty)$ *such that*

$$\left|\frac{\overline{y}(t) - \overline{x}(t)}{x_i(t)}\right| \leq \sigma(t) \text{ and } \left|\left(\frac{\overline{y}(t) - \overline{x}(t)}{x_i(t)}\right)'\right| \leq 2\sigma(t)\frac{\rho'(t)}{\rho(t)}. \tag{14}$$

Notice that $\mathcal{S}$ is a closed convex subset of $C^1[a,\infty)$. We will prove Theorem 1 by using Lemma 1 to show that there is a function $\overline{y}$ in $\mathcal{S}$ which is left fixed by one of the transformations $\mathcal{T}_1$ or $\mathcal{T}_2$ defined by (6) and (7).

For the case where $i = 1$ we need the following lemma, which was proved in [3].

Lemma 2. *Suppose that $u \in C[a,\infty)$ and $\int_a^\infty x_2(s)u(s)\,ds$ converges, and let*

$$\nu(t) = \sup_{\tau \geq t} \left| \int_\tau^\infty x_2(s)u(s)\,ds \right|.$$

Then $\int_t^\infty x_1(s)u(s)\,ds$ also converges, and

$$\left| \int_t^\infty \left[x_2(s) - x_1(s)\rho(t)\right] u(s)\,ds \right| \leq \nu(t), \quad \left| \int_t^\infty x_1(s)u(s)\,ds \right| \leq 2\nu(t)/\rho(t).$$

Proof of Theorem 1. Suppose that $i = 1$, $j = 2$ and $y \in \mathcal{S}$. From (6),

$$\frac{(\mathcal{T}_1 y)(t) - \overline{x}(t)}{x_1(t)} = \int_t^\infty \left[x_2(s) - x_1(s)\rho(t)\right] F(s;y)\,ds \tag{15}$$

and

$$\left(\frac{(\mathcal{T}_1 y)(t) - \overline{x}(t)}{x_1(t)} \right)' = -\rho'(t) \int_t^\infty x_1(s)F(s;y)\,ds. \tag{16}$$

Therefore, (10) and Lemma 2 with $u = F(\cdot;y)$ and $\nu = \sigma$ imply that

$$\left| \frac{(\mathcal{T}_1 y)(t) - \overline{x}(t)}{x_1(t)} \right| \leq \sigma(t) \text{ and } \left| \left(\frac{(\mathcal{T}_1 y)(t) - \overline{x}(t)}{x_1(t)} \right)' \right| \leq 2\sigma(t)\frac{\rho'(t)}{\rho(t)}. \tag{17}$$

Now (12) and (17) imply that $\mathcal{T}_1 y \in \mathcal{S}$; i.e., $\mathcal{T}_1(\mathcal{S}) \subset \mathcal{S}$.

Now suppose that $\{y_k\}$ is a sequence in $\mathcal{S}$ such that $y_k \to y$. If $\epsilon > 0$ choose $T > a$ such that $\sigma(t) < \epsilon/4$ if $t \geq T$. (This is possible because of (11).) Then (10) implies that

$$\left| \int_t^\infty x_2(s)\left[F(s;y_k) - F(s;y)\right]\,ds \right| < \epsilon/2,\ t \geq T, \tag{18}$$

for all k. With T now fixed, choose k_0 so that

$$\int_a^T x_2(s)\left|F(s;y_k) - F(s;y)\right|\,ds < \epsilon/2,\ k \geq k_0, \tag{19}$$

which is possible because of assumptions (ii) and (iii) and the bounded convergence theorem. Now (18) and (19) imply that

$$\left|\int_t^\infty x_2(s)\left[F(s;y_k)-F(s;y)\right]\,ds\right|<\epsilon,\; k\ge k_0. \tag{20}$$

