Linear Asymptotic Equilibrium of Nilpotent Systems of Linear Difference Equations

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Abstract

We give sufficient conditions for a $k\times k$ linear system of difference equations $\Delta x_n = A_n x_n,\ n = 0, 1, \dots,$ to have linear asymptotic equilibrium if $A_n = a_1(n)Q+\cdots+a_p(n)Q^p$, where $Q^{p+1} = 0$ for some $p \in \{1, 2, \dots, k-1\}$ and $\{a_1(n)\}_{n=0}^{\infty}, \dots, \{a_p(n)\}_{n=0}^{\infty}$ are sequences of scalars. The conditions involve convergence (perhaps conditional) of certain iterated sums involving these sequences.

Key words: difference equations, nilpotent, linear asymptotic equilibrium, conditional convergence

A $k \times k$ linear system of difference equations

$$\Delta x_n = A_n x_n \tag{1}$$

is said to have *linear asymptotic equilibrium* if $\lim_{n\to\infty} x_n$ exists and is nonzero whenever $x_0 \neq 0$. Since (1) can be written as

$$x_{n+1} = (I + A_n)x_n, \quad n = 0, 1, \dots,$$

its solution is $x_n = P_n x_0$, where

$$P_n = \prod_{m=0}^{n-1} (I + A_m) = (I + A_{n-1}) \cdots (I + A_0). \tag{2}$$

Therefore (1) has linear asymptotic equilibrium if and only if $I+A_n$ is invertible for every $n \geq 0$ and $\lim_{n\to\infty} P_n$ exists and is invertible. If it is assumed that $I + A_n$ is invertible for every n then the most well known sufficient condition for (1) to have linear asymptotic equilibrium is that

$$\sum^{\infty} \|A_n\| < \infty,$$

where $\|\cdot\|$ is any matrix norm such that $\|AB\| \leq \|A\|\|B\|$ for all square matrices A and B. (See [2, 4]). The following weaker condition for linear asymptotic equilibrium is given in [3]. For related results, see [1].

Theorem 1 *Suppose that $I + A_n$ is invertible for every $n \geq 0$ and either*

$$\sum_{n=0}^{\infty} \|A_n\| < \infty$$

or there is an integer $R \geq 1$ such that the sequences $\left\{B_n^{(r)}\right\}_{n=0}^{\infty}$ given by

$$B_n^{(r)} = \sum_{m=n}^{\infty} B_{m+1}^{(r-1)} A_m, \quad n = 0, 1, 2, \ldots \tag{3}$$

(with $B_n^{(0)} = I$) are defined for $1 \leq r \leq R$, and

$$\sum_{n=0}^{\infty} \|B_{n+1}^{(R)} A_n\| < \infty. \tag{4}$$

Then (1) has linear asymptotic equilibrium.

In [2] the author proved Theorem 1 with $R = 1$. In its full generality Theorem 1 is an analog of a result of Wintner [5] for a linear differential system $y' = A(t)y$.

In this paper we consider a class of systems that have linear asymptotic equilibrium under conditions that require only convergence – which may be conditional – of certain iterated series derived from $\{A_n\}$. As we point out at the end of the paper, whether or not the conditions that we impose here actually imply the hypotheses of Theorem 1 for these systems is an open question which is in a way irrelevant, since the results given here provide a constructive way to obtain $\lim_{n\to\infty} P_n$ for these systems, as opposed to Theorem 1, which is a nonconstructive existence theorem.

We assume henceforth that $k \geq 2$ and Q is a $k \times k$ nonzero nilpotent matrix; that is, there is an integer p in $\{1, 2, \ldots, k-1\}$ such that

$$Q^p \neq 0 \text{ and } Q^{p+1} = 0. \tag{5}$$

For example, any strictly upper triangular matrix is nilpotent.

We will say that (1) is a *nilpotent system* if

$$A_n = a_1(n)Q + \cdots + a_p(n)Q^p, \quad n = 0, 1, 2, \ldots, \tag{6}$$

where Q is $k \times k$ nilpotent matrix satisfying (5) and $\{a_1(n)\}_{n=0}^{\infty}, \ldots, \{a_p(n)\}_{n=0}^{\infty}$ are sequences of scalars. We will show that the nilpotent system (1) has linear asymptotic equilibrium if certain iterated sums involving these sequences converge. The convergence may be conditional.

