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Properties of Some Generalizations of Kac-Murdock-Szegö Matrices

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Abstract

We consider generalizations of the Kac-Murdock-Szegö matrices of the forms $L_n = (\rho^{|r-s|}\, c_{\min(r,s)})_{r,s=1}^n$ and $U_n = (\rho^{|r-s|}\, c_{\max(r,s)})_{r,s=1}^n$, where ρ and $c_1, c_2, \dots, c_n$ are real numbers. We obtain explicit expressions for the determinants and inverses of L_n and U_n, determine their inertias, and diagonalize their quadratic forms. We also consider the spectral distributions of two special cases.

Key words: Kac-Murdock-Szegö matrices; Toeplitz matrices; determinant; inverse; inertia; diagonalization; spectrum

1. Introduction.

The Kac-Murdock-Szegö (KMS) matrices [4] are the symmetric Toeplitz matrices

$$K_n(\rho) = \left(\rho^{|r-s|}\right)_{r,s=1}^n, \quad n = 1, 2, \dots,$$

where ρ is real. It is known [3, Section 7.2, Problems 12-13] that

$$\det(K_n(\rho)) = (1-\rho^2)^{n-1} \tag{1}$$

and, if $\rho \neq \pm 1$, then

$$K_n^{-1}(\rho) = \frac{1}{1-\rho^2} \begin{bmatrix} 1 & -\rho & 0 & \cdots & 0 & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\rho^2 & -\rho & 0 \\ 0 & 0 & 0 & \cdots & -\rho & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & \cdots & 0 & -\rho & 1 \end{bmatrix};$$

thus, except for its first and last rows, $K_n^{-1}(\rho)$ is a tridiagonal Toeplitz matrix.

In this paper we consider generalizations of the KMS matrices of the form

$$L_n = (\rho^{|r-s|}\, c_{\min(r,s)})_{r,s=1}^n \quad \text{and} \quad U_n = (\rho^{|r-s|}\, c_{\max(r,s)})_{r,s=1}^n,$$

where ρ and $c_1, c_2, \dots, c_n$ are real numbers; thus,

$$L_n = \begin{bmatrix} c_1 & \rho c_1 & \rho^2 c_1 & \cdots & \rho^{n-3}c_1 & \rho^{n-2}c_1 & \rho^{n-1}c_1 \\ \rho c_1 & c_2 & \rho c_2 & \cdots & \rho^{n-4}c_2 & \rho^{n-3}c_2 & \rho^{n-2}c_2 \\ \rho^2 c_1 & \rho c_2 & c_3 & \cdots & \rho^{n-5}c_3 & \rho^{n-4}c_3 & \rho^{n-3}c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \rho^{n-3}c_1 & \rho^{n-4}c_2 & \rho^{n-5}c_3 & \cdots & c_{n-2} & \rho c_{n-2} & \rho^2 c_{n-2} \\ \rho^{n-2}c_1 & \rho^{n-3}c_2 & \rho^{n-4}c_3 & \cdots & \rho c_{n-2} & c_{n-1} & \rho c_{n-1} \\ \rho^{n-1}c_1 & \rho^{n-2}c_2 & \rho^{n-3}c_3 & \cdots & \rho^2 c_{n-2} & \rho c_{n-1} & c_n \end{bmatrix} \tag{2}$$

and

$$U_n = \begin{bmatrix} c_1 & \rho c_2 & \rho^2 c_2 & \cdots & \rho^{n-3}c_{n-2} & \rho^{n-2}c_{n-1} & \rho^{n-1}c_n \\ \rho c_2 & c_2 & \rho c_3 & \cdots & \rho^{n-4}c_{n-2} & \rho^{n-3}c_{n-1} & \rho^{n-2}c_n \\ \rho^2 c_3 & \rho c_3 & c_3 & \cdots & \rho^{n-5}c_{n-2} & \rho^{n-4}c_{n-1} & \rho^{n-3}c_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \rho^{n-3}c_{n-2} & \rho^{n-4}c_{n-2} & \rho^{n-5}c_{n-2} & \cdots & c_{n-2} & \rho c_{n-1} & \rho^2 c_n \\ \rho^{n-2}c_{n-1} & \rho^{n-3}c_{n-1} & \rho^{n-4}c_{n-1} & \cdots & \rho c_{n-1} & c_{n-1} & \rho c_n \\ \rho^{n-1}c_n & \rho^{n-2}c_n & \rho^{n-3}c_n & \cdots & \rho^2 c_n & \rho c_n & c_n \end{bmatrix}.$$

Although we do not know of any practical applications in which these matrices occur, we believe that they have interesting properties. In particular, we hope to discover conditions on sequences $\{c_n\}_{n=1}^\infty$ which guarantee that the spectra of the family $\{L_n\}_{n=1}^\infty$ and/or the family $\{U_n\}_{n=1}^\infty$ have predictable distributions as $n \to \infty$. Theorems 5-8 provide a modest start in this direction.