Therefore, (15), (16), and Lemma 2 with $u=F(\cdot;y_k)-F(\cdot;y)$ and $\nu=\epsilon$ imply that

$$\left|\frac{(\mathcal{T}_1y_k)(t)-(\mathcal{T}_1y)(t)}{x_1(t)}\right|=\left|\int_t^\infty\left[x_2(s)-x_1(s)\rho(t)\right]\left(F(s;y_k)-F(s;y)\right)\,ds\right|\le\epsilon,\; k\ge k_0,$$

and

$$\left|\left(\frac{(\mathcal{T}_1y_k)(t)-(\mathcal{T}_1y)(t)}{x_1(t)}\right)'\right|=\rho'(t)\left|\int_t^\infty x_1(s)\left(F(s;y_k)-F(s;y)\right)\,ds\right|\le\frac{2\epsilon\rho'(t)}{\rho(t)},\; k\ge k_0.$$

From this an elementary argument shows that $\mathcal{T}_1y_k\to\mathcal{T}_1y$; hence, $\mathcal{T}_1$ is continuous.

Now suppose that $i=2$, $j=1$ and $y\in\mathcal{S}$. From (5) and (7),

$$\frac{(\mathcal{T}_2y)(t)-\overline{x}(t)}{x_2(t)}=-\frac{1}{\rho(t)}\int_a^t\rho'(\tau)\int_\tau^\infty x_1(s)F(s;y)\,ds\,d\tau,$$

and

$$\left(\frac{(\mathcal{T}_2y)(t)-\overline{x}(t)}{x_2(t)}\right)'=\frac{\rho'(t)}{\rho^2(t)}\int_a^t\rho'(\tau)\int_\tau^\infty x_1(s)F(s;y)\,ds\,d\tau$$
$$-\frac{\rho'(t)}{\rho(t)}\int_t^\infty x_1(s)F(s;y)\,ds,\; t\ge a.$$

Therefore, (10) implies that

$$\left|\frac{(\mathcal{T}_2y)(t)-\overline{x}(t)}{x_2(t)}\right|\le\frac{1}{\rho(t)}\int_a^t\rho'(\tau)\sigma(\tau)\,d\tau, \tag{21}$$

and

$$\left|\left(\frac{(\mathcal{T}_2 y)(t)-\overline{x}(t)}{x_2(t)}\right)'\right| \leq \frac{\rho'(t)}{\rho(t)}\left[\frac{1}{\rho(t)}\int_a^t \rho'(\tau)\sigma(\tau)\,d\tau+\sigma(t)\right]. \tag{22}$$

Now (13), (21), and (22) imply that $\mathcal{T}_2 y \in \mathcal{S}$; that is, $\mathcal{T}_2(\mathcal{S}) \subset \mathcal{S}$.

Now suppose that $\{y_k\}$ is a sequence in $\mathcal{S}$ such that $y_k \to y$, and let $\epsilon > 0$. The argument used to obtain (20) shows that there is an integer k_0 such that

$$\left|\int_t^\infty x_1(s)\left[F(s;y_k)-F(s,y)\right]\,ds\right| < \epsilon,\; k \geq k_0,$$

and therefore, from (7),

$$\begin{aligned}\left|\frac{(\mathcal{T}_2 y_k)(t)-(\mathcal{T}_2 y)(t)}{x_1(t)}\right| &= \int_a^t \rho'(\tau)\int_\tau^\infty x_1(s)\left[g_1(s;y_k)-g_1(s;y)\right]\,ds\,d\tau \\ &\leq \epsilon\left(\rho(t)-\rho(a)\right),\; k \geq k_0,\end{aligned}$$

and

$$\left|\left(\frac{(\mathcal{T}_2 y_k)(t)-(\mathcal{T}_2 y)(t)}{x_1(t)}\right)'\right| = \rho'(t)\int_t^\infty x_1(s)\left[g_1(s;y_k)-g_1(s;y)\right]\,ds \leq \epsilon\rho'(t),\; k \geq k_0.$$

From this an elementary argument shows that $\mathcal{T}_2 y_k \to \mathcal{T}_2 y$; hence, $\mathcal{T}_2$ is continuous.

