Some spectral properties of Hermitian Toeplitz matrices

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Abstract

Necessary conditions are given for the Hermitian Toeplitz matrix $T_n = (t_{r-s})_{r,s=1}^n$ to have a repeated eigenvalue λ with multiplicity $m > 1$, and for an eigenpolynomial of T_n associated with λ to have a given number of zeros off the unit circle $|z| = 1$. It is assumed that $t_r = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-ir\theta}\,d\theta\,(0 \le r \le n-1)$, where f is real–valued and in $L(-\pi,\pi)$. The conditions are given in terms of the number of changes in sign of $f(\theta) - \lambda$.

1 Introduction

We consider the Hermitian Toeplitz matrix

$$T_n = (t_{r-s})_{r,s=1}^n,$$

where

$$t_r = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-ir\theta}\,d\theta,\ r = 0, 1, \ldots, n-1, \tag{1}$$

and f is real–valued and Lebesgue integrable on $(-\pi, \pi)$, and not constant on a set of measure 2π.

Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of T_n, with associated orthonormal eigenvectors $x_1, x_2, \ldots, x_n$. Our first main result (Theorem 3) presents a necessary condition on f for λ_r to have multiplicity $m > 1$. To describe our second main result we first recall some well known properties of eigenvectors of Hermitian Toeplitz matrices. If J is the $n \times n$ matrix with ones on the secondary diagonal and zeros elsewhere, then $JT_nJ = \overline{T}_n$. This implies that a vector x_r is a λ_r-eigenvector of T_n if and only if $J\overline{x}_r$ is. It follows that if λ_r has multiplicity one then

$$J\overline{x}_r = \xi x_r, \tag{2}$$

where ξ is a complex constant with modulus one. A stronger result holds if T_n is real and symmetric: Cantoni and Butler [1] have shown that in this case

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(even if T_n has repeated eigenvalues) R^n has an orthonormal basis consisting of $\lceil n/2 \rceil$ eigenvectors of T_n for which (2) holds with $\xi = 1$ and $\lfloor n/2 \rfloor$ for which (2) holds with $\xi = -1$.

The polynomial

$$X_r(z) = [1, z, \dots, z^{n-1}]x_r \tag{3}$$

is said to be an *eigenpolynomial of* T_n *associated with* λ_r. The location of the zeros of the eigenpolynomials of Hermitian Toeplitz matrices is of interest in signal processing applications [2]-[5], [7]. If x_r satisfies (2) then

$$X_r(z) = \overline{\xi} z^{n-1} \overline{X_r(1/\overline{z})};$$

hence, zeros of $X_r(z)$ that are not on the unit circle must occur in pairs ζ and $1/\overline{\zeta}$.

Gueguen proved the following theorem in [5]. (See also [2] and [4].)

THEOREM 1 *Let* λ_r *be an eigenvalue of* T_n*, but not of* T_{n-1}*. Then its associated eigenpolynomial* $X_r(z)$ *has at least* $|n - 2r + 1|$ *zeros on the unit circle* $|z| = 1$.

Delsarte, Genin, and Kamp proved the following theorem in [3]. (See also [4].)

THEOREM 2 *Suppose that the eigenvalue* λ_r *of* T_n *has multiplicity* m *and let* s *be the largest integer* $< n$ *such that* λ_r *is not an eigenvalue of* T_s*. Then any eigenpolynomial* $X(z)$ *of* T_n *corresponding to* λ_r *has at least* $|n - m - 2r + 2|$ *and at most* $m + s - 1$ *zeros on the unit circle* $|z| = 1$.

Our second main result (Theorem 7) gives a necessary condition on f for an eigenpolynomial of T_n satisfying (2) to have a given number of zeros that are not on the unit circle.

2 A necessary condition for repeated eigenvalues.

Let α and β be the essential upper and lower bounds of f; that is, α is the largest number and β the smallest such that $\alpha \le f(\theta) \le \beta$ almost everywhere on $(-\pi, \pi)$. It is known ([6], p. 65) that all the eigenvalues of T are in (α, β). A proof of this is included naturally in the proof of the following theorem.

THEOREM 3 *If* λ_r *is an eigenvalue of* T_n *with multiplicity* m*, then* $f(\theta) - \lambda_r$ *must change sign at least* $2m - 1$ *times in* $(-\pi, \pi)$.

