On matrices with rotative symmetries

William F. Trench

Trinity University, San Antonio, Texas, USA

Note: This is paper is is poorly written and organized, so much so that I voluntarily withdrew it from consideration for publication in 2006. However, it is the origin of ideas that I developed successfully in later work.

Abstract

We say that a unitary matrix R is rotative (specifically, k-rotative) if its minimal polynomial is $x^k - 1$ for some $k \geq 2$. Let $R \in \mathbb{C}^{m\times m}$ and $S \in \mathbb{C}^{n\times n}$ be k-rotative, α, β, $\mu \in \{0, 1, \dots, k-1\}$, and $\alpha\beta \neq 0$. Let $\zeta = e^{2\pi i/k}$. We define $\mathcal{A}(R, S, \alpha, \beta, \mu)$ to be the class of matrices $A \in \mathbb{C}^{m\times n}$ such that $R^\alpha A S^\beta = \zeta^\mu A$. If $m = n$ and $S = R$, we denote the class by $\mathcal{A}(R, \alpha, \beta, \mu)$. We characterize the class $\mathcal{A}(R, S, \alpha, \beta, \mu)$ and discuss the problem of Moore–Penrose inversion of a wider class of matrices that includes $\mathcal{A}(R, S, \alpha, \beta, \mu)$. Under the additional assumption that$(\alpha, k) = (\beta, k) = 1$, we give a representation of a matrix A in $\mathcal{A}(R, S, \alpha, \beta, \mu)$ in terms of matrices $F_s \in \mathbb{C}^{c_s\times d_s}$, where $\sum_{s=0}^{k-1} c_s = m$ and $\sum_{s=0}^{k-1} d_s = n$, and show that Moore–Penrose inversion, singular value decomposition, and the least squares problem for such a matrix reduce respectively to the same problems for $F_0, \dots, F_{k-1}$. We consider the eigenvalue problem for matrices in $\mathcal{A}(R, \alpha, \beta, \mu)$. We study a class of generalized circulants generated by blocks $A_0, \dots, A_{k-1} \in \mathbb{C}^{d_1\times d_2}$, and show that they are in $\mathcal{A}(R, S, 1, \beta, \mu)$ for suitable choices of R, S, and μ. In this case we give explicit formulas for $F_0, \dots, F_{k-1}$ in terms of $A_0, \dots, A_{k-1}$, and for $A^\dagger$ in terms of $F_0^\dagger, \dots, F_{k-1}^\dagger$.

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1 Introduction

We say that a unitary matrix R is rotative (specifically, k-rotative) if its minimal polynomial is $x^k - 1$ for some $k \ge 2$. A rotative matrix is a special kind of circulation matrix, which was defined by Chen [5] to be a unitary matrix $R \ne I$ such that $R^k = I$ for some $k \ge 2$. The difference between the definitions is that ours requires the spectrum of R to be $\left\{e^{2\pi i r/k} \,\middle|\, 0 \le r \le k-1\right\}$, while Chen's requires only that the spectrum of R is some subset of $\left\{e^{2\pi i r/k} \,\middle|\, 0 \le r \le k-1\right\}$. Chen studied matrices A such that $A = e^{i\theta} R^* A R$, where R is a circulation matrix and $\theta \in [0, \pi)$. Fasino continued this study in [7].

Throughout this paper $R \in \mathbb{C}^{m\times m}$ and $S \in \mathbb{C}^{n\times n}$ are both k-rotative. We assume that $k > 2$, since if $k = 2$ our results do not improve on those already obtained in [12, 13, 14], of which this paper is an extension.

We assume throughout that $\alpha,\ \beta,\ \mu \in \mathbb{Z}_k = \{0, 1, \dots, k-1\}$ and $\alpha\beta \ne 0$. Let $\zeta = e^{2\pi i/k}$. We define $\mathcal{A}(R, S, \alpha, \beta, \mu)$ to be the class of matrices $A \in \mathbb{C}^{m\times n}$ such that $R^\alpha A S^\beta = \zeta^\mu A$. If $m = n$ and $S = R$, we denote the class by $\mathcal{A}(R, \alpha, \beta, \mu)$.