Now let $T > a$. Since $\mathcal{T}_i(\mathcal{S}) \subset \mathcal{S}$, the definition of $\mathcal{S}$ implies that the families $\mathcal{T}_i(\mathcal{S})$ and $\{(\mathcal{T}_i y)' : y \in \mathcal{S}\}$ are uniformly bounded on $[a,T]$. The uniform boundedness of the second family on $[a,T]$ implies the equicontinuity of the first on $[a,T]$. From (8), assumption (ii), and the boundedness of $\mathcal{T}_i(\mathcal{S})$ on $[a,T]$, the family $\{p(\mathcal{T}_i y)' : y \in \mathcal{S}\}$ is equicontinuous on $[a,T]$. Since p is bounded away from zero on $[a,T]$, this implies that the family $\{(\mathcal{T}_i y)' : y \in \mathcal{S}\}$ is equicontinuous on $[a,T]$.

We have now verified that $\mathcal{T}_i$ satisfies the hypotheses of the Lemma 1 on $\mathcal{S}$. Therefore $\mathcal{T}_i\overline{y} = \overline{y}$ for some $\overline{y} \in \mathcal{S}$. To verify (14) we set $y = \overline{y}$ in (17) if $i = 1$ or in (21) and (22) if $i = 2$. This completes the proof.

3. An Application to an Integro–Differential Equation.

We now consider the integro–differential equation

$$(p(t)y')' + q(t)y = (\rho(t))^{-\alpha} g_1(t) \int_a^t (y(\tau))^\gamma g_2(\tau)\, d\tau,\ t \geq a. \tag{23}$$

Theorem 2. *Suppose that Assumption A holds, γ $(\neq 0, 1)$ is real, and g_1 and g_2 are continuous and real–valued on $[a, \infty)$. Let the functions*

$$G_1(t) = \int_a^t x_1(u) g_1(u)\, du, \tag{24}$$

$$G_2(t) = \int_a^t (x_1(u))^\gamma g_2(u)\, du, \tag{25}$$

and

$$G_3(t) = \int_a^t (x_1(u))^\gamma G_1(u) g_2(u)\, du \tag{26}$$

be bounded on $[a, \infty)$ and c be a given positive constant. Suppose also that one of the following hypotheses holds:

(H1) $i = r = 1$ and $\lambda = \alpha - 1 > 0$.

(H2) $i = 1, r = 2$ and $\lambda = \alpha - \max\{\gamma + 1, 1\} > 0$.

(H3) $i = r = 2$ and $1 > \lambda = \alpha - \max\{\gamma, 0\} > 0$.

Then (23) has a solution $\overline{y}$ on $[a, \infty)$ such that

$$\frac{\overline{y}(t) - cx_r(t)}{x_i(t)} = O((\rho(t))^{-\lambda}) \text{ and } \left(\frac{\overline{y}(t) - cx_r(t)}{x_i(t)}\right)' = O((\rho(t))^{-\lambda-1}\rho'(t)),$$

provided that $c^{\gamma-1}$ is sufficiently small.

Proof. In (23) the functional F is

$$F(t; y) = (\rho(t))^{-\alpha} g_1(t) \int_a^t (y(\tau))^\gamma g_2(\tau)\, d\tau. \tag{27}$$

Let θ be a given constant in $(0,1)$, and define $\mathcal{S}$ to be the subset of $C^1[a,\infty)$ consisting of functions y such that

$$\left|\frac{y(t)-cx_r(t)}{x_i(t)}\right| \le \theta c(\rho(a))^{\lambda+r-i}(\rho(t))^{-\lambda} \tag{28}$$

and

$$\left|\left(\frac{y(t)-cx_r(t)}{x_i(t)}\right)'\right| \le 2\theta c(\rho(a))^{\lambda+r-i}(\rho(t))^{-\lambda-1}\rho'(t). \tag{29}$$

If y satisfies (28), then the monotonicity of ρ implies that

$$0 < c(1-\theta)(\rho(t))^{r-i} \le \left|\frac{y(t)}{x_i(t)}\right| \le c(1+\theta)(\rho(t))^{r-i},$$

which implies assumptions (i) and (ii) of Theorem 1. Moreover, the bounded convergence theorem implies assumption (iii) of Theorem 1.

