Conditional Convergence of Infinite Products

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In this article we revisit the classical subject of infinite products. For standard definitions and theorems on this subject see [1] or almost any textbook on complex analysis. We will restate parts of this material required to set the stage for our results, as follows.

The infinite product $P = \prod_{n=1}^{\infty} (1 + a_n)$ of complex numbers is said to converge if there is an integer $N$ such that $1 + a_n \neq 0$ for $n \geq N$ and $\lim_{n \to \infty} \prod_{m=N}^{n} (1 + a_m)$ is finite and nonzero. This occurs if and only if the series $\sum_{m=N}^{\infty} \log(1 + a_m)$ converges.

We say that $P$ converges absolutely if $\prod_{n=1}^{\infty} (1 + |a_n|)$ converges. If $P$ converges absolutely then $P$ converges, but the converse is false. The following theorem [1, p. 223] settles the question of absolute convergence of infinite products.

**Theorem 1** The infinite product $\prod_{n=1}^{\infty} (1 + a_n)$ converges absolutely if and only if $\sum_{n=1}^{\infty} |a_n| < \infty$.

If $P$ converges but $\prod_{n=1}^{\infty} (1 + |a_n|)$ does not, then we say that $P$ converges conditionally. Conditional convergence of $\sum_{n=1}^{\infty} a_n$ does not imply conditional convergence of $P$. The following theorem [1, p. 225] seems to be the only general result along these lines, at least in the textbook literature.

**Theorem 2** If $\sum_{n=1}^{\infty} |a_n|^2 < \infty$ then $\sum_{n=1}^{\infty} a_n$ and $\prod_{n=1}^{\infty} (1 + a_n)$ converge or diverge together.

Here we offer some other results concerning convergence of infinite products. Because of Theorem 1, these results are of interest only in the case where $\sum_{n=1}^{\infty} |a_n| = \infty$.

**Theorem 3** If there is a sequence $\{r_n\}$ such that

$$\lim_{n \to \infty} r_n = 1$$

and

$$\sum_{n=1}^{\infty} |r_n(1 + a_n) - r_{n+1}| < \infty,$$

then $\prod_{n=1}^{\infty} (1 + a_n)$ converges.
Proof: Let \( g_n = r_n(1 + a_n) - r_{n+1} \). Then

\[
\sum_{n=1}^{\infty} |g_n| < \infty
\]  

(3)

from (2), so \( \lim_{n \to \infty} g_n = 0 \) and therefore \( \lim_{n \to \infty} a_n = 0 \) by (1). Choose \( N \) so that \( r_n, 1 + a_n \) and \( 1 + g_n/r_{n+1} \) are nonzero if \( n \geq N \). Now define \( p_{N-1} = 1 \) and

\[
p_n = \prod_{m=N}^{n} (1 + a_m), \quad n \geq N.
\]

If \( n \geq N \) then \( 1 + a_n = p_n/p_{n-1} \), so \( g_n = (r_n p_n/p_{n-1}) - r_{n+1} \), and therefore \( p_n = r_{n+1} p_{n-1} (1 + g_n/r_{n+1})/r_n \), which implies that

\[
p_n = \frac{r_{n+1}}{r_N} \prod_{m=N}^{n} (1 + g_m/r_{m+1}).
\]  

(4)

Since (1) and (3) imply that \( \sum_{n=1}^{\infty} |g_m/r_{m+1}| < \infty \), Theorem 1 implies that the infinite product

\[
Q = \prod_{m=N}^{\infty} (1 + g_m/r_{m+1})
\]

converges; moreover \( Q \neq 0 \) because \( 1 + g_m/r_{m+1} \neq 0 \) if \( m \geq N \). Now (1) and (4) imply that \( \lim_{n \to \infty} p_n = Q/r_N \) is finite and nonzero. \( \blacksquare \)

To apply this theorem we must exhibit a sequence \( \{r_n\} \) that will enable us to obtain results even if \( \sum_{n=1}^{\infty} |a_n| = \infty \). The following theorem provides a way to do this.

**Theorem 4** Suppose that for some positive integer \( q \) the sequences

\[
a_n^{(k)} = \sum_{m=n}^{\infty} a_m a_m^{(k-1)}, \quad k = 1, \ldots, q \quad (\text{with} \quad a_0^{(0)} = 1),
\]

are all defined, and

\[
\sum_{n=1}^{\infty} |a_n a_n^{(q)}| < \infty.
\]  

(5)

Then \( \prod_{n=1}^{\infty} (1 + a_n) \) converges.