PROOF. Associate with each vector $v = [v_1, v_2, \cdots, v_n]^t$ in C^n the polynomial

$$V(z) = [1, z, \dots, z^{n-1}]v = \sum_{j=1}^{n} v_j z^{j-1}.$$

If u and v are in C^n then

$$(u, v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} U(z)\overline{V(z)}\, d\theta, \tag{4}$$

where $z = e^{i\theta}$ whenever z appears in an integral. Moreover, (1) implies that

$$(T_n u, v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)U(z)\overline{V(z)}\, d\theta. \tag{5}$$

Now let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of T_n, with corresponding orthonormal eigenvectors $x_1, x_2, \dots, x_n$, and let

$$X_i(z) = [1, z, \dots, z^{n-1}]x_i,\ 1 \le i \le n,$$

be the corresponding eigenpolynomials. From (4),

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} X_i(z)\overline{X_j(z)}\, d\theta = \delta_{ij},\ 1 \le i, j \le n, \tag{6}$$

and from (5),

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)X_i(z)\overline{X_j(z)}\, d\theta = \delta_{ij}\lambda_j,\ 1 \le i, j \le n. \tag{7}$$

The last two equations with $i = j$ show that the eigenvalues of T_n are in (α, β). Therefore, $f(\theta) - \lambda_r$ must change sign at some point in $(-\pi, \pi)$. This completes the proof if $m = 1$.

Now suppose that $m > 1$ and $f(\theta) - \lambda_r$ changes sign only at the points $\theta_1 < \theta_2 < \cdots < \theta_k$ in $(-\pi, \pi)$, where $k \le 2m - 2$. We will show that this assumption leads to a contradiction.

Define

$$g(\theta) = \frac{1}{2\pi} \left(f(\theta) - \lambda_r\right). \tag{8}$$

For reference below note that if $k = 2p$ then the function

$$g(\theta) \prod_{j=1}^{2p} \sin\left(\frac{\theta - \theta_j}{2}\right) \tag{9}$$

does not change sign in $(-\pi, \pi)$. This remains true if $k = 2p - 1$, if we define $\theta_{2p} = \pi$. Now suppose that λ_r has multiplicity m; that is,

$$\lambda_r = \lambda_{r+1} = \cdots = \lambda_{r+m-1}. \tag{10}$$

From (6), (7), and (10),

$$\int_{-\pi}^{\pi} g(\theta)X_i(z)\overline{X_j(z)}\, d\theta = 0\, (r \le i \le r + m - 1,\ 1 \le j \le n).$$

Therefore

$$\int_{-\pi}^{\pi} g(\theta)\left(\sum_{\ell=0}^{m-1} c_\ell X_{r+\ell}(z)\right)\overline{X_j(z)}\,d\theta = 0,\ 1 \le j \le n,$$

if $c_0, \dots, c_{m-1}$ are constants. This implies that

$$\int_{-\pi}^{\pi} g(\theta)\left(\sum_{\ell=0}^{m-1} c_\ell X_{r+\ell}(z)\right)\overline{Q(z)}\,d\theta = 0 \tag{11}$$

if Q is any polynomial of degree $\le n-1$, since any such polynomial can be written as a linear combination of $X_1(z), \dots, X_n(z)$. In particular, choose $c_0, \dots, c_{m-1}$ – not all zero – so that

$$\sum_{\ell=0}^{m-1} c_\ell X_{r+\ell}(e^{i\theta_j}) = 0,\ 1 \le j \le p,$$

(this is possible, since $p < m$), and let

$$Q(z) = \left(\sum_{\ell=0}^{m-1} c_\ell X_{r+\ell}(z)\right)\prod_{j=1}^{p} \frac{z - e^{i\theta_{p+j}}}{z - e^{i\theta_j}}.$$

Substituting this into (11) yields

$$\int_{-\pi}^{\pi} g(\theta)\left|\sum_{\ell=0}^{m-1} c_\ell X_{r+\ell}(z)\right|^2 \prod_{j=1}^{p} \frac{\overline{z} - e^{-i\theta_{p+j}}}{\overline{z} - e^{-i\theta_j}}\,d\theta = 0,$$

or, equivalently,

$$\int_{-\pi}^{\pi} g_1(\theta)\prod_{j=1}^{p}(z - e^{i\theta_j})(\overline{z} - e^{-i\theta_{p+j}})\,d\theta = 0, \tag{12}$$

where

$$g_1(\theta) = g(\theta)\left|\frac{\sum_{\ell=0}^{m-1} c_\ell X_{r+\ell}(z)}{\prod_{j=1}^{p}(z - e^{i\theta_j})}\right|^2.$$

If $z = e^{i\theta}$ then

$$(z - e^{i\theta_j})(\overline{z} - e^{-i\theta_{p+j}}) = 4e^{i(\theta_j - \theta_{p+j})/2}\sin\left(\frac{\theta - \theta_j}{2}\right)\sin\left(\frac{\theta - \theta_{p+j}}{2}\right);$$

hence, (12) implies that

$$\int_{-\pi}^{\pi} g_1(\theta)\prod_{j=1}^{2p}\sin\left(\frac{\theta - \theta_j}{2}\right)\,d\theta = 0,$$

which is impossible because of (8) and our observation that the function in (9) is sign constant on $(-\pi, \pi)$. ∎

Theorem 3 immediately implies the following theorems. Theorem 6 was proved in [8].