Now let $j=1$ if $i=2$ or $j=2$ if $i=1$, as in Theorem 1. We must now verify (10) for a suitable σ. First we consider

$$I(t) = \int_t^\infty x_j(s)F(s;x_r)\,ds = \int_t^\infty (\rho(s))^{j-\alpha-1}G_1'(s)\int_a^s (\rho(\tau))^{(r-1)\gamma}G_2'(\tau)\,d\tau. \tag{30}$$

We will show that

$$I(t) = O((\rho(t))^{-\lambda}). \tag{31}$$

For convenience, define

$$h(t) = \int_a^t (\rho(\tau))^{(r-1)\gamma}G_2'(\tau)\,d\tau. \tag{32}$$

From the boundedness of G_2 and integration by parts (if $r=2$)

$$h(t) = \begin{cases} O((\rho(t))^{(r-1)\gamma}) & \text{if } \gamma > 0, \\ O(1) & \text{if } \gamma \le 0. \end{cases} \tag{33}$$

Integrating (30) by parts yields

$$I(t) = -(\rho(t))^{j-\alpha-1}h(t)G_1(t) - (j-\alpha-1)\int_t^\infty (\rho(s))^{j-\alpha-2}\rho'(s)h(s)G_1(s)\,ds \\ - \int_t^\infty (\rho(s))^{j-\alpha-1}h'(s)G_1(s)\,ds. \tag{34}$$

Now it is important to note that under all three hypotheses (H1), (H2), and (H3),

$$\lambda = \begin{cases} \alpha - j + 1 - (r-1)\gamma & \text{if } \gamma > 0, \\ \alpha - j + 1 & \text{if } \gamma \le 0. \end{cases}$$

From (33) and the boundedness of G_1, the first two terms on the right of (34) are $O((\rho(t))^{-\lambda})$. From (24), (25), (26), and (32), the second integral on the right of (34) can be written as

$$\int_t^\infty (\rho(s))^{j-\alpha-1+(r-1)\gamma} G_3'(s)\,ds.$$

Integrating by parts and invoking the boundedness of G_3 shows that this integral is $O((\rho(t))^{j-\alpha-1+(r-1)\gamma})$. We have now verified (31) under all three hypotheses (H1), (H2), (H3).

We now consider $\int_t^\infty x_j(s)F(s;y)\,ds$ for arbitrary y in S. Because of (24), (27), and (30), this can be written as

$$\int_t^\infty x_j(s)F(s;y)\,ds = c^\gamma I(t) + \int_a^t (\rho(s))^{j-\alpha-1} G_1'(s) \int_a^s [(y(\tau))^\gamma - c^\gamma (x_r(\tau))^\gamma] g_2(\tau)\,d\tau. \tag{35}$$

If we introduce the new variable

$$z = y/x_i \tag{36}$$

and recall (25), then (35) can be rewritten as

$$\int_t^\infty x_j(s)F(s;y)\,ds = c^\gamma I(t) + W(t;z), \tag{37}$$

where

$$W(t;z) = \int_t^\infty (\rho(s))^{j-\alpha-1} G_1'(s) w(s;z)\, ds, \tag{38}$$

with

$$w(t;z) = \int_a^t \left[(z(\tau))^\gamma - c^\gamma (\rho(\tau))^{(r-i)\gamma} \right] (\rho(\tau))^{(i-1)\gamma} G_2'(\tau)\, d\tau. \tag{39}$$

From (28) and (36),

$$|z(t) - c(\rho(t))^{r-i}| \le \theta c (\rho(a))^{\lambda+r-i} (\rho(t))^{-\lambda}. \tag{40}$$

If z is any number between $z(t)$ and $c(\rho(t))^{(r-i)}$, then (29) and the monotonicity of ρ imply that

$$0 < (1-\theta) c (\rho(t))^{(r-i)} \le |z| \le (1+\theta) c (\rho(t))^{(r-i)},$$

and therefore, if k is any real number,

$$|z|^k \le (1 \pm \theta)^k c^k (\rho(t))^{(r-i)k}, \tag{41}$$

where the "$\pm$" is "$+$" if $k > 0$, or "$-$" if $k < 0$. Applying the mean value theorem to $A(z) = z^\gamma$ and invoking (40) and (41) shows that

$$|(z(t))^\gamma - c^\gamma (\rho(t))^{(r-i)\gamma}| \le K_1 c^\gamma (\rho(t))^{(r-i)(\gamma-1)-\lambda} \tag{42}$$