In Section 2 we obtain explicit expressions for the determinants and inverses of L_n and U_n. We also determine their inertias and diagonalize their quadratic forms. In Section 3 we discuss the distribution of the eigenvalues of the matrices

$$K_n(\rho, \gamma) = \left(\rho^{|r-s|} + \gamma\rho^{r+s}\right)_{r,s=1}^n$$

(which is of the form (2) with $c_r = 1 + \gamma\rho^{2r}$), where $0 < \rho < 1$ and γ is an arbitrary real number. In Section 4 we discuss the distribution of the eigenvalues of

$$L_n = (\min(r,s) - \gamma)_{r,s=1}^n$$

(which is of the form (2) with $c_r = r - \gamma$), where $\gamma \le 1/2$.

2. Properties of L_n and U_n.

Let A_n be the $n \times n$ matrix with 1's on the diagonal, $-\rho$'s on the super diagonal, and zeros elsewhere; thus,

$$A_n = \begin{bmatrix} 1 & -\rho & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -\rho & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\rho & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -\rho \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

It is straightforward to verify that

$$L_nA_n = \begin{bmatrix} c_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \rho c_1 & \alpha_1 & 0 & \cdots & 0 & 0 & 0 \\ \rho^2 c_1 & \rho\alpha_1 & \alpha_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \rho^{n-3}c_1 & \rho^{n-4}\alpha_1 & \rho^{n-5}\alpha_2 & \cdots & \alpha_{n-3} & 0 & 0 \\ \rho^{n-2}c_1 & \rho^{n-3}\alpha_1 & \rho^{n-4}\alpha_2 & \cdots & \rho\alpha_{n-3} & \alpha_{n-2} & 0 \\ \rho^{n-1}c_1 & \rho^{n-2}\alpha_1 & \rho^{n-1}\alpha_2 & \cdots & \rho^2\alpha_{n-3} & \rho\alpha_{n-2} & \alpha_{n-1} \end{bmatrix},$$

where

$$\alpha_i = c_{i+1} - \rho^2 c_i, \quad i = 1, \dots, n-1,$$

and that

$$\begin{aligned} A_n^T L_n A_n &= \operatorname{diag}(c_1, \alpha_1, \alpha_2, \dots, \alpha_{n-1}) \\ &= \operatorname{diag}(c_1, c_2 - \rho^2 c_1, c_3 - \rho^2 c_2, \dots, c_n - \rho^2 c_{n-1}). \end{aligned} \tag{3}$$

It is also straightforward to verify that

$$A_nU_n = \begin{bmatrix} \beta_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \rho\beta_2 & \beta_2 & 0 & \cdots & 0 & 0 & 0 \\ \rho^2\beta_3 & \rho\beta_3 & \beta_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \rho^{n-3}\beta_{n-2} & \rho^{n-4}\beta_{n-2} & \rho^{n-5}\beta_{n-2} & \cdots & \beta_{n-2} & 0 & 0 \\ \rho^{n-2}\beta_{n-1} & \rho^{n-3}\beta_{n-1} & \rho^{n-4}\beta_{n-1} & \cdots & \rho\beta_{n-1} & \beta_{n-1} & 0 \\ \rho^{n-1}c_n & \rho^{n-2}c_n & \rho^{n-1}c_n & \cdots & \rho^2 c_n & \rho c_n & c_n \end{bmatrix},$$

where

$$\beta_i = c_i - \rho^2 c_{i+1}, \quad i = 1, \dots, n-1,$$

and that

$$\begin{aligned} A_nU_nA_n^T &= \operatorname{diag}(\beta_1,\beta_2,\dots,\beta_{n-1},c_n) \\ &= \operatorname{diag}(c_1-\rho^2c_2,c_2-\rho^2c_3,\dots,c_{n-1}-\rho^2c_n,c_n). \end{aligned} \tag{4}$$

Since $\det(A_n)=1$, (3) and (4) imply that

$$\det(L_n)=c_1\prod_{i=1}^{n-1}(c_{i+1}-\rho^2c_i) \tag{5}$$

and

$$\det(U_n)=c_n\prod_{i=1}^{n-1}(c_i-\rho^2c_{i+1}). \tag{6}$$

Note that (5) and (6) both reduce to (1) when $c_1=c_2=\cdots=c_n=1$.

We will prove the following two theorems together.

THEOREM 1 *The inertia of L_n is (m,z,p), where m, z, and p are the numbers of negative, zero, and positive elements in the set*

$$\{c_1,c_2-\rho^2c_1,c_3-\rho^2c_2,\dots,c_n-\rho^2c_{n-1}\}.$$

Moreover,

$$\sum_{r,s=1}^{n}\rho^{|r-s|}c_{\min(r,s)}x_rx_s=c_1\left(\sum_{j=1}^{n}\rho^{j-1}x_j\right)^2+\sum_{i=2}^{n}(c_{i+1}-\rho^2c_i)\left(\sum_{j=i}^{n}\rho^{j-i}x_j\right)^2. \tag{7}$$

