SYSTEMS OF DIFFERENCE EQUATIONS WITH ASYMPTOTICALLY CONSTANT SOLUTIONS

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We consider the system

$$\Delta x_n = A_n x_n + f(n, x_n), \tag{1}$$

where x_n and f are k-vectors (real or complex) and A_n is a $k \times k$ matrix. We give conditions implying that (1) has a solution $\{\hat{x}_n\}$ such that $\lim_{n\to\infty} \hat{x}_n = c$, a given constant vector.

If u is a k-vector and B is a $k \times k$ matrix, then $|u|$ and $|A|$ are the ∞-norms of u and A.

THEOREM 1 *Let c be a given k-vector, and suppose there is a constant $M > 0$ and an integer N such that $f(n, x)$ is continuous with respect to x and*

$$|f(n, x) - f(n, c)| \le R(n, |x - c|) \tag{2}$$

on the set

$$S = \{(n, x) \mid n \ge N,\ |x - c| \le M\},$$

where $R = R(n, \lambda)$ is defined on the set

$$\{(n, x) \mid n \ge N,\ 0 \le \lambda \le M\}$$

and nondecreasing in λ for each n, and

$$\sum_{n=N}^{\infty} |R(n, M)| < \infty. \tag{3}$$

Suppose that either

$$\sum_{n=N}^{\infty} |A_n| < \infty \tag{4}$$

or there is a positive integer q such that the sequences

$$A_n^{(r)} = \sum_{m=n}^{\infty} A_{m+1}^{(r-1)} A_m, \quad r = 1, 2, \ldots, q \quad (\textit{with } A_m^{(0)} = I) \tag{5}$$

are all defined for $n \geq N$*, and*

$$\sum_{n=N}^{\infty} |A_{n+1}^{(q)} A_n| < \infty. \tag{6}$$

(*If* (4) *holds we let* $q = 0$ *and* (5) *is vacuous; note that* (4) *and* (6) *are equivalent in this case, since* $A_{m+1}^{(0)} = I$.)

Define

$$\Gamma_n = \sum_{r=0}^{q} A_n^{(r)}, \tag{7}$$

and suppose that $\sum_{n=N}^{\infty} \Gamma_{n+1} f(n,c)$ *converges* (*perhaps conditionally*).

Then, if n_0 *is sufficiently large, there is a solution* $\hat{X} = \{\hat{x}_n\}_{n=n_0}^{\infty}$ *of* (1) *such that*

$$|\hat{x}_n - c| \leq M, \quad n \geq n_0, \tag{8}$$

and

$$\lim_{n\to\infty} \hat{x}_n = c. \tag{9}$$

PROOF. Since $A_n^{(0)} = I$ and $\lim_{n\to\infty} A_n^{(r)} = 0$ if $r > 0$, $\lim_{n\to\infty} \Gamma_n = I$. Therefore Γ_n is invertible for large n. For now, choose $n_0 \geq N$ so that Γ_n is invertible if $n \geq n_0$; we will impose another condition on n_0 later. Define

$$h_n = (\Gamma_n^{-1} - I)c - \Gamma_n^{-1}\left(\sum_{m=n}^{\infty} A_{m+1}^{(q)} A_m c + \Gamma_{m+1} f(m,c)\right). \tag{10}$$

Let B be the Banach space of bounded sequences $U = \{u_n\}_{n_0}^{\infty}$ of k-vectors, with norm $\|U\| = \sup_{n\geq n_0} |u_n|$. Let B_M be the closed convex subset

$$B_M = \{U \in B \mid \|U\| \leq M\}$$

of B. From (2) and our assumption that $R(n,\lambda)$ is nondecreasing with respect to λ, if $U \in B_M$ then

$$|f(m, u_m + c) - f(m,c)| \leq R(m, |u_m|) \leq R(m, M). \tag{11}$$

Therefore (3) and (6) imply that if $U \in B_M$ then the sequence TU, with

$$(TU)_n = h_n - \Gamma_n^{-1} \sum_{m=n}^{\infty} \left[A_{m+1}^{(q)} A_m u_m + \Gamma_{m+1}[f(m, u_m + c) - f(m,c)] \right] \tag{12}$$

is well defined. We will show that if n_0 is sufficiently large then T is a continuous mapping of B_M into itself and $T(B_M)$ has compact closure. Given this, the Schauder-Tychonoff theorem [1, p. 405] implies that $T\hat{U} = \hat{U}$ for some $\hat{U} \in B_M$. We will then show that $\hat{X} = C + \hat{U}$ (with $C = \{c, c, c, \ldots,\}_{n_0}^{\infty}$) satisfies (1), (8), and (9).

