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TRENCH'S CANONICAL FORM FOR A DISCONJUGATE nTH-ORDER LINEAR DIFFERENCE EQUATION

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Abstract. We consider the disconjugate, nth order linear difference equation $l_nu(t) = u(t+n) + p_1(t)u(t+n-1) + \cdots + p_n(t)u(t) = 0$. We will prove the existence of a Trench factorization for $l_nu(t) = 0$. We will then use this factorization to find a set of solutions $\{u_0, u_1, \dots, u_{n-1}\}$ such that

$$\lim_{t\to\infty} \frac{u_i(t)}{u_{i+1}(t)} = 0 \text{ if } 0 \le i \le n-2.$$

A set of solutions with this property is called a *principal set* of solutions.

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1. Introduction

At the Midwest Differential Equations Conference in 1985, Bennette Harris presented a talk on the discrete Trench factorization of an nth order linear difference equation, but the material was never offered for publication. William Trench and later Robert Krueger worked independently on results related to the talk by Harris. This paper contains their combined results.

We consider the nth order linear difference equation

(1) $$l_nu(t) = u(t+n) + p_1(t)u(t+n-1) + \cdots + p_n(t)u(t) = 0$$

with the condition

(2) $$(-1)^n p_n(t) > 0$$

where t is defined on the discrete interval $[a, b] = \{a, a+1, \dots, b\}$. We will use the forward difference operator Δ, which is defined by $\Delta u(t) = u(t+1) - u(t)$.

Given that $u(t)$ is a real-valued function on $[a, b+n]$, we say $t_0 = a$ is a *generalized zero* for $u(t)$ if $u(a) = 0$ and that $t_0 > a$ is a *generalized zero* for $u(t)$ if either $u(t_0) = 0$ or there exists an integer k, $1 \le k \le t_0 - a$, such that $(-1)^k u(t_0-k)u(t_0) > 0$ and if $k > 1$, $u(t_0-k+1) = \cdots = u(t_0-1) = 0$.

The difference equation $l_nu(t) = 0$ is said to be *disconjugate* on $[a, b+n]$ if no nontrivial solution has n or more generalized zeros in $[a, b+n]$. We will usually assume that our equation (1) is disconjugate on $[a, b+n]$.

In 1922, Polya [7] considered the nth order linear differential equation

(3) $$L_ny = y^{(n)} + q_1(x)y^{(n-1)} + \cdots + q_n(x)y = 0$$

Date:

where the coefficient functions $q_i(x)$, $1 \le i \le n$, are assumed to be continuous on an interval I. If (3) is disconjugate on an open interval I, then certain Wronskians of solutions are positive on I. Polya [7] showed that if these conditions on the Wronskians are satisfied on an open interval I, then there are positive functions $s_i(x)$ on I of class $C^{n-i}(I)$ such that for any function u of class $C^n(I)$,

$$L_n u(x) = s_n(x)\frac{d}{dx}\left(s_{n-1}(x)\frac{d}{dx}\left(\dots\left(\frac{d}{dx}\left(s_0(x)u(x)\right)\right)\dots\right)\right) \tag{4}$$

for $x \in I$. Now (4) is called the Polya factorization of $L_n u(x)$. Coppel [2], Hartman [4], and Levin [6], showed this for an arbitrary interval. In 1974, Trench [8] proved that if $L_n u(x) = 0$ is disconjugate on $I = [a, b)$, $a < b \le \infty$, then there is a Polya factorization of $L_n u(x)$ of the form (4) where

$$\int^b \frac{1}{s_i(x)} dx = \infty, \tag{5}$$

$1 \le i \le n-1$. When the conditions (5) hold, (4) is called a Trench factorization of (3). (See, for example, page 5 in [3].)

In 1978, Hartman [5] proved that if (1) is disconjugate on $[a, b+n]$ and there are solutions $u_1(t), \dots, u_n(t)$ such that the Wronskian (Casoratian) defined by

$$w_k(t) := \begin{vmatrix} u_1(t) & u_2(t) & \cdots & u_k(t) \\ u_1(t+1) & u_2(t+1) & \cdots & u_k(t+1) \\ \vdots & \vdots & \ddots & \vdots \\ u_1(t+k-1) & u_2(t+k-1) & \cdots & u_k(t+k-1) \end{vmatrix} > 0 \tag{6}$$

on $[a, b+n-k+1]$ for $1 \le k \le n$, then for any $u(t)$ defined on $[a, b+n]$, we obtain the Polya factorization of $l_n u$,

$$l_n u(t) = \rho_n(t)\Delta(\rho_{n-1}(t)\Delta(\dots\Delta(\rho_0(t)u(t))\dots) \tag{7}$$

for $t \in [a, b]$, where

$$\rho_0(t) = \frac{1}{u_1(t)} > 0, \quad t \in [a, b+n]$$
$$\rho_i(t) = \frac{w_i(t)\ w_i(t+1)}{w_{i-1}(t+1)\ w_{i+1}(t)} > 0,$$
$$t \in [a, b+n-i], \quad 1 \le i \le n-1$$
$$\rho_n(t) = \frac{w_n(t+1)}{w_{n-1}(t+1)} > 0, \quad t \in [a, b].$$

Our results are the discrete analogues of the work done by Trench in [8]. In particular, we will show that (7) can be written as

$$l_n u = \frac{1}{\beta_n(t)}\Delta\left[\dots\frac{1}{\beta_1(t)}\Delta\left(\frac{u(t)}{\beta_0(t)}\right)\right] \tag{8}$$

where

$$\sum_{t=a}^{\infty} \beta_i(t) = \infty$$

for $1 \le i \le n-1$.