THEOREM 2 *The inertia of U_n is (m,z,p), where m, z, and p are the numbers of negative, zero, and positive elements in the set*

$$\{c_1-\rho^2c_2,c_2-\rho^2c_3,\dots,c_{n-1}-\rho^2c_n,c_n\}.$$

Moreover,

$$\sum_{r,s=1}^{n}\rho^{|r-s|}c_{\max(r,s)}x_rx_s=\sum_{i=1}^{n-1}(c_i-\rho^2c_{i+1})\left(\sum_{j=1}^{i}\rho^{i-j}x_j\right)^2+c_n\left(\sum_{j=1}^{n}\rho^{n-j}x_j\right)^2. \tag{8}$$

PROOF: By Sylvester's theorem, (3) and (4) imply the statements concerning inertia. From (3),

$$L_n=(A_n^{-1})^T\operatorname{diag}(c_1,c_2-\rho^2c_1,c_3-\rho^2c_2,\dots,c_n-\rho^2c_{n-1})A_n^{-1}. \tag{9}$$

From (4),

$$U_n = A_n^{-1}\,\mathrm{diag}(c_1 - \rho^2 c_2, c_2 - \rho^2 c_3, \dots, c_{n-1} - \rho^2 c_n, c_n)(A_n^{-1})^T. \tag{10}$$

Since

$$A_n^{-1} = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-3} & \rho^{n-2} & \rho^{n-1} \\ 0 & 1 & \rho & \cdots & \rho^{n-4} & \rho^{n-3} & \rho^{n-2} \\ 0 & 0 & 1 & \cdots & \rho^{n-5} & \rho^{n-4} & \rho^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \rho & \rho^2 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \rho \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix},$$

(9) implies (7) and (10) implies (8). ■

Example 1. With $\rho = 1$ and and $c_r = r$, (7) and (8) reduce to

$$\sum_{r,s=1}^{n} \min(r,s) x_r x_s = \sum_{i=1}^{n} \left(\sum_{j=i}^{n} x_j \right)^2 \tag{11}$$

and

$$\sum_{r,s=1}^{n} \max(r,s) x_r x_s = -\sum_{i=1}^{n-1} \left(\sum_{j=1}^{i} x_j \right)^2 + n \left(\sum_{j=1}^{n} x_j \right)^2.$$

These diagonalizations have recently been obtained by T. Y. Lam [5], who observed that (11) was previously stated in [2].

Example 2. With $c_r = 1$, (7) and (8) provide distinct diagonalizations of the quadratic form associated with $K_n(\rho)$:

$$\sum_{r,s=1}^{n} \rho^{|r-s|} x_r x_s = \left(\sum_{j=1}^{n} \rho^{j-1} x_j \right)^2 + (1-\rho^2) \sum_{i=2}^{n} \left(\sum_{j=i}^{n} \rho^{j-i} x_j \right)^2,$$

$$\sum_{r,s=1}^{n} \rho^{|r-s|} x_r x_s = (1-\rho^2) \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i} \rho^{i-j} x_j \right)^2 + \left(\sum_{j=1}^{n} \rho^{n-j} x_j \right)^2.$$

Theorem 3 *If* $\det(L_n) \neq 0$ *define*

$$\sigma_i = \frac{1}{c_{i+1} - \rho^2 c_i}, \quad i = 1, \dots, n-1.$$

Then $L_n^{-1} = (u_{rs})_{r,s=1}^n$ *is the symmetric tridiagonal matrix with*

$$u_{11} = 1/c_1 + \rho^2\sigma_1, \quad u_{nn} = \sigma_{n-1},$$

$$u_{rr} = \sigma_{r-1} + \rho^2\sigma_r, \quad r = 2, \dots, n-1,$$

and

$$u_{r+1,r} = u_{r,r+1} = -\rho\sigma_r, \quad r = 1, \dots, n-1.$$

For example,

$$L_5^{-1} = \begin{bmatrix} 1/c_1 + \rho^2\sigma_1 & -\rho\sigma_1 & 0 & 0 & 0 \\ -\rho\sigma_1 & \sigma_1 + \rho^2\sigma_2 & -\rho\sigma_2 & 0 & 0 \\ 0 & -\rho\sigma_2 & \sigma_2 + \rho^2\sigma_3 & -\rho\sigma_3 & 0 \\ 0 & 0 & -\rho\sigma_3 & \sigma_3 + \rho^2\sigma_4 & -\rho\sigma_4 \\ 0 & 0 & 0 & -\rho\sigma_4 & \sigma_4 \end{bmatrix}.$$