This paper is influenced by the work of Ablow and Brenner [1], who considered the case where $m = n = k$, $R = S =$ the circulant with first row $\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$, $\mu = 0$, $\alpha = 1$, and $\beta = k - g$, where $1 \le g \le k-1$. They showed that $A \in \mathbb{C}^{k\times k}$ is a g-circulant (i.e., $A = [a_{(s-gr)(\bmod k)}]_{r,s=0}^{k-1}$) if and only if $RAR^{k-g} = A$, and used this to find the Jordan canonical form for A in the case where $(g, k) = 1$. They also considered the case where $(g, k) \ne 1$, and obtained results for a class of square block g-circulants. Other authors (see, e.g., [4, 6, 8, 9]) have considered spectral decompositions of various kinds of circulant-like matrices. Moore–Penrose inversion of such matrices has also been studied (see, e.g.,[3, 10, 11]).

In Section 2 we characterize the class $\mathcal{A}(R, S, \alpha, \beta, \mu)$ assuming only that $\alpha\beta \ne 0$, and we discuss Moore–Penrose inversion of a wider class of matrices that includes $\mathcal{A}(R, S, \alpha, \beta, \mu)$. Most of our results in Sections 3–7 require that $(\alpha, k) = (\beta, k) = 1$. In Section 3, under this assumption, we give a more specific representation A in $\mathcal{A}(R, S, \alpha, \beta, \mu)$ in terms of matrices $F_s \in \mathbb{C}^{c_s\times d_s}$, where $\sum_{s=0}^{k-1} c_s = m$ and $\sum_{s=0}^{k-1} d_s = n$, and show that $A^\dagger$ can be written in terms of $F_0^\dagger,\ \dots,\ F_{k-1}^\dagger$ and a singular value decomposition of A can be written in terms of singular value decompositions of $F_0, \dots, F_{k-1}$. In Section 4 it is shown that the least squares problem for A reduces to k independent least squares problems for $F_0, \dots, F_{k-1}$. In Section 5 we consider the eigenvalue problem for matrices in $\mathcal{A}(R, \alpha, \beta, \mu)$. In Section 6 we study the eigenvalue problem for $\mathcal{A}(R, 1, k-1, 0)$, which is the set of matrices $A \in \mathbb{C}^{n\times n}$ such that $RAR^* = A$. In Section 7 we study a class of generalized circulants generated by blocks $A_0, \dots, A_{k-1} \in \mathbb{C}^{d_1\times d_2}$, and show that they are in $\mathcal{A}(R, S, 1, \beta, \mu)$ for suitable choices of R, S, and μ. Under the assumption that $(\beta, k) = 1$, we give explicit formulas for $F_0, \dots, F_{k-1}$ in terms of $A_0, \dots, A_{k-1}$, and for $A^\dagger$ in terms of $F_0^\dagger, \dots, F_{k-1}^\dagger$.

2 Preliminary considerations

Throughout this paper $\mathcal{E}_B(\lambda)$ denotes the λ-eigenspace of B. Let c_s and d_s be the dimensions of $\mathcal{E}_R(\zeta^s)$ and $\mathcal{E}_S(\zeta^s)$, respectively. Then $\sum_{s=0}^{k-1} c_s = m$, $\sum_{s=0}^{k-1} d_s = n$, and there are matrices $P_s \in \mathbb{C}^{m\times c_s}$ and $Q_s \in \mathbb{C}^{n\times d_s}$ such that

$$RP_s = \zeta^s P_s, \quad SQ_s = \zeta^s Q_s, \quad 0 \le s \le k-1, \tag{1}$$

$$P_r^* P_s = \delta_{rs} I_{c_s} \quad \text{and} \quad Q_r^* Q_s = \delta_{rs} I_{d_s}, \quad 0 \le r, s \le k-1. \tag{2}$$