We will use this Trench factorization to find a set of solutions $\{u_0, u_1, \dots, u_{n-1}\}$ such that

$$\lim_{t\to\infty} \frac{u_i(t)}{u_{i+1}(t)} = 0 \text{ if } 0 \le i \le n-2.$$

A set of solutions with this property is called a *principal set* of solutions.

2. Preliminaries

The following Lemmas will be the $n = 2$ and $n = 3$ cases for our induction in the proof of Theorem 3.

Lemma 1. *If*

$$M = \frac{1}{\alpha_2}\Delta\left(\frac{1}{\alpha_1}\Delta\left(\frac{\cdot}{\alpha_0}\right)\right)$$

with $\sum_{t=a}^{\infty}\alpha_1(t) < \infty$*, then* M *can be rewritten as*

$$M = \frac{1}{\beta_2}\Delta\left(\frac{1}{\beta_1}\Delta\left(\frac{\cdot}{\beta_0}\right)\right)$$

such that $\sum_{t=a}^{\infty}\beta_1(t) = \infty$.

Proof. Let

$$\beta_0(t) = \alpha_0(t)\sum_{s=t}^{\infty}\alpha_1(s), \tag{9}$$

$$\beta_1(t) = \frac{\alpha_1(t)}{\sum_{s=t}^{\infty}\alpha_1(s)\sum_{s=t+1}^{\infty}\alpha_1(s)}, \tag{10}$$

$$\beta_2(t) = \alpha_2(t)\sum_{s=t+1}^{\infty}\alpha_1(s), \tag{11}$$

and let

$$\xi(t) = \sum_{s=t}^{\infty}\alpha_1(s). \tag{12}$$

Then

$$\Delta\xi(t) = -\alpha_1(t), \tag{13}$$

$$\Delta\left(\frac{1}{\xi(t)}\right) = \frac{-\Delta\xi(t)}{\xi(t)\xi(t+1)}. \tag{14}$$

Using (10), consider

$$\sum_{t=a}^{\infty}\beta_1(t) = \sum_{t=a}^{\infty}\left(\frac{\alpha_1(t)}{\sum_{s=t}^{\infty}\alpha_1(s)\sum_{s=t+1}^{\infty}\alpha_1(s)}\right).$$

By (12) and (13),

$$\sum_{t=a}^{\infty}\beta_1(t) = \sum_{t=a}^{\infty}\frac{-\Delta\xi(t)}{\xi(t)\xi(t+1)}.$$

By (14),

$$\begin{aligned}\sum_{t=a}^{\infty}\beta_1(t) &= \sum_{t=a}^{\infty}\Delta\left(\frac{1}{\xi(t)}\right)\\ &= \lim_{b\to\infty}\sum_{t=a}^{b}\Delta\left(\frac{1}{\xi(t)}\right)\\ &= \lim_{b\to\infty}\left(\frac{1}{\xi(b+1)}-\frac{1}{\xi(a)}\right)\\ &= \lim_{b\to\infty}\left(\frac{1}{\sum_{s=b+1}^{\infty}\alpha_1(s)}-\frac{1}{\sum_{s=a}^{\infty}\alpha_1(s)}\right)\\ &= \infty.\end{aligned}$$

Now we must show that the two factorizations define the same operator. If $u(t)$ is an arbitrary function defined on $[a,\infty)$ then

$$\begin{aligned}\Delta\left(\frac{u(t)}{\beta_0(t)}\right) &= \Delta\left(\frac{\alpha_0^{-1}(t)u(t)}{\sum_{s=t}^{\infty}\alpha_1(s)}\right)\\ &= \frac{\sum_{s=t}^{\infty}\alpha_1(s)\Delta(\alpha_0^{-1}(t)u(t))-\alpha_0^{-1}(t)u(t)(-\alpha_1(t))}{\sum_{s=t}^{\infty}\alpha_1(s)\sum_{s=t+1}^{\infty}\alpha_1(s)}.\end{aligned}$$

Then

$$\begin{aligned}\frac{1}{\beta_1(t)}\Delta\left(\frac{u(t)}{\beta_0(t)}\right) &= \frac{\sum_{s=t}^{\infty}\alpha_1(s)\sum_{s=t+1}^{\infty}\alpha_1(s)}{\alpha_1(t)}\left(\frac{\sum_{s=t}^{\infty}\alpha_1(s)\Delta(\alpha_0^{-1}(t)u(t))-\alpha_0^{-1}(t)u(t)(-\alpha_1(t))}{\sum_{s=t}^{\infty}\alpha_1(s)\sum_{s=t+1}^{\infty}\alpha_1(s)}\right)\\ &= \left[\sum_{s=t}^{\infty}\alpha_1(s)\right]\left[\alpha_1^{-1}(t)\Delta(\alpha_0^{-1}(t)u(t))\right]+\alpha_0^{-1}(t)u(t).\end{aligned}$$

It follows that

$$\Delta\left(\frac{1}{\beta_1(t)}\Delta\left(\frac{u(t)}{\beta_0(t)}\right)\right) = \left[\sum_{s=t+1}^{\infty}\alpha_1(s)\right]\Delta\left[\alpha_1^{-1}(t)\Delta(\alpha_0^{-1}(t)u(t))\right].$$

Finally,

$$\begin{aligned}\frac{1}{\beta_2(t)}\Delta\left(\frac{1}{\beta_1(t)}\Delta\left(\frac{u(t)}{\beta_0(t)}\right)\right) &= \frac{1}{\alpha_2(t)\sum_{s=t+1}^{\infty}\alpha_1(s)}\left[\sum_{s=t+1}^{\infty}\alpha_1(s)\right]\Delta\left[\alpha_1^{-1}(t)\Delta(\alpha_0^{-1}(t)u(t))\right]\\ &= \frac{1}{\alpha_2(t)}\Delta\left(\frac{1}{\alpha_1(t)}\Delta\left(\frac{u(t)}{\alpha_0(t)}\right)\right).\end{aligned}$$