Proof: From (9),

$$L_n^{-1} = A_n \operatorname{diag}(1/c_1, \sigma_1, \sigma_2, \dots, \sigma_{n-1}) A_n^T,$$

and routine manipulations verify the stated result. ■

Example 3. Let $c_r = 1 + \gamma\rho^{2r}$, where γ is an arbitrary real number. Then

$$\rho^{|r-s|} c_{\min(r,s)} = \rho^{|r-s|} + \gamma\rho^{r+s}.$$

We denote L_n by $K_n(\rho, \gamma)$, since we will return to this matrix in Section 3; thus

$$K_n(\rho, \gamma) = \left(\rho^{|r-s|} + \gamma\rho^{r+s}\right)_{r,s=1}^n.$$

In this case $c_{i+1} - \rho^2 c_i = 1 - \rho^2$, so (5) implies that

$$\det(K_n(\rho, \gamma)) = (1 + \gamma\rho^2)(1 - \rho^2)^{n-1}. \tag{12}$$

Since

$$\sigma_i = \frac{1}{1 - \rho^2}, \dots, i = 1, \dots, n-1,$$

Theorem 3 implies that

$$K_n^{-1}(\rho, \gamma) = \frac{1}{1 - \rho^2} \begin{bmatrix} \dfrac{1 + \gamma\rho^4}{1 + \gamma\rho^2} & -\rho & 0 & \cdots & 0 & 0 & 0 \\ -\rho & 1 + \rho^2 & -\rho & \cdots & 0 & 0 & 0 \\ 0 & -\rho & 1 + \rho^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 + \rho^2 & -\rho & 0 \\ 0 & 0 & 0 & \cdots & -\rho & 1 + \rho^2 & -\rho \\ 0 & 0 & 0 & \cdots & 0 & -\rho & 1 \end{bmatrix}$$

if $\rho \neq \pm 1$ and $\gamma\rho^2 \neq -1$.

Example 4. If $\rho = 1$ and $c_r = r - \gamma$ then

$$L_n = (\min(r,s) - \gamma)_{r,s=1}^n.$$

Since $\sigma_i = 1$, $i = 1, \ldots, n-1$, Theorem 3 implies that

$$L_n^{-1} = \begin{bmatrix} \dfrac{2-\gamma}{1-\gamma} & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & 1 \end{bmatrix} \tag{13}$$

if $\gamma \neq 1$.

THEOREM 4 *If* $\det(U_n) \neq 0$ *define*

$$\tau_i = \frac{1}{c_i - \rho^2 c_{i+1}}, \quad i = 1, \ldots, n-1.$$

Then $U_n^{-1} = (v_{rs})_{r,s=1}^n$ *is the symmetric tridiagonal matrix with*

$$v_{11} = \tau_1, \quad v_{nn} = \rho^2 \tau_{n-1} + 1/c_n,$$

$$v_{rr} = \rho^2 \tau_{r-1} + \tau_r, \quad r = 2, \ldots, n-1,$$

and

$$v_{r+1,r} = v_{r,r+1} = -\rho\tau_r, \quad r = 1, \ldots, n-1.$$

For example,

$$U_5^{-1} = \begin{bmatrix} \tau_1 & -\rho\tau_1 & 0 & 0 & 0 \\ -\rho\tau_1 & \rho^2\tau_1 + \tau_2 & -\rho\tau_2 & 0 & 0 \\ 0 & -\rho\tau_2 & \rho^2\tau_2 + \tau_3 & -\rho\tau_3 & 0 \\ 0 & 0 & -\rho\tau_3 & \rho^2\tau_3 + \tau_4 & -\rho\tau_4 \\ 0 & 0 & 0 & -\rho\tau_4 & \rho^2\tau_4 + 1/c_5 \end{bmatrix}.$$

PROOF: From (10),

$$U_n^{-1} = A_n^T \operatorname{diag}(\tau_1, \tau_2, \tau_3, \ldots, \tau_{n-1}, 1/c_n) A_n,$$

and routine manipulations verify the stated result. ■

Example 5. Let $c_r = 1 + \gamma\rho^{-2r}$, where γ is real. Then

$$U_n = \left(\rho^{|r-s|} + \gamma\rho^{-r-s}\right)_{r,s=1}^{n}.$$

In this case $c_i - \rho^2 c_{i+1} = 1 - \rho^2$, so (6) implies that

$$\det(U_n) = (1 + \gamma\rho^{-2n})(1 - \rho^2)^{n-1}.$$

Since

$$\tau_i = \frac{1}{1-\rho^2}, \dots, i = 1, \dots, n-1,$$

Theorem 4 implies that

$$U_n^{-1} = \frac{1}{1-\rho^2}\begin{bmatrix} 1 & -\rho & 0 & \cdots & 0 & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\rho^2 & -\rho & 0 \\ 0 & 0 & 0 & \cdots & -\rho & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & \cdots & 0 & -\rho & \dfrac{\rho^{2n}+\gamma\rho^2}{\rho^{2n}+\gamma} \end{bmatrix}$$

if $\rho \neq \pm 1$ and $\gamma \neq -\rho^{2n}$.

Example 6. If $\rho = 1$ and $c_r = r - \gamma$ then

$$U_n = (\max(r,s) - \gamma)_{r,s=1}^{n}.$$

Since $\tau_i = -1$, $i = 1, \dots, n-1$, Theorem 4 implies that

$$U_n^{-1} = \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \dfrac{-n+1+\gamma}{n-\gamma} \end{bmatrix}$$

if $\gamma \neq n$.