THEOREM 4 *If f is monotonic on $(-\pi, \pi)$ or there is a number ϕ in $(-\pi, \pi)$ such that f is monotonic on $(-\pi, \phi)$ and (ϕ, π) then all eigenvalues of T_n have multiplicity one.*

THEOREM 5 *Suppose that $f(-\theta) = f(\theta)$, so that T_n is a real symmetric Toeplitz matrix. If λ_r is an eigenvalue of T_n with multiplicity m then $f(\theta) - \lambda_r$ must change sign at least m times in $(0, \pi)$*

THEOREM 6 *Suppose that $f(-\theta) = f(\theta)$ and f is monotonic on $(0, \pi)$. Then all the eigenvalues of T_n have multiplicity one.*

3 Location of the zeros of eigenpolynomials

The following theorem is the main result of this section.

THEOREM 7 *Suppose that the eigenvalue λ_r has an associated eigenvector x_r such that $J\overline{x}_r = \xi x_r$, where ξ is a constant, and the eigenpolynomial $X_r(z)$ defined in (3) has $2m$ zeros ($m \geq 1$) that are not on the unit circle. Then $f(\theta) - \lambda_r$ must change sign at least $2m + 1$ times in $(-\pi, \pi)$.*

PROOF. The proof is by contradiction. Suppose $f(\theta) - \lambda_r$ changes sign only at the points $\theta_1 < \cdots < \theta_k$ in $(-\pi, \pi)$, where $1 \leq k \leq 2m$. Then, as in the proof of Theorem 3, the function (9) does not change sign in $(-\pi, \pi)$. (Again, $k = 2p$ if k is even, and we define $\theta_{2p} = \pi$ if $k = 2p - 1$.) From among the $2m$ zeros of $X_r(z)$ not on the unit circle choose $2p$ distinct zeros $\zeta_1, \ldots, \zeta_p, 1/\overline{\zeta}_1, \ldots, 1/\overline{\zeta}_p$, and define g as in (8).

From (6) and (7),

$$\int_{-\pi}^{\pi} g(\theta) X_r(z) \overline{X_s(z)}\, d\theta = 0 \, (1 \leq s \leq n),$$

which implies that

$$\int_{-\pi}^{\pi} g(\theta) X_r(z) \overline{Q(z)}\, d\theta = 0 \tag{13}$$

if Q is any polynomial of degree $\leq n - 1$.

Now define

$$q_j(z) = \frac{(z - e^{i\theta_j})(1 - e^{-i\theta_{p+j}} z)}{(z - \zeta_j)(1 - \overline{\zeta}_j z)},\ 1 \leq j \leq p,$$

and let

$$Q(z) = X_r(z) q_1(z) \cdots q_p(z).$$

Then (13) implies that

$$\int_{-\pi}^{\pi} g(\theta) |X_r(z)|^2 \overline{q_1(z)} \cdots \overline{q_p(z)}\, d\theta = 0. \tag{14}$$

However, if $z = e^{i\theta}$ then

$$q_j(z) = \frac{4e^{i(\theta_j - \theta_{p+j})/2}}{|1 - \overline{\zeta}_j e^{i\theta}|^2} \sin\left(\frac{\theta - \theta_j}{2}\right) \sin\left(\frac{\theta - \theta_{p+j}}{2}\right).$$

This and (14) imply that

$$\int_{-\pi}^{\pi} \frac{g(\theta)|X_r(z)|^2}{\prod_{j=1}^{p} |1 - \overline{\zeta}_j e^{i\theta}|^2} \prod_{j=1}^{2p} \sin\left(\frac{\theta - \theta_j}{2}\right) d\theta = 0, \tag{15}$$

which is impossible, since the function (9) is sign constant in $(-\pi, \pi)$. □

Theorem 7 immediately implies the following theorem.

THEOREM 8 *If f satisfies the hypotheses of either Theorem 4 or Theorem 6 then all zeros of the eigenpolynomials of T_n are on the unit circle $|z| = 1$.*

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