Since $R^* = R^{-1}$ and $S^* = S^{-1}$, (1) implies that

$$R^* P_s = \zeta^{-s} P_s \quad \text{and} \quad S^* Q_s = \zeta^{-s} Q_s, \quad 0 \le s \le k-1. \tag{3}$$

Let

$$P = \begin{bmatrix} P_0 & P_1 & \cdots & P_{k-1} \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} Q_0 & Q_1 & \cdots & Q_{k-1} \end{bmatrix}. \tag{4}$$

Then (1) implies that

$$R = P(I_{c_0} \oplus \zeta I_{c_1} \oplus \cdots \oplus \zeta^{k-1} I_{c_{k-1}})P^* \tag{5}$$

and

$$S = Q(I_{d_0} \oplus \zeta I_{d_1} \oplus \cdots \oplus \zeta^{k-1} I_{d_{k-1}})Q^*. \tag{6}$$

Theorem 1 *$A \in \mathcal{A}(R, S, \alpha, \beta, \mu)$ if and only if*

$$A = PCQ^* \quad \textit{with} \quad C = [C_{rs}]_{r,s=0}^{k-1}, \tag{7}$$

where $C_{rs} \in \mathbb{C}^{c_r \times d_s}$ and

$$C_{rs} = 0 \quad \textit{if} \quad \alpha r + \beta s \not\equiv \mu \pmod{k}, \quad 0 \le r, s \le k-1. \tag{8}$$

PROOF. Any A in $\mathbb{C}^{n\times n}$ can be written as in (7) with $C = P^*AQ$. From (1) and (3),

$$R^\alpha P = \begin{bmatrix} P_0 & \zeta^\alpha P_1 & \cdots & \zeta^{(k-1)\alpha} P_{k-1} \end{bmatrix} \quad \text{and} \quad Q^* S^\beta = \begin{bmatrix} Q_0^* \\ \zeta^\beta Q_1^* \\ \vdots \\ \zeta^{(k-1)\beta} Q_{k-1}^* \end{bmatrix},$$

so

$$R^\alpha A S^\beta = P\left(\left[\zeta^{\alpha r+\beta s} C_{rs}\right]_{r,s=0}^{k-1}\right) Q^* = \zeta^\mu A = P\,[\zeta^\mu C_{rs}]_{r,s=0}^{k-1}\, Q^*$$

if and only if (8) holds. $\square$

It can be seen from this proof that $\left\{A \in \mathbb{C}^{m\times n} \,\middle|\, R^\alpha A S^\beta = cA\right\} = \{0_{mn}\}$ unless $c = \zeta^\mu$ for some $\mu \in \mathbb{Z}_k$.

The following theorem is valid for a class of matrices that includes $\mathcal{A}(R, S, \alpha, \beta, \mu)$ and the matrices studied by Chen [5] and Fasino [7].

Theorem 2 *Let $c_0, \dots, c_{k-1}, d_0, \dots, d_{k-1}$ be positive integers with $k \ge 2$ and let μ, $p_0, \dots, p_{k-1}$, and $q_0, \dots, q_{k-1}$ be integers such that the set*

$$\mathcal{T} = \left\{(r,s) \in \mathbb{Z}_k \times \mathbb{Z}_k \,\middle|\, p_r + q_s \equiv \mu \pmod{k}\right\} \tag{9}$$

is nonempty. Suppose that $C = [C_{rs}]_{r,s=0}^{k-1}$ with $C_{rs} \in \mathbb{C}^{c_r \times d_s}$ and

$$C_{rs} = 0 \quad \text{if} \quad (r,s) \notin \mathcal{T}. \tag{10}$$

Then

$$C^\dagger = [D_{rs}]_{r,s=0}^{k-1} \quad \text{with} \quad D_{rs} \in \mathbb{C}^{d_r \times c_s} \tag{11}$$