3. Spectral Properties of $K_n(\rho,\gamma)$.

If $0<\rho<1$ then

$$\sum_{n=-\infty}^{\infty}\rho^{|n|}e^{in\theta}=F(\theta)=\frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2}, \tag{14}$$

and it is known that the eigenvalues $\lambda_{1n}<\lambda_{2n}<\cdots<\lambda_{nn}$ of $K_n(\rho)=(\rho^{|r-s|})_{r,s=1}^n$ are given by

$$\lambda_{jn}=F(\phi_{n-j+1,n}),$$

where

$$\frac{(j-1)\pi}{n+1}<\phi_{jn}<\frac{j\pi}{n+1},\quad j=1,2,\dots n.$$

(See [6] for more on this.) This illustrates a theorem of Szegö [1, Chapter 5] which implies that if $\{c_r\}_{r=-\infty}^{\infty}$ are the Fourier coefficients of a bounded real-valued even function $f\in L[-\pi,\pi]$ then the spectra of the symmetric Toeplitz matrices $T_n=(c_{r-s})_{r,s=1}^n$, $n=1,2,\dots$, are equally distributed in the sense of H. Weyl [1, p. 62] with values of f at n equally spaced points in $[0,\pi]$, as $n\to\infty$. We will now obtain related results on the spectrum of $K_n(\rho,\gamma)$ as $n\to\infty$, assuming that $0<\rho<1$. We also assume temporarily that $\gamma\rho^2\neq -1$, so $K_n(\rho,\gamma)$ is invertible.

We begin by considering the spectrum of

$$V_n=(1-\rho^2)K_n^{-1}(\rho,\gamma)=\begin{bmatrix}\frac{1+\gamma\rho^4}{1+\gamma\rho^2} & -\rho & 0 & \cdots & 0 & 0 & 0\\ -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 & 0\\ 0 & -\rho & 1+\rho^2 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1+\rho^2 & -\rho & 0\\ 0 & 0 & 0 & \cdots & -\rho & 1+\rho^2 & -\rho\\ 0 & 0 & 0 & \cdots & 0 & -\rho & 1\end{bmatrix}.$$

It is straightforward to verify that if $x_0,x_1,\dots,x_n,x_{n+1}$ (not all zero) satisfy

$$-\rho x_{r-1}+[1+\rho^2-\mu]x_r-\rho x_{r+1}=0,\quad 1\le r\le n, \tag{15}$$

and the boundary conditions

$$(1+\gamma\rho^2)x_0=\rho(1+\gamma)x_1\quad\text{and}\quad x_{n+1}=\rho x_n, \tag{16}$$

then $x=[x_1\,x_2\,\cdots\,x_n]^T$ is a μ-eigenvector of V_n. The solutions of (15) are of the form

$$x_r=c_1\zeta^r+c_2\zeta^{-r}, \tag{17}$$

where ζ and $1/\zeta$ are the zeros of the reciprocal polynomial

$$P(z) = -\rho z^2 + (1+\rho^2-\mu)z - \rho. \tag{18}$$

The boundary conditions (16) require that

$$\begin{array}{rcl} (1+\gamma\rho^2)(c_1+c_2) & = & \rho(1+\gamma)(c_1\zeta + c_2/\zeta) \\ c_1\zeta^{n+1} + c_2\zeta^{-n-1} & = & \rho(c_1\zeta^n + c_2\zeta^{-n}). \end{array} \tag{19}$$

The determinant of this system is

$$\begin{array}{rcl} D_n(\zeta) & = & \left| \begin{array}{cc} 1+\gamma\rho^2-\rho(1+\gamma)\zeta & 1+\gamma\rho^2-\rho(1+\gamma)/\zeta \\ \zeta^{n+1}(1-\rho/\zeta) & \zeta^{-n-1}(1-\rho\zeta) \end{array} \right| \\ & = & (1+\gamma\rho^2)(\zeta^{-n-1}-\zeta^{n+1}) - \rho(2+\gamma(1+\rho^2))(\zeta^{-n}-\zeta^n) \\ & & +\rho^2(1+\gamma)(\zeta^{-n+1}-\zeta^{n-1}). \end{array} \tag{20}$$

With $\zeta = \pm 1$, (19) has the nontrivial solution $(1,-1)$, but (17) yields $x_r = 0$ for all r. Therefore the zeros ± 1 of D_n are not associated with eigenvalues of V_n. The remaining $2n$ zeros of D_n occur in reciprocal pairs $(\zeta, 1/\zeta)$. Corresponding to a given pair, x as defined in (17) is an eigenvector of V_n, and therefore of $K_n(\rho,\gamma)$. To determine the eigenvalue μ of V_n with which it is associated, we note that since

$$P(z) = -\rho(z-\zeta)(z-1/\zeta) = -\rho(z^2-(\zeta+1/\zeta)z+1),$$

(18) implies that

$$\mu = 1-\rho(\zeta+1/\zeta)+\rho^2.$$

Therefore

$$\lambda = G(\zeta) = \frac{1-\rho^2}{1-\rho(\zeta+1/\zeta)+\rho^2}$$

is an eigenvalue of $K_n(\rho,\gamma)$. In particular, if $\zeta = e^{i\theta}$ then $F(\theta)$ (see (14)) is an eigenvalue of $K_n(\rho,\gamma)$.