□

Lemma 2. *If*

$$N = \frac{1}{\mu_3}\Delta\left(\frac{1}{\mu_2}\Delta\left(\frac{1}{\mu_1}\Delta\left(\frac{\cdot}{\mu_0}\right)\right)\right)$$

with $\sum_{t=a}^{\infty}\mu_1(t)=\infty$ *and* $\sum_{t=a}^{\infty}\mu_2(t)<\infty$, *then* N *can be rewritten as*

$$N = \frac{1}{\nu_3}\Delta\left(\frac{1}{\nu_2}\Delta\left(\frac{1}{\nu_1}\Delta\left(\frac{\cdot}{\nu_0}\right)\right)\right)$$

where $\sum_{t=a}^{\infty} \nu_1(t) = \infty$ *and* $\sum_{t=a}^{\infty} \nu_2(t) = \infty$.

Proof. Apply Lemma 1 to obtain

$$N = \frac{1}{\tilde{\nu_3}} \Delta \left(\frac{1}{\tilde{\nu_2}} \Delta \left(\frac{1}{\tilde{\nu_1}} \Delta \left(\frac{\cdot}{\tilde{\nu_0}} \right) \right) \right)$$

where

$$\tilde{\nu_0}(t) = \mu_0(t), \tag{15}$$

$$\tilde{\nu_1}(t) = \mu_1(t) \sum_{s=t}^{\infty} \mu_2(s), \tag{16}$$

$$\tilde{\nu_2}(t) = \frac{\mu_2(t)}{\sum_{s=t}^{\infty} \mu_2(s) \sum_{s=t+1}^{\infty} \mu_2(s)}, \tag{17}$$

$$\tilde{\nu_3}(t) = \mu_3(t) \sum_{s=t+1}^{\infty} \mu_2(s), \tag{18}$$

and $\sum_{t=a}^{\infty} \tilde{\nu_2}(t) = \infty$.

If $\sum_{t=a}^{\infty} \tilde{\nu_1}(t) = \infty$, then there is nothing to show, so assume $\sum_{t=a}^{\infty} \tilde{\nu_1}(t) < \infty$. Now apply Lemma 1 again to obtain

$$N = \frac{1}{\nu_3} \Delta \left(\frac{1}{\nu_2} \Delta \left(\frac{1}{\nu_1} \Delta \left(\frac{\cdot}{\nu_0} \right) \right) \right)$$

where

$$\nu_0(t) = \tilde{\nu_0}(t) \sum_{s=t}^{\infty} \tilde{\nu_1}(s), \tag{19}$$

$$\nu_1(t) = \frac{\tilde{\nu_1}(t)}{\sum_{s=t}^{\infty} \tilde{\nu_1}(s) \sum_{s=t+1}^{\infty} \tilde{\nu_1}(s)}, \tag{20}$$

$$\nu_2(t) = \tilde{\nu_2}(t) \sum_{s=t+1}^{\infty} \tilde{\nu_1}(s), \tag{21}$$

$$\nu_3(t) = \tilde{\nu_3}(t), \tag{22}$$

and $\sum_{t=a}^{\infty} \nu_1(t) = \infty$. We must show $\sum_{t=a}^{\infty} \nu_2(t) = \infty$. Consider

$$\sum_{t=a}^{b-1} \nu_2(t) = \sum_{t=a}^{b-1} \left[\tilde{\nu_2}(t) \sum_{s=t+1}^{\infty} \tilde{\nu_1}(s) \right]$$

by (21). Now by summation by parts, we obtain

$$\sum_{t=a}^{b-1} \nu_2(t) = \left(\sum_{s=t+1}^{\infty} \tilde{\nu_1}(s) \right) \left(\sum_{s=a}^{t-1} \tilde{\nu_2}(s) \right) \Bigg|_{t=a}^{t=b} + \sum_{t=a}^{b-1} \left(\tilde{\nu_1}(t+1) \sum_{s=a}^{t} \tilde{\nu_2}(s) \right).$$

Note: In the first term when $t = a$, the sum (by convention) equals zero. Now from (17) we obtain,

$$\sum_{t=a}^{b-1} \nu_2(t) = \left(\sum_{t=b+1}^{\infty} \tilde{\nu_1}(t)\right)\left(\sum_{t=a}^{b-1} \frac{\mu_2(t)}{\sum_{s=t}^{\infty}\mu_2(s)\sum_{s=t+1}^{\infty}\mu_2(s)}\right) + \sum_{t=a}^{b-1}\left(\tilde{\nu_1}(t+1)\sum_{s=a}^{t}\frac{\mu_2(s)}{\sum_{j=s}^{\infty}\mu_2(j)\sum_{j=s+1}^{\infty}\mu_2(j)}\right).$$

Using the same idea as in (12)-(14) yields

$$\begin{aligned}\sum_{t=a}^{b-1} \nu_2(t) &= \left(\sum_{t=b+1}^{\infty} \tilde{\nu_1}(t)\right)\sum_{t=a}^{b-1}\Delta_t\left[\frac{1}{\sum_{s=t}^{\infty}\mu_2(s)}\right] + \sum_{t=a}^{b-1}\left(\tilde{\nu_1}(t+1)\sum_{s=a}^{t}\Delta_s\left[\frac{1}{\sum_{j=s}^{\infty}\mu_2(j)}\right]\right)\\ &= \left(\sum_{t=b+1}^{\infty} \tilde{\nu_1}(t)\right)\left(\frac{1}{\sum_{t=b}^{\infty}\mu_2(t)} - \frac{1}{\sum_{t=a}^{\infty}\mu_2(t)}\right)\\ &\quad + \sum_{t=a}^{b-1}\left(\tilde{\nu_1}(t+1)\left(\frac{1}{\sum_{s=t+1}^{\infty}\mu_2(s)} - \frac{1}{\sum_{s=a}^{\infty}\mu_2(s)}\right)\right).\end{aligned}$$