**Proof:** Define

\[
r_n^{(k)} = 1 + \sum_{j=1}^{k} (-1)^j a_n^{(j)}, \quad 1 \leq k \leq q.
\]

We show by finite induction on \( k \) that

\[
r_n^{(k)}(1 + a_n) - r_{n+1}^{(k)} = (-1)^k a_n a_n^{(k)}
\]  

(6)
for \(1 \leq k \leq q\). Since \(\lim_{n \to \infty} r_n^{(q)} = 1\) we can then set \(k = q\) and conclude from (5) and Theorem 3 with \(r_n = r_n^{(q)}\) that \(\prod_{n=1}^{\infty} (1 + a_n)\) converges.

Since \(r_n^{(1)} = 1 - a_n^{(1)}\) the left side of (6) with \(k = 1\) is

\[
(1 - a_n^{(1)})(1 + a_n) - (1 - a_{n+1}^{(1)}) = a_n - a_n^{(1)} - a_n a_n^{(1)} + a_n^{(1)} = -a_n a_n^{(1)},
\]

since \(a_n^{(1)} + a_n = a_n^{(1)}\). This proves (6) for \(k = 1\).

Now suppose that (6) holds if \(1 \leq k < q\). Since \(r_n^{(k)} = r_n^{(k+1)} + (-1)^k a_n^{(k+1)}\), (6) implies that

\[
\left( r_n^{(k+1)} + (-1)^k a_n^{(k+1)} \right) (1 + a_n) - r_n^{(k+1)} - (-1)^k a_n^{(k+1)} = (-1)^k a_n a_n^{(k)}.
\]

Therefore

\[
r_n^{(k+1)} (1 + a_n) - r_n^{(k+1)} = (-1)^k \left( a_n a_n^{(k)} - a_n a_n^{(k+1)} + a_n^{(k+1)} \right)
\]

\[
= (-1)^{(k+1)} a_n a_n^{(k+1)},
\]

since \(a_n^{(k+1)} + a_n a_n^{(k)} = a_n^{(k+1)}\). This completes the induction.

We now prepare for a specific application of Theorem 4. Henceforth \(\Delta\) is the forward difference operator; thus, if \(\{g_m\}\) is a sequence, then \(\Delta g_m = g_{m+1} - g_m\), while if \(G\) is a function of the continuous variable \(x\) then \(\Delta G(x) = G(x + 1) - G(x)\). Higher order forward differences are defined inductively; thus, if \(\nu \geq 2\) is an integer, then

\[
\Delta^{\nu} g_m = \Delta^{\nu-1} g_{m+1} - \Delta^{\nu-1} g_m = \sum_{r=0}^{\nu} (-1)^{r-\nu} \begin{pmatrix} \nu \\ r \end{pmatrix} g_{m+r}.
\]

A similar definition yields \(\Delta^{\nu} G(x)\).

**Lemma 1** Suppose that \(t\) is a real number, not an integral multiple of \(2\pi\), and \(\{g_m\}_{m=0}^{\infty}\) is a sequence such that \(\lim_{m \to \infty} g_m = 0\) and

\[
\sum_{m=0}^{\infty} |\Delta^{\nu} g_m| < \infty
\]

for some positive integer \(\nu\). Then \(\sum_{m=0}^{\infty} g_m e^{i \nu t}\) converges and

\[
\sum_{m=0}^{\infty} g_m e^{i \nu t} = (1 - e^{i t})^{-\nu} \left[ \sum_{s=0}^{\nu-1} A_s g_s + e^{i \nu t} \sum_{m=0}^{\infty} (\Delta^{\nu} g_m) e^{i \nu t} \right],
\]

where

\[
A_s = \sum_{m=s}^{\nu-1} (-1)^{m-s} \begin{pmatrix} \nu \\ m - s \end{pmatrix} e^{i \nu t}, \quad 0 \leq s \leq \nu - 1.
\]
Proof: Suppose that $M > 2\nu$ and let

$$S_M = (1 - e^{it})^\nu \sum_{m=0}^{M} g_m e^{imt}. \quad (10)$$

Since

$$(1 - e^{it})^\nu e^{imt} = \sum_{r=0}^{\nu} (-1)^r \binom{\nu}{r} e^{i(m+r)t},$$

we have

$$S_M = \sum_{m=0}^{M} g_m \sum_{r=0}^{\nu} (-1)^r \binom{\nu}{r} e^{i(m+r)t} = \sum_{r=0}^{\nu} (-1)^r \binom{\nu}{r} \sum_{m=0}^{M} g_m e^{i(m+r)t}$$

$${} = \sum_{r=0}^{\nu} (-1)^r \left( \binom{\nu}{r} \sum_{m-r}^{M} g_{m-r} e^{imt} \right).$$