for some constant K which does not depend upon c or z. (Similar constants introduced in the rest of this proof are also independent of c and z, but we will refrain from stating this each time, to avoid repetition.) Also, (29) and (36) imply that

$$|z'(t)| \le 2\theta c (\rho(a))^\lambda (\rho(t))^{-\lambda-1} \rho'(t) \text{ if } i = r, \tag{43}$$

and

$$|z'(t) - c\rho'(t)| \le 2\theta c (\rho(a))^{\lambda+1} (\rho(t))^{-\lambda-1} \rho'(t) \text{ if } i = 1,\ r = 2. \tag{44}$$

From (41) and (43),

$$|(z(t))^{\gamma-1} z'(t)| \le K_2 c^\gamma (\rho(t))^{-\lambda-1} \rho'(t) \text{ if } i = r, \tag{45}$$

for some constant K_2. Applying the mean value theorem to $B(z, z') = z^{\gamma-1} z'$ and invoking (40), (41), and (44) shows that

$$|(z(t))^{\gamma-1} z' - c^\gamma (\rho(t))^{\gamma-1} \rho'(t)| \le K_2 c^\gamma (\rho(t))^{\gamma-\lambda-2} \rho'(t) \text{ if } i = 1,\ r = 2. \tag{46}$$

We now consider the hypotheses (H1), (H2), and (H3) separately. In each case we will show that

$$|W(t;z)| \leq Jc^{\gamma}(\rho(t))^{-\lambda}, \tag{47}$$

for some constant J. Once (47) is established then (31) and (37) will imply that

$$\left| \int_t^{\infty} x_j(s)F(s;y)\,ds \right| \leq Hc^{\gamma}(\rho(t))^{-\lambda}$$

for some constant H. Then we can apply Theorem 1 with

$$\phi(t) = \theta c(\rho(a))^{\lambda+r-i}(\rho(t))^{-\lambda} \tag{48}$$

(compare (9) with (28) and (29)), and

$$\sigma(t) = Hc^{\gamma}(\rho(t))^{-\lambda}. \tag{49}$$

If $i = 1$ then (48) and (49) imply (12) if $c^{\gamma-1}$ is sufficiently small. If $i = 2$ then (48), (49), and our assumption that $\lambda < 1$ imply (13) if $c^{\gamma-1}$ is sufficiently small. Thus, our proof will be complete when we have established (47).

Case 1. Let $i = r = 1$ (and therefore $j = 2$). Integrating (39) by parts yields

$$w(t;z) = \left[(z(\tau))^{\gamma} - c^{\gamma}\right] G_2(\tau)\Big|_a^t - \gamma \int_a^t (z(\tau))^{\gamma-1} z'(\tau) G_2(\tau)\,d\tau,$$

and therefore (42), (45), and the boundedness of G_2 imply that

$$|w(t;z)| \leq Kc^{\gamma} \tag{50}$$

for some constant K. Integrating (38) by parts and recalling that $\lambda = \alpha - 1$ yields

$$W(t;z) = I_1(t;z) + I_2(t;z) + I_3(t;z), \tag{51}$$

with

$$I_1(t;z) = -(\rho(t))^{-\lambda} G_1(t) w(t;z),$$

$$I_2(t;z) = \lambda \int_t^{\infty} (\rho(s))^{-\lambda-1} \rho'(s) G_1(s) w(s;z)\,ds,$$

and

$$I_3(t;z) = -\int_t^\infty (\rho(s))^{-\lambda} \left[(z(s))^\gamma - c^\gamma\right] G_3'(s)\, ds.$$

From (50) and the boundedness of G_1,

$$|I_k(t;z)| \le J_k c^\gamma (\rho(t))^{-\lambda} \; (k = 1, 2) \tag{52}$$

for some constants J_1 and J_2. Integrating $I_3(t;z)$ by parts shows that

$$I_3(t;z) = (\rho(t))^{-\lambda} \left[(z(t))^\gamma - c^\gamma\right] G_3(t) - \lambda \int_t^\infty (\rho(s))^{-\lambda-1} \rho'(s) \left[(z(s))^\gamma - c^\gamma\right] G_3(s)\, ds$$

$$+ \gamma \int_t^\infty (\rho(s))^{-\lambda} (z(s))^{\gamma-1} z'(s) G_3(s)\, ds;$$

hence, (42), (45), and the boundedness of G_3 imply that

$$|I_3(t;z)| \le J_3 c^\gamma (\rho(t))^{-2\lambda}. \tag{53}$$

Now (51), (52), and (53) imply (47).