Using (16), we obtain

$$\begin{aligned}\sum_{t=a}^{b-1} \nu_2(t) &= \left(\sum_{t=b+1}^{\infty} \tilde{\nu_1}(t)\right)\left(\frac{1}{\sum_{t=b}^{\infty}\mu_2(t)} - \frac{1}{\sum_{t=a}^{\infty}\mu_2(t)}\right)\\ &\quad + \sum_{t=a}^{b-1}\left[\mu_1(t+1)\left(\sum_{s=t+1}^{\infty}\mu_2(s)\right)\left(\frac{1}{\sum_{s=t+1}^{\infty}\mu_2(s)} - \frac{1}{\sum_{s=a}^{\infty}\mu_2(s)}\right)\right]\\ &= \left(\sum_{t=b+1}^{\infty} \tilde{\nu_1}(t)\right)\left(\frac{1}{\sum_{t=b}^{\infty}\mu_2(t)} - \frac{1}{\sum_{t=a}^{\infty}\mu_2(t)}\right)\\ &\quad + \sum_{t=a}^{b-1}\mu_1(t+1) - \sum_{t=a}^{b-1}\frac{\mu_1(t+1)\sum_{s=t+1}^{\infty}\mu_2(s)}{\sum_{s=a}^{\infty}\mu_2(s)}.\end{aligned}$$

Changing the index of summation in the last two terms,

$$\begin{aligned}\sum_{t=a}^{b-1} \nu_2(t) &= \left(\sum_{t=b+1}^{\infty} \tilde{\nu_1}(t)\right)\left(\frac{1}{\sum_{t=b}^{\infty}\mu_2(t)} - \frac{1}{\sum_{t=a}^{\infty}\mu_2(t)}\right)\\ &\quad + \sum_{t=a+1}^{b}\mu_1(t) - \sum_{t=a+1}^{b}\frac{\mu_1(t)\sum_{s=t}^{\infty}\mu_2(s)}{\sum_{s=a}^{\infty}\mu_2(s)}.\end{aligned}$$

So by (16),

$$\sum_{t=a}^{b-1} \nu_2(t) = \frac{\sum_{t=b+1}^{\infty}\tilde{\nu_1}(t)}{\sum_{t=b}^{\infty}\mu_2(t)} - \frac{\sum_{t=b+1}^{\infty}\tilde{\nu_1}(t)}{\sum_{t=a}^{\infty}\mu_2(t)} + \sum_{t=a+1}^{b}\mu_1(t) - \frac{\sum_{t=a+1}^{b}\tilde{\nu_1}(t)}{\sum_{s=a}^{\infty}\mu_2(s)}.$$

Combining the second and last terms, we obtain

$$\sum_{t=a}^{b-1} \nu_2(t) = \frac{\sum_{t=b+1}^{\infty}\tilde{\nu_1}(t)}{\sum_{t=b}^{\infty}\mu_2(t)} - \frac{\sum_{t=a+1}^{\infty}\tilde{\nu_1}(t)}{\sum_{t=a}^{\infty}\mu_2(t)} + \sum_{t=a+1}^{b}\mu_1(t).$$

Letting $b \to \infty$, we obtain the desired result. Using a method similar to the proof of Lemma 1, we can show the two factorizations define the same operator. □

3. Main Results

Note that the operator l_n in Theorem 3 below is in a Polya factorization form where $\rho_i(t) = \frac{1}{\alpha_i(t)}$ for $0 \le i \le n$.

Theorem 3. *Any operator*

$$l_n = \frac{1}{\alpha_n}\Delta\left(\frac{1}{\alpha_{n-1}}\Delta \ldots \Delta\left(\frac{1}{\alpha_1}\Delta\left(\frac{\cdot}{\alpha_0}\right)\right)\right) \tag{23}$$

with $\alpha_i(t) > 0$ on $[a, \infty)$, $0 \le i \le n$, can be written as

$$l_n = \frac{1}{\beta_n}\Delta\left(\frac{1}{\beta_{n-1}}\Delta \ldots \Delta\left(\frac{1}{\beta_1}\Delta\left(\frac{\cdot}{\beta_0}\right)\right)\right) \tag{24}$$

with $\sum_{t=a}^{\infty} \beta_i(t) = \infty$ for $1 \le i \le n-1$ and $\beta_i(t) > 0$ on $[a, \infty)$, $0 \le i \le n$.

Proof. Proof is by induction. Lemma 1 and Lemma 2 imply the desired results for $n = 2$ and $n = 3$, respectively. Suppose $n \ge 4$ and assume the theorem is satisfied for any $(n-1)$st order operator l_n. This implies

$$\sum_{t=a}^{\infty} \alpha_j(t) = \infty \tag{25}$$

for $1 \le j \le n-2$. If $\sum_{t=a}^{\infty} \alpha_{n-1}(t) = \infty$, then we are done, so assume $\sum_{t=a}^{\infty} \alpha_{n-1}(t) < \infty$.