Let

$$\mu(n_0) = \sup_{m\geq n_0} |\Gamma_m^{-1}| \quad \text{and} \quad \nu(n_0) = \sup_{m\geq n_0} |\Gamma_{m+1}|.$$

From (11) and (12), if $U \in B_M$ then

$$|(TU)_n| \le |h_n| + \mu(n_0) \sum_{m=n}^{\infty} \Big[|A_{m+1} A_m^{(q)}| M + \nu(n_0) R(m, M)| \Big]. \qquad (13)$$

Since $\lim_{n_0 \to \infty} \mu(n_0) = \lim_{n_0 \to \infty} \nu(n_0) = 1$, (3) and (6) enable us to choose n_0 so that the quantity on the right side of (13) is less than M if $n \ge n_0$. Then $T(B_M) \subset B_M$.

We will now show that T is continuous on B_M. Suppose that $U = \lim_{r \to \infty} U^{(r)}$ where $\{U^{(r)}\} \subset B_M$. Let $V = TU$ and $V^{(r)} = TU^{(r)}$. Then

$$v_n^{(r)} - v_n = \Gamma_n^{-1} \sum_{m=n}^{\infty} \Big[A_{m+1}^{(q)} A_m (u_m - u_m^{(r)}) + \Gamma_{m+1} \Big(f(m, u_m + c) - f(m, u_m^{(r)} + c) \Big) \Big].$$

Therefore

$$\|v^{(r)} - v\| \le \mu(n_0) \sum_{m=n_0}^{\infty} \sigma_m^{(r)}, \qquad (14)$$

where

$$\sigma_m^{(r)} = |A_{m+1}^{(q)} A_m| |u_m^{(r)} - u_m| + \nu(n_0) \Big| f(m, u_m^{(r)} + c) - f(m, u_m + c) \Big|.$$

Note that

$$\lim_{r \to \infty} \sigma_m^{(r)} = 0, \quad m \ge n_0,$$

because of the continuity assumption on f, and

$$\sigma_m^{(r)} \le \sigma_m = 2 \Big(M |A_{m+1}^{(q)} A_m| + |\nu(n_0)| R(m, M) \Big) \qquad (15)$$

(see (11), applied to U and $U^{(r)}$) because U and $U^{(r)}$ are in B_M. Because of (3) and (6), $\sum_{m=n_0}^{\infty} \sigma_m < \infty$. Given $\epsilon > 0$, choose $n_1 \ge n_0$ so that $\sum_{m=n_1+1}^{\infty} \sigma_m < \epsilon$. Then (14) and (15) imply that

$$\|v^{(r)} - v\| \le \mu(n_0) \left(\sum_{m=n_0}^{n_1} \sigma_m^{(r)} + \epsilon \right). \qquad (16)$$

Now choose r_0 so that

$$\sigma_m^{(r)} < \frac{\epsilon}{(n_1 - n_0 + 1)} \text{ for } m = n_0, \dots, n_1 \text{ if } r \ge r_0.$$

Then (16) implies that

$$\|v^{(r)} - v\| < 2\mu(n_0)\epsilon \text{ if } r \ge r_0,$$

which shows that T is continuous on B_M.

We will now show that $\overline{T(B_M)}$ (the closure of $T(B_M)$) is compact. From (11) and (12), $\overline{T(B_M)}$ is a subset of

$$A = \{v \in B \mid |v_n| \le \rho(n),\ n \ge n_0\},$$

where

$$\rho(n) = |h_n| + \mu(n_0)\left(M \sum_{m=n}^{\infty} |A^{(q)}_{m+1} A_m| + \sum_{m=n}^{\infty} \nu(n_0) R(m, M)\right).$$

Therefore, it suffices to show that A is compact. From [2, pp. 51-53], this is true if A is totally bounded; that is, for every $\epsilon > 0$ there is a finite subset A_ϵ of B such that for each $v \in A$ there is a $\tilde{v} \in A_\epsilon$ that satisfies the inequality $\|v - \tilde{v}\| < \epsilon$. To establish the existence of A_ϵ, choose an integer $n_1 \ge n_0$ such that $\rho(n) < \epsilon$ if $n > n_1$, and let p be an integer such that $p\epsilon > M$. Then, since $|v_n| \le M$ for all $n \ge n_0$, the finite set A_ϵ consisting of sequences of the form

$$a = (a_{n_0}, \dots, a_{n_1}, 0, 0, \dots)$$

where the components of the k-vectors $\{a_{n_0}, \dots, a_{n_1}\}$ are all in the set

$$\{-p\epsilon, -(p-1)\epsilon, \dots, 0, \dots (p-1)\epsilon, p\epsilon\}, \quad n = n_0, \dots, n_1,$$

has the desired property.