THEOREM 5 *Let ρ and γ be real numbers, with $0<\rho<1$. Then:*

(a) *$K_n(\rho,\gamma)$ has eigenvalues of the form $F(\theta_{jn})$, $j=2,\dots,n-1$, where*

$$\frac{(j-1)\pi}{n} < \theta_{jn} < \frac{j\pi}{n}, \quad j=2,\dots,n-1. \tag{21}$$

(b) *If $\gamma \le 1/\rho$ then $K_n(\rho,\gamma)$ has an eigenvalue of the form $F(\theta_{1n})$, where*

$$0 < \theta_{1n} < \frac{\pi}{n}.$$

(c) *If $\gamma \ge -1/\rho$ then $K_n(\rho,\gamma)$ has an eigenvalue of the form $F(\theta_{nn})$, where*

$$\frac{(n-1)\pi}{n} < \theta_{nn} < \pi.$$

PROOF: It suffices to prove (a) and (b) under the additional assumption that $\gamma \neq -1/\rho^2$ (so that V_n is defined), since the conclusions will then follow in the case where $\gamma = -1/\rho^2$ by a continuity argument. We first isolate the zeros $\zeta = e^{i\theta}$ of D_n with $0 < \theta < \pi$. Define

$$S_n(\theta) = (1+\gamma\rho^2)\frac{\sin(n+1)\theta}{\sin\theta} - \rho(2+\gamma(1+\rho^2))\frac{\sin n\theta}{\sin\theta} + \rho^2(1+\gamma)\frac{\sin(n-1)\theta}{\sin\theta}$$

on $[0,\pi]$, where the definition at the endpoints is by continuity; then $D_n(e^{i\theta}) = D_n(e^{-in\theta}) = 0$ if and only if $S_n(\theta) = 0$.

It is routine to verify that

$$S_n(0) = (1-\rho)[1+\rho+n(1-\rho)(1-\gamma\rho)], \tag{22}$$

$$S_n\left(\frac{j\pi}{n}\right) = (-1)^j(1-\rho^2), \quad j = 1,\ldots,n-1, \tag{23}$$

and

$$S_n(\pi) = (-1)^n(1-\rho^2+n(1+\rho)^2(1+\gamma\rho)). \tag{24}$$

From (23), S_n changes sign on $((j-1)\pi/n, j\pi/n)$, $j = 2,\ldots,n-1$. This implies (a). If $\gamma \leq 1/\rho$ then (22), and (23) with $j = 1$ imply that S_n changes sign on $(0,\pi/n)$. This implies (b). If $\gamma \geq -1/\rho$ then (23) with $j = n-1$ and (24) imply that S_n changes sign on $((n-1)\pi/n, \pi)$. This implies (c).

Now let $\lambda_{1n} < \lambda_{2n} < \ldots < \lambda_{nn}$ be the eigenvalues of $K_n(\rho,\gamma)$. Let

$$\alpha = \frac{1-\rho}{1+\rho} = \min_{0\leq\theta\leq\pi} F(\theta) \quad \text{and} \quad \beta = \frac{1+\rho}{1-\rho} = \max_{0\leq\theta\leq\pi} F(\theta),$$

and define

$$\chi_{jn} = F\left(\frac{(2n-2j+1)\pi}{2n}\right) \quad j = 1,\ldots,n. \tag{25}$$

THEOREM 6 *Suppose that* $0 < \rho < 1$, $|\gamma| \leq 1/\rho$, *and* H *is continuous on* $[\alpha,\beta]$. *Then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} |H(\lambda_{jn}) - H(\chi_{jn})| = 0. \tag{26}$$

According to a definition given in [8], the sets $\{\lambda_{jn}\}_{j=1}^n$ and $\{\chi_{jn}\}_{j=1}^n$ are *absolutely equally distributed as* $n \to \infty$. This is stronger than Weyl's definition of *equally distributed as* $n \to \infty$, which does not require the absolute value signs in (26). The proof that we are about to give is similar to the proof of Theorem 4 in [7]. We repeat the proof here because there were minor – but potentially confusing – errors in the enumeration of $\{\lambda_{jn}\}_{j=1}^n$ and $\{\chi_{jn}\}_{j=1}^n$ in [7].

Proof: Since F is decreasing, Theorem 5 implies that

$$\lambda_{jn} = F(\theta_{n-j+1,n}), \quad j = 1, \dots, n.$$

Therefore (21), (25), and the mean value theorem imply that

$$|\lambda_{kn} - \chi_{kn}| \le \frac{K\pi}{2n}, \tag{27}$$

where $K = \max_{0 \le \theta \le \pi} |F'(\theta)|$. Let

$$W_n(H) = \sum_{k=1}^{n} |H(\lambda_{kn}) - H(\chi_{kn})|.$$

If H is constant then $W_n(H) = 0$. If N is a positive integer then (27) and the mean value theorem imply that

$$\left|\lambda_{kn}^N - \chi_{kn}^N\right| \le N\beta^{N-1}|\lambda_{kn} - \chi_{kn}| \le \frac{N\beta^{N-1}K\pi}{2n},$$

so (26) holds if H is a polynomial.