Case 2. Let $i = 1$ and $r = 2$ (so that $j = 2$). Integrating (39) by parts yields

$$w(t;z) = \left[(z(\tau))^\gamma - c^\gamma (\rho(\tau))^\gamma\right] G_2(\tau)\Big|_a^t - \gamma \int_a^t \left[(z(\tau))^{\gamma-1} z'(\tau) - c^\gamma (\rho(\tau))^{\gamma-1} \rho'(\tau)\right] G_2(\tau)\, d\tau,$$

and therefore (42) and (46) imply that

$$|w(t;z)| \le \begin{cases} K c^\gamma (\rho(t))^{\gamma-\lambda-1} & \text{if } \gamma > \lambda + 1, \\ K c^\gamma & \text{if } \gamma \le \lambda + 1, \end{cases} \tag{54}$$

for some constant K. Integrating (38) by parts yields (51) with

$$I_1(t;z) = -(\rho(t))^{-\alpha+1} w(t;z) G_1(t),$$

$$I_2(t;z) = -(1-\alpha) \int_t^\infty (\rho(s))^{-\alpha} \rho'(s) w(s;z) G_1(s)\, ds,$$

and

$$I_3(t;z) = -\int_t^\infty (\rho(s))^{-\alpha+1} \left[(z(s))^\gamma - c^\gamma(\rho(s))^\gamma\right] G_3'(s)\,ds.$$

From (54) and the boundedness of G_1,

$$|I_k(t;z)| \le \begin{cases} J_1 c^\gamma (\rho(t))^{\gamma-\alpha-\lambda} & \text{if } \gamma > \lambda+1, \\ J_1 c^\gamma (\rho(t))^{-\alpha+1} & \text{if } \gamma \le \lambda+1, \end{cases} \quad (k=1,2). \tag{55}$$

Integrating $I_3(t;z)$ by parts shows that

$$\begin{aligned} I_3(t;z) = {} & (\rho(t))^{-\alpha+1}\left[(z(t))^\gamma - c^\gamma(\rho(t))^\gamma\right] G_3(t) \\ & + (1-\alpha)\int_t^\infty (\rho(s))^{-\alpha}\rho'(s)\left[(z(s))^\gamma - c^\gamma(\rho(s))^\gamma\right] G_3(s)\,ds \\ & + \gamma\int_t^\infty (\rho(s))^{-\alpha+1}\left[(z(s))^{\gamma-1}z'(s) - c^\gamma(\rho(s))^{\gamma-1}\rho'(s)\right] G_3(s)\,ds; \end{aligned}$$

hence, (42), (46), and the boundedness of G_3 imply that

$$|I_3(t;z)| \le J_3 c^\gamma (\rho(t))^{\gamma-\alpha-\lambda} \tag{56}$$

for some constant J_3. Recalling that $\lambda = \alpha - \max\{\gamma+1, 1\} > 0$, we can now infer (47) from (51), (55), and (56).

Case 3. Let $i = r = 2$ (so that $j = 1$). Integrating (39) by parts yields

$$\begin{aligned} w(t;z) = {} & \left[(z(\tau))^\gamma - c^\gamma\right](\rho(\tau))^\gamma G_2(\tau)\Big|_a^t - \gamma\int_a^t (z(\tau))^{\gamma-1}z'(\tau)(\rho(\tau))^\gamma G_2(\tau)\,d\tau \\ & - \gamma\int_a^t \left[(z(\tau))^\gamma - c^\gamma\right](\rho(\tau))^{\gamma-1}\rho'(\tau)G_2(\tau)\,d\tau, \end{aligned}$$