Now the Schauder-Tychonoff theorem implies that T has a fixed point $\hat{U}$. Since $\hat{U} = T\hat{U}$, (10) and (12) imply that if $\hat{X} = C + \hat{U}$ then

$$\hat{x}_n = \Gamma_n^{-1}\left(c - \sum_{m=n}^{\infty}\left[A^{(q)}_{m+1} A_m \hat{x}_m + \Gamma_{m+1} f(m, \hat{x}_m)\right]\right). \tag{17}$$

Therefore, $\lim_{n\to\infty} \hat{x}_n = c$. If $q = 0$ then (17) reduces to

$$\hat{x}_n = c - \sum_{m=n}^{\infty} (A_m \hat{x}_m + f(m, x_m)),$$

so

$$\Delta \hat{x}_n = A_n \hat{x}_n + f(n, \hat{x}_n). \tag{18}$$

If $q > 0$ then (17) implies that

$$\Delta \hat{x}_n = \Gamma_{n+1}^{-1} A^{(q)}_{n+1} A_n \hat{x}_n + f(n, \hat{x}_n) + (\Delta \Gamma_n^{-1}) \Gamma_n \hat{x}_n. \tag{19}$$

Since $\Delta \Gamma_n^{-1} = -\Gamma_{n+1}^{-1} (\Delta \Gamma_n) \Gamma_n^{-1}$, (19) implies that

$$\Delta \hat{x}_n = \Gamma_{n+1}^{-1}\left[A^{(q)}_{n+1} A_n - \Delta \Gamma_n\right] \hat{x}_n + f(n, \hat{x}_n). \tag{20}$$

However, (5) and (7) imply that

$$\Delta \Gamma_n = -\sum_{r=1}^{q} A^{(r-1)}_{n+1} A_n,$$

so

$$A_{n+1}^{(q)} A_n - \Delta \Gamma_n = \Gamma_{n+1} A_n,$$

and therefore (20) implies (18). ■

The hypotheses of Theorem 1 may hold for some constant vectors c and fail to hold for others. In the following corollary c may be chosen arbitrarily.

COROLLARY 1 *Let A_n satisfy the hypotheses of Theorem* 1. *Suppose there is an integer N such that $f(n, x)$ is continuous with respect to x for all $n \geq N$ and all x, and*

$$|f(n, x_1) - f(n, x_2)| \leq R(n, |x_1 - x_2|)$$

where $R = R(n, \lambda)$ is defined on

$$\{(n, x) \mid n \geq N,\ 0 \leq \lambda \leq \infty\}$$

and nondecreasing in λ for each n, and $\sum_{n=N}^{\infty} |R(n, M)| < \infty$ for some constant $M > 0$. Suppose also that $\sum_{n=N}^{\infty} \Gamma_{n+1} f(n, c)$ converges (*perhaps conditionally*) *for every constant vector c. Let c be a given constant vector. Then, if n_0 is sufficiently large, there is a solution $\hat{X} = \{\hat{x}_n\}_{n=n_0}^{\infty}$ of* (1) *that satisfies* (8) *and* (9).

The following corollary applies to the linear system

$$\Delta x_n = (A_n + B_n) x_n + g_n, \tag{21}$$

where A_n and B_n are $k \times k$ matrices and g_n is a k-vector.

COROLLARY 2 *Suppose that A_n satisfies the hypotheses of Theorem* 1, *while $\sum^{\infty} |B_n| < \infty$ and $\sum^{\infty} \Gamma_{n+1} g_n$ converges* (*perhaps conditionally*). *Let c be an arbitrary vector. Then* (21) *has a solution $\hat{X}$ such that* $\lim_{n\to\infty} \hat{x}_n = c$.

References

[1] P. Hartman, *Ordinary Differential Equations*, John Wiley & Sons, Inc., New York, London, Sydney, 1964.

[2] A. N. Kolmogorov and S. V. Fomin, *Elements of the Theory of Functions and Functional Analysis*, v. 1 (translated from the 1954 Russian edition by L. F . Boron), Graylock Press, Rochester, N.Y., 1957.