Reversing the order of summation in the last sum yields

$$S_M = \sum_{m=0}^{\nu-1} \left( \sum_{r=0}^{m} (-1)^r \binom{\nu}{r} g_{m-r} \right) e^{imt} + \sum_{m=0}^{\nu-1} \left( \sum_{r=M+1}^{\nu} (-1)^r \binom{\nu}{r} g_{m-r} \right) e^{imt}.$$

Since $\lim_{m \to \infty} g_m = 0$ the last sum on the right converges to 0 as $M \to \infty$. The second sum on the right is

$$\sum_{m=0}^{M} (\Delta^\nu g_{m-\nu}) e^{imt} = e^{i\nu t} \sum_{m=0}^{M-\nu} (\Delta^\nu g_m) e^{imt},$$

which converges as $M \to \infty$ because of (7). Therefore

$$\lim_{M \to \infty} S_M = S = \sum_{m=0}^{\nu-1} \left( \sum_{r=0}^{m} (-1)^r \binom{\nu}{r} g_{m-r} \right) e^{imt} + e^{i\nu t} \sum_{m=0}^{\infty} (\Delta^\nu g_m) e^{imt},$$

which can also be written as

$$S = \sum_{s=0}^{\nu-1} A_s g_s + e^{i\nu t} \sum_{m=0}^{\infty} (\Delta^\nu g_m) e^{imt},$$

with $A_s$ as in (9). This and (10) imply (8).

Therefore we write $G(x) = O(x^{-\alpha})$ to indicate that $x^{\alpha}G(x)$ remains bounded as $x \to \infty$. \hfill \blacksquare
**Definition 1**  Let $\mathcal{F}_\alpha$ be the set of infinitely differentiable functions $F$ on $[1, \infty)$ such that

$$F^{(\nu)}(x) = O(x^{-\alpha-\nu}), \quad \nu = 0, 1, \ldots.$$  \hfill (11)

For example, let $F(x) = u^\gamma(x)$, where $u$ is a rational function with positive values on $[1, \infty)$ and a zero of order $p > 0$ at $\infty$; then $F$ satisfies (11) with $\alpha = p\gamma$. To see this, we first recall that if $f = f(u)$ and $u = u(x)$, the formula of Faa di Bruno [2] for the derivatives of a composite function says that

$$\frac{d^\nu}{dx^\nu}f(u(x)) = \sum_{r=1}^{\nu} \frac{d^\nu}{du^\nu}f(u) \prod_{r_1, \ldots, r_\nu} \left( \frac{u'}{1!} \right)^{r_1} \left( \frac{u''}{2!} \right)^{r_2} \cdots \left( \frac{u^{(\nu)}}{\nu!} \right)^{r_\nu},$$  \hfill (12)

where the prime denotes differentiation with respect to $x$. We are assuming here that the derivatives on the right of (12) exist. Here $u, \ldots, u^{(\nu)}$ are evaluated at $x$, and $\sum_r$ is over all partitions of $r$ as a sum of nonnegative integers,

$$r_1 + r_2 + \cdots + r_\nu = r,$$  \hfill (13)

such that

$$r_1 + 2r_2 + \cdots + \nu r_\nu = \nu.$$  \hfill (14)

Applying (12) with $f(u) = u^\gamma$ yields

$$F^{(\nu)}(x) = \sum_{r_1, \ldots, r_\nu} \frac{d^\nu}{du^\nu}f(u) \prod_{r_1, \ldots, r_\nu} \left( \frac{u'(x)}{1!} \right)^{r_1} \left( \frac{u''(x)}{2!} \right)^{r_2} \cdots \left( \frac{u^{(\nu)}(x)}{\nu!} \right)^{r_\nu},$$

where $(\gamma)^{(r)} = \gamma(\gamma - 1) \cdots (\gamma - r + 1)$. Since $u^{(0)}(x) = O(x^{-p-1})$, it follows that

$$u^{\gamma-r}(x)(u'(x))^{r_1}(u''(x))^{r_2} \cdots (u^{(\nu)}(x))^{r_\nu} = O(x^{-\lambda}),$$

where

$$\lambda = p(\gamma - r) + (p + 1)r_1 + (p + 2)r_2 + \cdots + (p + \nu)r_\nu = p\gamma + \nu$$

because of (13) and (14). This verifies (11) with $\alpha = p\gamma$.