Any matrix of the form

$$Z = I + u_1 Q + \cdots + u_p Q^p$$

is invertible, and any two matrices of this form commute. Therefore, the product P_n in (2) is of the form

$$P_n = \prod_{m=0}^{n-1}(I + A_m) = I + b_1(n)Q + \cdots + b_p(n)Q^p,$$

with

$$b_1(0) = b_2(0) = \cdots = b_p(0) = 0,$$

and $\lim_{n\to\infty} P_n$ exists if and only if the limits

$$\lim_{n\to\infty} b_i(n) = b_i,\ \ i = 1, \ldots, p,$$

exist (finite), in which case

$$\prod_{m=0}^{\infty}(I + A_m) = I + b_1 Q + \cdots + b_p Q^p.$$

Hence, (1) has linear asymptotic equilbrium if and only if $b_1, \ldots, b_p$ exist (finite).

We will now introduce some notation which will enable us to give recursive formulas for $b_1(n), \ldots, b_p(n)$. If $i_1, i_2, \ldots$ are integers in $\{1, 2, \ldots, r\}$ and n is an arbitrary positive integer, let

$$\begin{aligned}
\sigma(n; i_1) &= \sum_{m=0}^{n-1} a_{i_1}(m), \\
\sigma(n; i_2, i_1) &= \sum_{m=0}^{n-1} a_{i_2}(m)\sigma(m; i_1), \\
&\vdots \\
\sigma(n; i_s, i_{s-1}, \ldots, i_1) &= \sum_{m=0}^{n-1} a_{i_s}(m)\sigma(m; i_{s-1}, \ldots, i_1).
\end{aligned}$$

We say that an s-tuple $(i_s, i_{s-1}, \ldots, i_1)$ of positive integers is an *ordered partition* of r if

$$i_s + i_{s-1} + \cdots + i_1 = r.$$

We denote an ordered partition of r by ψ_r. For each positive integer r let $\mathcal{O}_r$ be the set of all ordered partitions of r.

LEMMA 1 *If the sequence* $\{A_n\}$ *is as defined in* (6) *then*

$$P_n = \prod_{m=0}^{n-1}(I + A_m) = I + b_1(n)Q + \cdots + b_p(n)Q^p,$$

with

$$b_r(n) = \sum_{\psi_r \in \mathcal{O}_r} \sigma(n; \psi_r) \tag{7}$$

for $1 \leq r \leq p$.

Proof: We will establish (7) by finite induction on r. Since $P_{n+1} = (I + A_n)P_n$,

$$P_{n+1} = [I + a_1(n)Q + \cdots + a_p(n)Q^p]\,[I + b_1(n)Q + \cdots + b_p(n)Q^p]\,.$$

Since $Q^{p+1} = 0$ we are interested in

$$b_i(n+1) = b_i(n) + a_i(n) + \sum_{q=1}^{i-1} a_q(n) b_{i-q}(n), \quad 1 \leq i \leq p,$$

or

$$\Delta b_i(n) = a_i(n) + \sum_{q=1}^{i-1} a_q(n) b_{i-q}(n), \quad 1 \leq i \leq p,$$

so

$$b_i(n) = \sigma(n; i) + \sum_{q=1}^{i-1} \sum_{m=0}^{n-1} a_q(m) b_{i-q}(m), \quad 1 \leq i \leq p.$$

Letting $i = 1$ here yields

$$b_1(n) = \sigma(n; 1),$$

which confirms (7) with $r = 1$. Now suppose that $2 \leq i < p$ and (7) has been established for $1 \leq r \leq i - 1$. Then

$$b_{i-q}(m) = \sum_{\psi_{i-q} \in \mathcal{O}_{i-q}} \sigma(m; \psi_{i-q}), \quad 1 \leq q \leq i - 1,$$

so

$$b_i(n) = \sigma(n, i) + \sum_{q=1}^{i-1} \sum_{\psi_{i-q} \in \mathcal{O}_{i-q}} \sum_{m=0}^{n-1} a_q(m) \sigma(m, \psi_{i-q}). \tag{8}$$

For a given q,

$$\sum_{\psi_{i-q} \in \mathcal{O}_{i-q}} \sum_{m=0}^{n-1} a_q(m) \sigma(m, \psi_{i-q}) = \sum_{\psi_i \in \mathcal{O}_i^q} \sigma(n; \psi_i)$$

where $\mathcal{O}_i^q$ is the set of all ordered partitions $(i_s, i_{s-1}, \ldots, i_1)$ of i for which $i_s = q$. Therefore (8) implies that

$$b_i(n) = \sigma(n, i) + \sum_{q=1}^{i-1} \sum_{\psi_i \in \mathcal{O}_i^q} \sigma(n; \psi_i)$$

Since $\mathcal{O}_i = \bigcup_{q=1}^{i} \mathcal{O}_i^q$ and $\mathcal{O}_i^i = \{(i)\}$, it now follows that

$$b_i(n) = \sum_{q=1}^{i} \sum_{\psi_i \in \mathcal{O}_i^q} \sigma(n;\psi_i) = \sum_{\psi_i \in \mathcal{O}_i} \sigma(n;\psi_i),$$

which implies (7) with $r = i$, completing the induction. ∎

The following theorem is our main result.