Construct a sequence of operators

$$l_n = \frac{1}{\alpha_{n,i}}\Delta\left(\frac{1}{\alpha_{n-1,i}}\Delta \ldots \Delta\left(\frac{1}{\alpha_{1,i}}\Delta\left(\frac{\cdot}{\alpha_{0,i}}\right)\right)\right) \tag{26}$$

with $\alpha_{j,0} = \alpha_j$ for $0 \le j \le n$. So for $i \ge 1$,

$$\alpha_{j,i}(t) = \alpha_{j,i-1}(t), \quad j \ne n-i+1,\ n-i,\ n-i-1, \tag{27}$$

$$\alpha_{n-i+1,i}(t) = \alpha_{n-i+1,i-1}(t) \sum_{\tau=t+1}^{\infty} \alpha_{n-i,i-1}(\tau), \tag{28}$$

$$\alpha_{n-i,i}(t) = \frac{\alpha_{n-i,i-1}(t)}{\sum_{\tau=t}^{\infty} \alpha_{n-i,i-1}(\tau) \sum_{\tau=t+1}^{\infty} \alpha_{n-i,i-1}(\tau)}, \tag{29}$$

$$\alpha_{n-i-1,i}(t) = \alpha_{n-i-1,i-1}(t) \sum_{\tau=t}^{\infty} \alpha_{n-i,i-1}(\tau). \tag{30}$$

This process stops at the ith step if

$$\sum_{t=a}^{\infty} \alpha_{j,i}(t) = \infty \tag{31}$$

for all $j \in \{1, \ldots, n-1\}$.

Now (31) holds by the induction hypothesis and (27)-(30) except possibly for $j = n-i-1$. Hence, this process terminates when

$$\sum_{t=a}^{\infty} \alpha_{n-i-1,i}(t) = \infty \tag{32}$$

or when $i = n-1$, whichever is first. So if the process terminates at $i = r$, then $\beta_j = \alpha_{j,r}$, $0 \le j \le n$.

Finally, we can show that the two factorizations define the same operator using the same method as in the proof of Lemma 1. □

Corollary 4. *If $l_n u = 0$ is disconjugate on $[a, \infty)$, then the operator l_n has a Trench factorization.*

We say $\{u_0(t), u_1(t), \ldots, u_{n-1}(t)\}$ is a *principal set of solutions* of $l_n u(t) = 0$ on $[a, \infty)$ provided, for each i, $0 \le i \le n-1$, $u_i(t) > 0$ in a neighborhood of infinity and

$$\lim_{t\to\infty} \frac{u_i(t)}{u_{i+1}(t)} = 0 \ \text{ if } \ 0 \le i \le n-2.$$

To show the Trench factorization leads to a principal set of solutions, first we need a theorem which is Theorem 1.7.9 in Agarwal [1].

Theorem 5. **(Discrete L'Hospital's Rule)** *Let $u(k)$ and $v(k)$ be defined on $[a, \infty)$ and assume $v(k) > 0$ and $\Delta v(k) > 0$ for all large $k \in [a, \infty)$, then, for $0 \le c \le \infty$, if*

$$\lim_{k\to\infty} v(k) = \infty$$

and

$$\lim_{k\to\infty} \frac{\Delta u(k)}{\Delta v(k)} = c,$$

then

$$\lim_{k\to\infty} \frac{u(k)}{v(k)} = c.$$

We now prove the existence of a principal set of solutions of (1).

Theorem 6. *If $l_n u(t) = 0$ is disconjugate on $[a, \infty)$, then there exists a principal set of solutions on $[a, \infty)$.*

Proof. By Corollary 4, we have a Trench factorization (8). So assume

$$\frac{1}{\beta_n(t)}\Delta\left(\frac{1}{\beta_{n-1}(t)}\Delta\ldots\Delta\left(\frac{1}{\beta_1(t)}\Delta\left(\frac{1}{\beta_0(t)}u(t)\right)\right)\ldots\right) = 0 \tag{33}$$

is a Trench factorization of $l_n u = 0$.

First, if we set $u_0(t) = \beta_0(t)$, then $u_0(t)$ is a solution of (33).

Second, we take $u_1(t)$ to be the solution of the IVP

$$\frac{1}{\beta_1}\Delta\left(\frac{1}{\beta_0}u\right) = 1, \ u(a) = 0.$$

That would imply

$$u_1(t) = \beta_0(t)\sum_{s=a}^{t-1}\beta_1(s).$$

Finally, we take $u_k(t)$, $1 \le k \le n-1$, to be the solution of the IVP

$$\frac{1}{\beta_k}\Delta\left(\frac{1}{\beta_{k-1}}\Delta\ldots\Delta\left(\frac{1}{\beta_0}u\right)\ldots\right) = 1$$
$$u(a) = \Delta u(a) = \cdots = \Delta^{k-1}u(a) = 0.$$

Solving this IVP yields

$$u_k(t) = \beta_0(t)\sum_{s_1=a}^{t-1}\left(\beta_1(s_1)\sum_{s_2=a}^{s_1-1}\cdots\sum_{s_{k-1}=a}^{s_{k-2}-1}\left(\beta_{k-1}(s_{k-1})\sum_{s_k=a}^{s_{k-1}-1}\beta_k(s_k)\right)\ldots\right). \tag{34}$$

To show that $\{u_0(t), u_1(t), \ldots, u_{n-1}(t)\}$ is a principal set of solutions for (1), first we must show $u_k(t) > 0$ in a neighborhood of infinity for $0 \le k \le n-1$.