Now suppose H is an arbitrary continuous function on $[\alpha, \beta]$ and let $\epsilon > 0$ be given. From the Weierstrass approximation theorem, there is a polynomial P such that $|H(u) - P(u)| < \epsilon$ for all u in $[\alpha, \beta]$. Therefore $W_n(H) < W_n(P) + 2n\epsilon$, and

$$\limsup_{n\to\infty} \frac{W_n(H)}{n} \le \lim_{n\to\infty} \frac{W_n(P)}{n} + 2\epsilon = 2\epsilon.$$

Now let $\epsilon \to 0$ to conclude that $\lim_{n\to\infty} W_n(H)/n = 0$. ■

Theorem 7 *Suppose that* $0 < \rho < 1$ *and* H *is continuous on* $[\alpha, \beta]$. *Then:*

(a) *If* $\gamma > 1/\rho$ *then*

$$\lim_{n\to\infty} \lambda_{nn} = \frac{(1+\gamma)(1+\gamma\rho^2)}{\gamma(1-\rho^2)} \tag{28}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n-1} |H(\lambda_{jn}) - H(\chi_{jn})| = 0. \tag{29}$$

(b) *If* $\gamma < -1/\rho$ *then*

$$\lim_{n\to\infty} \lambda_{1n} = \frac{(1+\gamma)(1+\gamma\rho^2)}{\gamma(1-\rho^2)} \tag{30}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=2}^{n} |H(\lambda_{jn}) - H(\chi_{jn})| = 0. \tag{31}$$

PROOF: If $\gamma = -1/\rho^2$ then (12) implies that $\lambda_{1n} = 0$, which verifies (30) in this case. Henceforth we assume that $|\gamma| > 1/\rho$, but $\gamma \neq -1/\rho^2$. For all values of n and γ, Theorem 5 implies that at least $n-1$ eigenvalues of $K_n(\rho,\gamma)$ are values of $F(\theta)$ and therefore in (α,β). This and the fact that $D_n(1) = D_n(-1) = 0$ account for at least $2n$ zeros of D_n. If $\gamma > 1/\rho$ then $S_n(0)$ and $S_n(\pi/n)$ are both negative for n sufficiently large, while if $\gamma < -1/\rho^2$ then $S_n(\pi)$ and $S_n((n-1)\pi/n)$ and $S_n(\pi)$ have the same sign for n sufficiently large. Therefore, there is an N such that if $n \geq N$ then D_n has exactly one pair $(\zeta_n, 1/\zeta_n)$ of zeros which are not on the unit circle.

Hence, ζ_n is real, and we may assume without loss of generality that $|\zeta_n| > 1$. We denote the eigenvalue corresponding to ζ_n by ν_n; thus,

$$\nu_n = G(\zeta_n) = \frac{1-\rho^2}{1-\rho(\zeta_n + 1/\zeta_n) + \rho^2}. \tag{32}$$

Since ζ_n is not on the unit circle, $\nu_n \notin [\alpha,\beta]$. Therefore the Cauchy interlacement theorem implies that $\nu_n = \lambda_{nn}$ for all $n \geq N$ or $\nu_n = \lambda_{1n}$ for every $n \geq N$, and that $|\nu_{n+1}| > |\nu_n|$. Therefore (32) implies that $|\zeta_{n+1}| > |\zeta_n|$.

Now it is convenient to rewrite (20) as

$$D_n(\zeta) = \zeta^{-n+1}H(1/\zeta) - \zeta^{n-1}H(\zeta), \tag{33}$$

with

$$\begin{aligned} H(\zeta) &= (1+\gamma\rho^2)\zeta^2 - \rho(2+\gamma(1+\rho^2)\zeta + \rho^2(1+\gamma) \\ &= (1+\gamma\rho^2)(\zeta-\rho)(\zeta-\zeta_\infty), \end{aligned} \tag{34}$$

where

$$\zeta_\infty = \frac{\rho(1+\gamma)}{1+\gamma\rho^2}.$$

Since $D_n(\zeta_n) = 0$, (33) and (34) imply that

$$\zeta_n - \zeta_\infty = \frac{\zeta_n^{-2n+2}H(1/\zeta_n)}{(1+\gamma\rho^2)(\zeta_n-\rho)}.$$

Since $|\zeta_n|$ is increasing and greater than 1, this implies that $\lim_{n\to\infty}\zeta_n = \zeta_\infty$. Therefore

$$\lim_{n\to\infty} \nu_n = G(\zeta_\infty) = \frac{(1+\gamma)(1+\gamma\rho^2)}{\gamma(1-\rho^2)}.$$

Since the quantity on the right is greater than β if $\gamma > 1/\rho$, or less than α if $\gamma < -1/\rho$, this implies (28) if $\gamma > 1/\rho$, or (30) if $\gamma < -1/\rho$.