For our purposes it is important to note that $\mathcal{F}_\alpha$ is a vector space over the complex numbers. Moreover, if $F_i \in \mathcal{F}_{\alpha_i}, i = 1, 2$, then $F_1 F_2 \in \mathcal{F}_{\alpha_1 + \alpha_2}$.

**Lemma 2** If $F \in \mathcal{F}_\alpha$ then

$$\Delta^\nu F(x) = O(x^{-\alpha-\nu}), \quad \nu = 0, 1, \ldots.$$  \hfill (15)

**Proof:** We show that

$$|\Delta^\nu F(x)| \leq K \max_{x < \xi < x + \nu} |F^{(\nu)}(\xi)|,$$  \hfill (15)
where \( K \) is a constant independent of \( F \). Since \( F^{(\nu)}(x) = O(x^{\alpha-\nu}) \) this implies the conclusion.

To verify (15), we note that if \( x > 1 \) and \( r > 0 \) then Taylor’s theorem implies that
\[
F(x + r) = \sum_{m=0}^{\nu-1} \frac{F^{(m)}(x)}{m!} r^m + \frac{F^{(\nu)}(\xi_r)}{\nu!} r^\nu,
\]
where \( x < \xi < x + r \). Since \( \Delta^{\nu} F(x) = \sum_{r=0}^{\nu} (-1)^{r-\nu} F(x + r) \), it follows that
\[
\Delta^{\nu} F(x) = \sum_{m=0}^{\nu-1} \frac{F^{(m)}(x)}{m!} \left( \sum_{r=0}^{\nu} (-1)^{r-\nu} \binom{\nu}{r} r^m \right) + \frac{1}{\nu!} \sum_{r=0}^{\nu} (-1)^{r-\nu} \binom{\nu}{r} r^\nu F^{(\nu)}(\xi_r).
\]
Since \( \sum_{r=0}^{\nu} (-1)^{r-\nu} \binom{\nu}{r} r^m = 0 \) for \( m = 0, \ldots, \nu - 1 \), we can now infer (15) with \( K = \frac{\sum_{r=0}^{\nu} (-1)^{r-\nu} \binom{\nu}{r} r^m}{\nu!} \).

**Lemma 3** Suppose that \( F \in \mathcal{F}_\alpha \). Let \( \nu \) be a fixed positive integer and let \( t \) be a real number, not an integral multiple of \( 2\pi \). Then
\[
\sum_{m=n}^{\infty} F^{(m)}(x) e^{int} = G(n)e^{int} + O(n^{\alpha-\nu+1}),
\]
where \( G \in \mathcal{F}_\alpha \) (and \( G \) depends upon \( \nu \)).

**Proof:** We write
\[
\sum_{m=n}^{\infty} F^{(m)}(x) e^{int} = e^{int} \sum_{m=n}^{\infty} F^{(m)}(x) e^{imt}.
\]
From Lemma 2, \( \Delta^{\nu} F(n + m) = O((n + m)^{-\alpha-\nu}) \); that is, there is a constant \( A \) such that \( |\Delta^{\nu} F(n + m)| < A(n + m)^{-\alpha-\nu} \) if \( n + m > 0 \). Therefore, if \( n > 2 \),
\[
\sum_{m=0}^{\infty} |\Delta^{\nu} F(n + m)| < A \sum_{m=0}^{\infty} \frac{1}{(n + m)^\alpha} < A \sum_{m=0}^{\infty} \int_{n+m-1}^{n+m} \frac{dx}{(x + \alpha)^\nu} = A \int_{n-1}^{\infty} \frac{dx}{(x + \alpha)^\nu} = O(n^{\alpha-\nu+1}).
\]
Applying Lemma 1 (specifically, (8)) with \( g_m = F(n + m) \) and \( n \) fixed shows that
\[
\sum_{m=0}^{\infty} F(n + m) e^{imt} = G(n) + O(n^{\alpha-\nu+1})
\]
with
\[
G(x) = (1 - e^{it})^{-\nu} \sum_{s=0}^{\nu-1} A_s F(x + s),
\]
so $G \in F_n$. Now (16) implies the conclusion.