When $s_{k-1} = a$ in equation (34), then (by convention)

$$\sum_{s_k=a}^{a-1} \beta_k(s_k) = 0.$$

Therefore,

$$\sum_{s_{k-1}=a}^{s_{k-2}-1} (\dots) = \sum_{s_{k-1}=a+1}^{s_{k-2}-1} (\dots)\,. \tag{35}$$

When $s_{k-2} = a+1$, we obtain a similar expression which is zero. Eventually, we can rewrite (34) as

$$\begin{aligned}
u_k(t) &= \beta_0(t) \sum_{s_1=a+k-1}^{t-1} \left(\beta_1(s_1) \sum_{s_2=a+k-2}^{s_1-1} \cdots \sum_{s_{k-1}=a+1}^{s_{k-2}-1} \left(\beta_{k-1}(s_{k-1}) \sum_{s_k=a}^{s_{k-1}-1} \beta_k(s_k) \right) \dots \right) \\
&\geq \beta_0(t) \sum_{s_1=a+k-1}^{t-1} \left(\beta_1(s_1) \sum_{s_2=a+k-2}^{s_1-1} \cdots \sum_{s_{k-1}=a+1}^{s_{k-2}-1} (\beta_{k-1}(s_{k-1}) \cdot \beta_k(a)) \dots \right) \\
&\geq \beta_0(t) \sum_{s_1=a+k-1}^{t-1} \left(\beta_1(s_1) \sum_{s_2=a+k-2}^{s_1-1} \dots (\beta_{k-1}(a+1) \cdot \beta_k(a)) \dots \right) \\
&\geq \beta_k(a) \cdot \beta_{k-1}(a+1) \dots \beta_2(a+k-2) \cdot \beta_0(t) \sum_{s_1=a+k-1}^{t-1} (\beta_1(s_1)) \\
&\to \infty \text{ as } t \to \infty
\end{aligned}$$

because (33) is a Trench factorization. Therefore, $u_k(t) > 0$ near infinity and we will be able to use Theorem 5 since $u_k(t) \to \infty$ as $t \to \infty$.

Now consider

$$\begin{aligned}
\lim_{t\to\infty} \frac{u_1(t)}{u_0(t)} &= \lim_{t\to\infty} \frac{\beta_0(t) \sum_{s=a}^{t-1} \beta_1(s)}{\beta_0(t)} \\
&= \lim_{t\to\infty} \sum_{s=a}^{t-1} \beta_1(s) \\
&= \sum_{s=a}^{\infty} \beta_1(s) = \infty
\end{aligned}$$

since (33) is a Trench factorization. Therefore,

$$\lim_{t\to\infty} \frac{u_0(t)}{u_1(t)} = 0.$$

Similarly, consider

$$\lim_{t\to\infty} \frac{u_k(t)}{u_{k-1}(t)} = \lim_{t\to\infty} \frac{\beta_0(t) \sum_{s_1=a}^{t-1} \left(\beta_1(s_1) \sum_{s_2=a}^{s_1-1} \cdots \sum_{s_{k-1}=a}^{s_{k-2}-1} \left(\beta_{k-1}(s_{k-1}) \sum_{s_k=a}^{s_{k-1}-1} \beta_k(s_k) \right) \dots \right)}{\beta_0(t) \sum_{s_1=a}^{t-1} \left(\beta_1(s_1) \sum_{s_2=a}^{s_1-1} \cdots \sum_{s_{k-1}=a}^{s_{k-2}-1} (\beta_{k-1}(s_{k-1})) \dots \right)}.$$

By cancellation and using Theorem 5 $k-1$ times, we obtain

$$\lim_{t\to\infty}\frac{u_k(t)}{u_{k-1}(t)} = \lim_{t\to\infty}\frac{\beta_{k-1}(t)\sum_{s=a}^{t-1}\beta_k(s)}{\beta_{k-1}(t)}$$

$$= \sum_{s=a}^{\infty}\beta_k(s) = \infty$$

for $1\le k\le n-1$ because (33) is a Trench factorization. Therefore,

$$\lim_{t\to\infty}\frac{u_{k-1}(t)}{u_k(t)} = 0 \text{ for } 1\le k\le n-1.$$

□

Example 7. $l_3u(t) = u(t+3) - 9u(t+2) + 26u(t+1) - 24u(t) = 0$

The third order difference equation $l_3u(t) = 0$ has solutions

$$u_1(t) = 3^t,\ u_2(t) = -2^t,\ u_3(t) = 4^t$$

such that

$$w_1(t) = 3^t > 0,$$
$$w_2(t) = 6^t > 0,$$
$$w_3(t) = 2(24)^t > 0.$$

Thus, we have the Polya factorization

$$l_3u(t) = 8(4)^t\Delta\left(\left(\frac{1}{2}\right)^t\Delta\left(3\left(\frac{3}{2}\right)^t\Delta\left(\left(\frac{1}{3}\right)^t u(t)\right)\right)\right) \tag{36}$$

where

$$\rho_0(t) = \left(\frac{1}{3}\right)^t > 0,$$
$$\rho_1(t) = 3\left(\frac{3}{2}\right)^t > 0,$$
$$\rho_2(t) = \left(\frac{1}{2}\right)^t > 0,$$
$$\rho_3(t) = 8(4)^t > 0.$$

Consider

$$\alpha_0(t) = \frac{1}{\rho_0(t)} = 3^t,$$
$$\alpha_1(t) = \frac{1}{\rho_1(t)} = \frac{1}{3}\left(\frac{2}{3}\right)^t,$$
$$\alpha_2(t) = \frac{1}{\rho_2(t)} = 2^t,$$
$$\alpha_3(t) = \frac{1}{\rho_3(t)} = \frac{1}{8}\left(\frac{1}{4}\right)^t.$$