and therefore (42) and (45) imply that

$$|w(t;z)| \le \begin{cases} Kc^\gamma(\rho(t))^{\gamma-\lambda} & \text{if } \gamma > \lambda, \\ Kc^\gamma & \text{if } \gamma \le \lambda, \end{cases} \tag{57}$$

for some constant K. Integrating (38) by parts yields (51) with

$$I_1(t;z) = -(\rho(t))^{-\alpha} G_1(t) w(t;z),$$

$$I_2(t;z) = \alpha \int_t^\infty (\rho(s))^{-\alpha-1} \rho'(s) G_1(s) w(s;z)\, ds,$$

and

$$I_3(t;z) = -\int_t^\infty (\rho(s))^{-\alpha+\gamma} \left[(z(s))^\gamma - c^\gamma \right] G_3'(s)\, ds.$$

From (57) and the boundedness of G_1,

$$|I_k(t;z)| \leq \begin{cases} J_k c^\gamma (\rho(t))^{\gamma-\lambda-\alpha} & \text{if } \gamma > \lambda, \\ J_k c^\gamma (\rho(t))^{-\alpha} & \text{if } \gamma \leq \lambda, \end{cases} \quad (k = 1, 2) \tag{58}$$

for some constants J_1 and J_2. Integrating $I_3(t;z)$ by parts shows that

$$I_3(t;z) = (\rho(t))^{-\alpha+\gamma} \left[(z(t))^\gamma - c^\gamma \right] G_3(t) + \gamma \int_t^\infty (\rho(s))^{-\alpha+\gamma} (z(s))^{\gamma-1} z'(s) G_3(s)\, ds$$

$$+ (\gamma - \alpha) \int_t^\infty (\rho(s))^{-\alpha+\gamma-1} \rho'(s) \left[(z(s))^\gamma - c^\gamma \right] G_3(s)\, ds;$$

hence, (42), (43), and the boundedness of G_3 imply that

$$|I_3(t;z)| \leq J_3 c^\gamma (\rho(t))^{\gamma-\lambda-\alpha}. \tag{59}$$

Recalling that $\lambda = \alpha - \max\{\gamma, 0\} > 0$, we can now infer (47) from (51), (58), and (59). This completes the proof of Theorem 2.

4. An Example.

Consider the equation

$$(t^{-2} y')' + 2t^{-4} y = t^{-\alpha-1} \sin t \int_a^t \tau^{-\gamma} (y(\tau))^\gamma \sin \tau\, d\tau \ (a > 0). \tag{60}$$

The solutions of the unperturbed equation

$$(t^{-2}x')' + 2t^{-4}x = 0$$

are $x_1 = t$ and $x_2 = t^2$; hence $\rho(t) = t$, and (60) is of the form (23), with

$$g_1(t) = \frac{\sin t}{t} \text{ and } g_2(t) = t^{-\gamma}\sin t.$$

Therefore,

$$G_1(t) = G_2(t) = \cos a - \cos t$$

and

$$G_3(t) = \cos a(\cos a - \cos t) + \frac{1}{4}(\cos 2t - \cos 2a)$$

are all bounded, and Theorem 2 implies the following results if $c^{\gamma-1}$ is sufficiently small.

(i) If $\alpha > 1$ then (60) has a solution $\overline{y}$ on $[a, \infty)$ such that

$$\overline{y}(t) = ct + O(t^{-\alpha+2}) \text{ and } \left(\frac{\overline{y}(t)}{t}\right)' = O(t^{-\alpha}).$$

(ii) If $\lambda = \alpha - \max\{\gamma + 1, 1\} > 0$ then (60) has a solution $\overline{y}$ on $[a, \infty)$ such that

$$\overline{y}(t) = ct^2 + O(t^{-\lambda+1}) \text{ and } \left(\frac{\overline{y}(t) - ct^2}{t}\right)' = O(t^{-\lambda-1}).$$

(iii) If $1 > \lambda = \alpha - \max\{\gamma, 0\} > 0$ then (60) has a solution $\overline{y}$ on $[a, \infty)$ such that

$$\overline{y}(t) = ct^2 + O(t^{-\lambda+2}) \text{ and } \left(\frac{\overline{y}(t)}{t^2}\right)' = O(t^{-\lambda-1}).$$

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