The following theorem shows that Theorem 4 has nontrivial applications for every positive integer $q$.

**Theorem 5** Suppose that

$$a_n = f(n)e^{i\alpha}, \quad n = 1, 2, 3, \ldots, \quad (17)$$

where $f \in F_\gamma$ for some $\gamma \in (0, 1]$, and let $q$ be the smallest integer such that

$$(q + 1)\gamma > 1. \quad (18)$$

Then the infinite product $P = \prod_{n=1}^\infty (1 + a_n)$ converges if $\theta$ is not of the form $2k\pi/r$ with $k$ an integer and $r \in \{1, \ldots, q\}$.

**Proof:** We show by finite induction on $p$ that if $p = 1, \ldots, q$ then

$$a_n a_n^{(p)} = f_p(n)e^{i(p+1)n\theta} + O(n^{-(p+1)\gamma - q + p}) \quad (19)$$

where $f_p \in F_{(p+1)\gamma}$. In particular, (19) with $p = q$ implies that $a_n a_n^{(q)} = O(n^{-(q+1)\gamma})$, so (18) implies (5) and $P$ converges, by Theorem 4.

From (17) and Lemma 3 with $t = \theta$, $F = f$, $\alpha = \gamma$, and $\nu = q$,

$$a_n^{(1)} = \sum_{m=n}^\infty f(m)e^{im\theta} = G_1(n)e^{i\theta} + O(n^{-\gamma + 1}),$$

with $G_1 \in F_\gamma$. Therefore $a_n a_n^{(1)} = f(n)e^{i\alpha} (G_1(n)e^{i\theta} + O(n^{-\gamma + 1}))$. Since $f \in F_\gamma$, this can be rewritten as $a_n a_n^{(1)} = f_1(n)e^{2i\theta} + O(n^{-2\gamma + 1})$, with $f_1 = f G_1 \in F_2\gamma$. This establishes (19) with $p = 1$, so we are finished if $q = 1$.

Now suppose that $q > 1$ and (19) holds if $1 \leq p < q$. Since $(p + 1)\theta$ is by assumption not an integral multiple of $2\pi$, Lemma 3 with $t = (p+1)\theta$, $F = f_p$, $\alpha = (p+1)\gamma$, and $\nu = q - p$ implies that

$$\sum_{n=m}^\infty f_p(m)e^{i(p+1)n\theta} = G_p(n)e^{i(p+1)n\theta} + O(n^{-(p+1)\gamma - q + p + 1}),$$

where $G_p \in F_{(p+1)\gamma}$. This and (19) imply that

$$a_n^{(p+1)} = \sum_{m=n}^\infty a_m a_m^{(p)} = G_p(n)e^{i(p+1)n\theta} + O(n^{-(p+1)\gamma - q + p + 1}),$$

so

$$a_n a_n^{(p+1)} = f(n)e^{i\alpha} \left(G_p(n)e^{i(p+1)n\theta} + O(n^{-(p+1)\gamma - q + p + 1})\right).$$

Since $f \in F_\gamma$, this can be rewritten as

$$a_n a_n^{(p+1)} = f_{p+1}(n)e^{i(p+2)n\theta} + O(n^{-(p+2)\gamma - q + p + 1}),$$

with $f_{p+1} = f G_p \in F_{(p+2)\gamma}$. This completes the induction.
Corollary 1 Suppose that $\{a_n\}^\infty$ is as defined in Theorem 5. Then the infinite product $\prod^\infty (1 + a_n)$ converges if $\theta$ is not a rational multiple of $2\pi$.

Corollary 2 Suppose that $\alpha > 0$ and $R$ is a rational function such that $R(x) > 0$ on $[N, \infty)$ ($N = \text{integer}$) and $\lim_{n \to \infty} R(x) = 0$. Then the infinite product $\prod_{n=N}^\infty (1 + (R(n))^\alpha e^{in\theta})$ converges if $\theta$ is not a rational multiple of $2\pi$.

Corollary 3 The infinite product $\prod^\infty (1 + n^\alpha e^{in\theta})$ converges if $\alpha > 0$ and $\theta$ is not a rational multiple of $2\pi$.

REFERENCES