Notice that

$$\sum_{t=a}^{\infty}\alpha_1(t) = \left(\frac{2}{3}\right)^a < +\infty.$$

Therefore, (36) is not a Trench factorization. So by the proof of Lemma 2, define

$$\beta_0(t) := \alpha_0(t) \sum_{s=t}^{\infty} \alpha_1(s) = 2^t,$$

$$\beta_1(t) := \frac{\alpha_1(t)}{\sum_{s=t}^{\infty} \alpha_1(s) \sum_{s=t+1}^{\infty} \alpha_1(s)} = \frac{1}{2}\left(\frac{3}{2}\right)^t,$$

$$\beta_2(t) := \alpha_2(t) \sum_{s=t+1}^{\infty} \alpha_1(s) = \frac{2}{3}\left(\frac{4}{3}\right)^t,$$

$$\beta_3(t) := \alpha_3(t) = \frac{1}{8}\left(\frac{1}{4}\right)^t.$$

Since

$$\sum_{s=t}^{\infty} \beta_1(s) = \infty \text{ and } \sum_{s=t}^{\infty} \beta_2(s) = \infty,$$

$$l_3 u(t) = 8(4)^t \Delta\left(\frac{3}{2}\left(\frac{3}{4}\right)^t \Delta\left(2\left(\frac{2}{3}\right)^t \Delta\left(\left(\frac{1}{2}\right)^t u(t)\right)\right)\right)$$

is a Trench factorization of l_3.

We know from the proof in Theorem 6 that $l_3 u = 0$ has solutions of the form

$$u_0(t) = \beta_0(t) = \boxed{2^t}$$

$$u_1(t) = \beta_0(t) \sum_{s=a}^{t-1} \beta_1(s) = \boxed{3^t + C_0 2^t}$$

$$\begin{aligned} u_2(t) &= \beta_0(t) \sum_{s_1=a}^{t-1} \left(\beta_1(s_1) \sum_{s_2=a}^{s_1-1} \beta_2(s_2)\right) \\ &= \boxed{4^t + C_1 3^t + C_2 2^t} \end{aligned}$$

where the C_i are known quantities. These solutions form a principal set of solutions.

Now we will explore the essential uniqueness of the Trench factorization. First, a theorem about principal sets of solutions for the operator l_n.

Theorem 8. *If $\{x_0, \ldots, x_{n-1}\}$ and $\{y_0, \ldots, y_{n-1}\}$ are principal sets of solutions for $l_n u(t) = 0$ on $[a, \infty)$, then*

$$y_i = \sum_{j=0}^{i} a_{ij} x_j$$

where a_{ij} is a constant and $a_{ii} > 0$ for $0 \le j \le i$ and $0 \le i \le n-1$.

Proof. Since $\{x_0, \ldots, x_{n-1}\}$ is a linearly independent set of solutions to the nth order difference equation $l_n u(t) = 0$, We can write the solutions

$$y_i = \sum_{j=0}^{n-1} a_{ij} x_j$$

for $0 \le i \le n-1$ where a_{ij} is a constant for $0 \le j \le i$ and $0 \le i \le n-1$. Consider

$$\frac{y_i(t)}{y_{n-1}(t)} = \frac{a_{i,0}x_0(t) + \cdots + a_{i,n-2}x_{n-2}(t) + a_{i,n-1}x_{n-1}(t)}{a_{n-1,0}x_0(t) + \cdots + a_{n-1,n-2}x_{n-2}(t) + a_{n-1,n-1}x_{n-1}(t)}$$

for $0 \le i \le n-2$. Dividing through the numerator and the denominator by $x_{n-1}(t)$ and letting $t \to \infty$, we obtain

$$0 = \lim_{t\to\infty} \frac{y_i(t)}{y_{n-1}(t)} = \frac{a_{i,n-1}}{a_{n-1,n-1}}.$$

Therefore $a_{i,n-1} = 0$ for $0 \le i \le n-2$ and $a_{n-1,n-1} \ne 0$. Next consider

$$\frac{y_i(t)}{y_{n-2}(t)} = \frac{a_{i,0}x_0(t) + \cdots + a_{i,n-3}x_{n-3}(t) + a_{i,n-2}x_{n-2}(t)}{a_{n-2,0}x_0(t) + \cdots + a_{n-2,n-3}x_{n-3}(t) + a_{n-2,n-2}x_{n-2}(t)}$$

for $0 \le i \le n-3$. Dividing through the numerator and the denominator by $x_{n-2}(t)$ and letting $t \to \infty$, we obtain

$$0 = \lim_{t\to\infty} \frac{y_i(t)}{y_{n-2}(t)} = \frac{a_{i,n-2}}{a_{n-2,n-2}}.$$

Therefore $a_{i,n-2} = 0$ for $0 \le i \le n-3$ and $a_{n-2,n-2} \ne 0$. Continuing this process yields

$$y_i = \sum_{j=0}^{i} a_{ij}x_j \tag{37}$$

for $0 \le i \le n-1$ where $a_{ii} \ne 0$.

By assumption, $x_0(t) > 0$ and $y_0(t) > 0$ near infinity. Thus, by (37), $a_{00} > 0$. Similarily, by assumption and (37), $x_i(t) > 0$ and

$$y_i(t) = a_{i,0}x_0(t) + \cdots + a_{i,i-1}x_{i-1}(t) + a_{i,i}x_i(t) > 0$$

near infinity. Thus,

$$\frac{a_{i,0}x_0(t)}{x_i(t)} + \cdots + \frac{a_{i,i-1}x_{i-1}(t)}{x_i(t)} + \frac{a_{i,i}x_i(t)}{x_i(t)} > 0$$

near infinity. Letting $t \to \infty$, we obtain $a_{ii} > 0$ for $0 \le i \le n-1$. □

In the following theorem, we will obtain the essential uniqueness of the Trench factorization.