THEOREM 2 *Let Q be a $k \times k$ matrix such that*

$$Q^p \neq 0 \text{ and } Q^{p+1} = 0.$$

Let

$$A_n = a_1(n)Q + \cdots + a_p(n)Q^p,$$

where $\{a_1(n)\}_{n=0}^{\infty}, \ldots, \{a_p(n)\}_{n=0}^{\infty}$ are sequences of scalars such that all the limits

$$\lim_{n\to\infty} \sigma(n;\psi_r), \quad \psi_r \in \mathcal{O}_r, \quad 1 \leq r \leq p,$$

exist. (Convergence may be conditional.) Then the system

$$\Delta x_n = A_n x_n, \quad n = 0, 1, \ldots, \tag{9}$$

has linear asymptotic equilibrium, and

$$\prod_{n=0}^{\infty} (I + A_n) = I + b_1 Q + \cdots + b_p Q^p, \tag{10}$$

where

$$b_r = \lim_{n\to\infty} \sum_{\psi_r \in \mathcal{O}_r} \sigma(n;\psi_r), \quad 1 \leq r \leq p. \tag{11}$$

The following example will motivate our discussion of the connection between Theorem 1 (as it applies to nilpotent systems) and Theorem 2.

EXAMPLE. Suppose that $k \geq 3$ and let

$$A_n = (-1)^n \alpha_n Q + (-1)^n \beta_n Q^2,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are nonincreasing null sequences and

$$\sum^{\infty} \alpha_n^2 < \infty. \tag{12}$$

Then the series $S_\alpha = \sum_{n=0}^{\infty} (-1)^n \alpha_n$ and $S_\beta = \sum_{n=0}^{\infty} (-1)^n \beta_n$ converge by the alternating series test. Now consider

$$S_{\alpha\alpha} = \sum_{n=0}^{\infty} (-1)^n \alpha_n \sum_{m=0}^{n-1} (-1)^m \alpha_m = S_\alpha^2 - \sum_{n=0}^{\infty} (-1)^n \alpha_n \sum_{m=n}^{\infty} (-1)^m \alpha_m. \tag{13}$$

Since

$$0 < (-1)^n \sum_{m=n}^{\infty} (-1)^m \alpha_m < \alpha_n,$$

(12) implies that the last series in (13) converges. Therefore Theorem 2 implies that (9) has linear asymptotic equilibrium, and that

$$\prod_{n=0}^{\infty} (I + A_n) = I + S_\alpha Q + (S_\beta + S_{\alpha\alpha}) Q^2. \tag{14}$$

We note that S_α, S_β, and $S_{\alpha\alpha}$ may all converge conditionally.

Theorem 1 is also applicable under the assumptions of this example, since (see (3)),

$$B_n^{(1)} = \left(\sum_{m=n}^{\infty} (-1)^m \alpha_m \right) Q + \left(\sum_{m=n}^{\infty} (-1)^m \beta_m \right) Q^2$$

is defined, as is

$$B_n^{(2)} = \left(\sum_{m=n}^{\infty} (-1)^m \alpha_m \sum_{l=m+1}^{\infty} (-1)^l \alpha_l \right) Q^2.$$

(Recall that $Q^3 = 0$.) Moreover, (4) obviously holds, since $B_{n+1}^{(2)} = 0$. However, Theorem 1 does not imply (14).

It is natural to ask whether the hypotheses of Theorem 2 in general imply those of Theorem 1. To establish this it would be sufficient to verify that the hypotheses of Theorem 2 imply that the sequences $\{B_n^{(r)}\}_{n=0}^{\infty}$, $r = 1, \ldots, p$ in (3) are all defined. However, to verify this – if it is true – would surely be at least as difficult as the proof of Lemma 1. If it is true then these sequences would necessarily be of the form

$$B_n^{(r)} = \sum_{s=r}^{p} c_{rs}(n) Q^s, \quad 1 \le r \le p,$$

so $B_n^{(p)} = 0$ for all n, and (4) would hold automatically with $R = p$. Therefore, we could conclude that (9) has linear asymptotic equilibrium, but we would not have proved (10) and (11).

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