Now Theorem 5 implies that if $\gamma > 1/\rho$ then $\lambda_{jn} = F(\theta_{n-j+1,n})$, $j = 1,\dots,n-1$, while if $\gamma < -1/\rho$ then $\lambda_{jn} = F(\theta_{n-j+1,n})$, $j = 2,\dots,n$, and arguments similar to the proof of Theorem 6 yield (29) and (31). ■

4. Spectral Properties of $L_n = (\min(r,s) - \gamma)_{r,s=1}^n$.

We now consider the spectrum of $L_n = (\min(r,s)-\gamma)_{r,s=1}^n$ in the case where $\gamma \le 1/2$. We begin by considering the spectrum of L_n^{-1} (see (13)). It is straightforward to verify that if $x_0, x_1, \dots, x_n, x_{n+1}$ (not all zero) satisfy the difference equation

$$x_{r-1} - (2-\mu)x_r + x_{r+1} = 0, \quad 1 \le r \le n, \tag{35}$$

and the boundary conditions

$$(1-\gamma)x_0 + \gamma x_1 = 0 \quad \text{and} \quad x_n - x_{n+1} = 0, \tag{36}$$

then $x = [x_1\, x_2\, \cdots\, x_n]^T$ satisfies $L_n^{-1}x = \mu x$; therefore, μ is an eigenvalue of L_n^{-1} if and only if (35) has a nontrivial solution satisfying (36), in which case x is μ-eigenvector of L_n^{-1}.

The solutions of (35) are of the form

$$x_r = c_1\zeta^r + c_2\zeta^{-r}, \tag{37}$$

where ζ and $1/\zeta$ are the zeros of the reciprocal polynomial

$$P(z) = z^2 - (2-\mu)z + 1. \tag{38}$$

The boundary conditions (36) require that

$$\begin{aligned} (1-\gamma)(c_1+c_2) + \gamma(c_1\zeta + c_2/\zeta) &= 0 \\ (c_1\zeta^n + c_2\zeta^{-n}) - (c_1\zeta^{n+1} + c_2\zeta^{-n-1}) &= 0. \end{aligned} \tag{39}$$

The determinant of this system is

$$\begin{aligned} D_n(\zeta) &= \begin{vmatrix} 1-\gamma+\gamma\zeta & 1-\gamma+\gamma/\zeta \\ \zeta^n - \zeta^{n+1} & 1/\zeta^n - 1/\zeta^{n+1} \end{vmatrix} \\ &= \zeta^{-n-1}(\zeta-1)[(1-\gamma)(\zeta^{2n+1}+1) + \gamma(\zeta^{2n}+\zeta)]. \end{aligned}$$

With $\zeta = 1$, (39) has the nontrivial solution $(1,-1)$, but (37) yields $x_r = 0$ for all r. Therefore $\zeta = 1$ is not associated with an eigenvalue of L_n^{-1}. The remaining $2n$ zeros of D_n occur in reciprocal pairs $(\zeta, 1/\zeta)$. Corresponding to a given pair, x as defined in (37) is an eigenvector of L_n^{-1} (and therefore of L_n). To determine the eigenvalue μ of L_n^{-1} with which it is associated, we note that since

$$P(z) = (z-\zeta)(z-1/\zeta) = z^2 - (\zeta + 1/\zeta)z + 1,$$

(38) implies that

$$\mu = \left(2 - \zeta - \frac{1}{\zeta}\right).$$

Therefore

$$\lambda = \frac{1}{2-\zeta-1/\zeta} \tag{40}$$

is an eigenvalue of L_n.

THEOREM 8 *If $\gamma \le 1/2$ then the eigenvalues $\lambda_{1n} < \lambda_{2n} < \cdots < \lambda_{nn}$ of*

$$L_n = (\min(r,s) - \gamma)_{r,s=1}^n$$

are of the form

$$\lambda_{jn} = \frac{1}{4}\csc^2\frac{\theta_{n-j+1,n}}{2},$$

where

$$\frac{2(j-1)\pi}{2n+1} < \theta_{jn} < \frac{2j\pi}{2n+1}.$$

PROOF: It suffices to isolate the zeros $\zeta = e^{i\theta}$ of D_n with $0 < \theta < \pi$. Define

$$C_n(\theta) = (1-\gamma)\cos(n+1/2)\theta + \gamma\cos(n-1/2)\theta.$$

Then $D_n(e^{in\theta}) = D_n(e^{-in\theta}) = 0$ if $C_n(\theta) = 0$. If $\gamma \le 1/2$ then S_n changes sign on each interval

$$I_{jn} = \left(\frac{2(j-1)\pi}{2n+1}, \frac{2j\pi}{2n+1}\right), \quad j = 1, \dots, n.$$

This implies that $S_n(\theta_{jn}) = 0$ for some θ_{jn} in I_{jn}. From (40), $(1/4)\csc^2(\theta_{jn}/2)$ is an eigenvalue of L_n. Since $\csc^2(\theta/2)$ is decreasing on $(0,\pi)$, the conclusion follows. ■

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