Theorem 9. *If*

$$l_n u(t) = \frac{1}{\alpha_n(t)}\Delta\left(\frac{1}{\alpha_{n-1}(t)}\Delta \dots \Delta\left(\frac{1}{\alpha_1(t)}\Delta\left(\frac{u(t)}{\alpha_0(t)}\right)\right)\dots\right) \tag{38}$$

and

$$l_n u(t) = \frac{1}{\beta_n(t)}\Delta\left(\frac{1}{\beta_{n-1}(t)}\Delta \dots \Delta\left(\frac{1}{\beta_1(t)}\Delta\left(\frac{u(t)}{\beta_0(t)}\right)\right)\dots\right) \tag{39}$$

are Trench factorizations for l_n, then

$$\alpha_i(t) = d_i\beta_i(t)$$

where the d_i are positive constants for $0 \le i \le n$.

Proof. Let $\{y_0, \dots, y_{n-1}\}$ be the principal set of solutions for $l_n u(t) = 0$ using the operator (38) as in the proof of Theorem 6 and let $\{x_0, \dots, x_{n-1}\}$ be the principal set of solutions for $l_n u(t) = 0$ using the operator (39) as in the proof of Theorem 6. Hence,

$$y_0(t) = \alpha_0(t) \text{ and } x_0(t) = \beta_0(t).$$

By Theorem 8,

$$y_0(t) = a_{00}x_0(t).$$

Thus we can conclude that

$$\alpha_0(t) = a_{00}\beta_0(t).$$

Furthermore,

$$y_1(t) = \alpha_0(t) \sum_{s_1=a}^{t-1} \alpha_1(s_1) \text{ and } x_1(t) = \beta_0(t) \sum_{s_1=a}^{t-1} \beta_1(s_1).$$

By Theorem 8,

$$y_1(t) = a_{10}x_0(t) + a_{11}x_1(t).$$

Thus we obtain

$$\begin{aligned}
\alpha_0(t) \sum_{s_1=a}^{t-1} \alpha_1(s_1) &= a_{10}\beta_0(t) + a_{11}\beta_0(t) \sum_{s_1=a}^{t-1} \beta_1(s_1) \\
a_{00}\beta_0(t) \sum_{s_1=a}^{t-1} \alpha_1(s_1) &= a_{10}\beta_0(t) + a_{11}\beta_0(t) \sum_{s_1=a}^{t-1} \beta_1(s_1) \\
a_{00} \sum_{s_1=a}^{t-1} \alpha_1(s_1) &= a_{10} + a_{11} \sum_{s_1=a}^{t-1} \beta_1(s_1).
\end{aligned}$$

Taking the difference of both sides yields

$$a_{00}\alpha_1(t) = a_{11}\beta_1(t).$$

Since $a_{00} > 0$,

$$\alpha_1(t) = \frac{a_{11}}{a_{00}}\beta_1(t).$$

Continuing this process, we get

$$\alpha_i(t) = \frac{a_{i,i}}{a_{i-1,i-1}}\beta_i(t)$$

for $0 \leq i \leq n-1$.

Thus, we could rewrite (38) as

$$l_n u(t) = \frac{1}{a_{n-1,n-1}} \frac{1}{\alpha_n(t)} \Delta \left(\frac{1}{\beta_{n-1}(t)} \Delta \dots \Delta \left(\frac{1}{\beta_1(t)} \Delta \left(\frac{u(t)}{\beta_0(t)} \right) \right) \dots \right). \tag{40}$$

Define

$$x_n(t) = \beta_0(t) \sum_{s_1=a}^{t-1} \left(\beta_1(s_1) \sum_{s_2=a}^{s_1-1} \cdots \sum_{s_{n-1}=a}^{s_{n-2}-1} \left(\beta_{n-1}(s_{n-1}) \sum_{s_n=a}^{s_{n-1}-1} \beta_n(s_n) \right) \dots \right).$$

Using (39), $l_n x_n(t) = 1$, while from (40),

$$l_n x_n(t) = \frac{1}{a_{n-1,n-1}} \frac{\beta_n(t)}{\alpha_n(t)}.$$

Hence, $\alpha_n(t) = \dfrac{1}{a_{n-1,n-1}}\beta_n(t)$.

Therefore, letting

$$\begin{aligned}
d_0 &= a_{00}, \\
d_i &= \frac{a_{i,i}}{a_{i-1,i-1}}, \text{ for } 1 \leq i \leq n-1, \\
d_n &= \frac{1}{a_{n-1,n-1}},
\end{aligned}$$

we obtain the desired result. □

References

[1] R. Agarwal, *Difference Equations and Inequalities: Theory, Methods, and Applications*, Marcel Dekker, New York, 1992.

[2] W.A. Coppel, *Disconjugacy*, Lecture notes in Mathematics, 220, Springer-Verlag, Berlin, 1971.

[3] U. Elias, *Oscillation theory of two-term differential equations*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.

[4] P. Hartman, *Principal solutions of disconjugate nth order linear differential equations,* Amer. J. Math. 91 (1969), 306-362.

[5] P. Hartman, *Difference equations: disconjugacy, principal solutions, Green's functions, complete monotonicity,* Trans. Amer. Math. Soc. 246 (1978), 1-29.

[6] A. Ju. Levin, *Non-oscillation of solutions of the equation* $x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = 0$, Russian Math. Surveys 24 (1969), 43-99.

[7] G. Polya, *On the mean-value theorem corresponding to a given linear homogeneous differential equation,* Trans. Amer. Math. Soc. 24 (1922), 312-324.

[8] W. Trench, *Canonical forms and principal systems for general disconjugate equations,* Trans. Amer. Math. Soc. 189 (1974), 319-327.