and

$$D_{rs} = 0 \quad \text{if} \quad (s,r) \notin \mathcal{T}. \tag{12}$$

Moreover, if $r \ne r'$ and $s \ne s'$ whenever (r,s) and (r',s') are distinct pairs in $\mathcal{T}$, then

$$D_{rs} = C_{sr}^\dagger, \quad (s,r) \in \mathcal{T}. \tag{13}$$

PROOF. In any case, $C^\dagger$ can be written as in (11). Let

$$R_0 = \zeta^{p_0} I_{c_0} \oplus \zeta^{p_1} I_{c_1} \oplus \cdots \oplus \zeta^{p_{k-1}} I_{c_{k-1}}$$

and

$$S_0 = \zeta^{-q_0} I_{d_0} \oplus \zeta^{-q_1} I_{d_1} \oplus \cdots \oplus \zeta^{-q_{k-1}} I_{d_{k-1}}.$$

Then

$$R_0 C S_0^* = \left[\zeta^{p_r+q_s} C_{rs}\right]_{r,s=0}^{k-1} = \zeta^\mu C,$$

where (9) and(10) imply the second equality; hence $C = \zeta^{-\mu} R_0 C S_0^*$. Now let $D = \zeta^\mu S_0 C^\dagger R_0^*$. We will show that C and D satisfy the Penrose conditions. Since R_0 and S_0 are unitary,

$$CD = R_0 C C^\dagger R_0^* = (CD)^*, \quad DC = S_0 C^\dagger C S_0^* = (DC)^*,$$
$$CDC = \zeta^{-\mu} R_0 C C^\dagger C S_0^* = \zeta^{-\mu} R_0 C S_0^* = C,$$

and

$$DCD = \zeta^\mu S_0 C^\dagger C C^\dagger R_0^* = \zeta^\mu S_0 C^\dagger R_0^* = D.$$

Hence $D = C^\dagger$, so $C^\dagger = \zeta^\mu S_0 C^\dagger R_0^*$. Hence,

$$D_{rs} = \zeta^{\mu - q_r - p_s} D_{rs}, \quad 0 \le r, s \le k-1.$$

This and (9) imply (12).

If the second assumption holds, there is a permutation $\{r_0, r_1, \dots, r_{k-1}\}$ of $\mathbb{Z}_k$ such that

$$\mathcal{T} \subset \{(r_0,0), (r_1,1), \quad (r_{k-1}, k-1)\}.$$

Since $C_{rs}^\dagger = 0_{sr}$ if $C_{rs} = 0_{rs}$, (10), (12), and (13) imply that

$$C = U\left(C_{r_0,0} \oplus C_{r_1,1} \oplus \cdots \oplus C_{r_{k-1},k-1}\right)$$

and

$$D = \left(C_{r_0,0}^\dagger \oplus C_{r_1,1}^\dagger \oplus \cdots \oplus C_{r_{k-1},k-1}^\dagger\right) U^T,$$

where U is a permutation matrix. It is straightforward to verify that C and D satisfy the Penrose conditions. $\square$

Theorem 3 *If* $A \in \mathcal{A}(R,S,\alpha,\beta,\mu)$, *then* $(A^{\dagger})^* \in \mathcal{A}(R,S,\alpha,\beta,\mu)$.

PROOF. Let

$$\mathcal{T} = \left\{(r,s) \in \mathbb{Z}_k \times \mathbb{Z}_k \,\middle|\, \alpha r + \beta s \equiv \mu \pmod{k}\right\}.$$

From Theorem 1, (7) holds with $C_{rs} = 0$ if $(r,s) \notin \mathcal{T}$. Hence Theorem 2 implies that $(C^{\dagger})^* = [E_{rs}]_{r,s=0}^{k-1}$, where $E_{rs} = D_{sr}^* = 0$ if $(r,s) \notin \mathcal{T}$. Now apply Theorem 1 to $(A^{\dagger})^* = P(C^{\dagger})^* Q^*$ to obtain the conclusion. □

3 The case where $(\alpha,k) = (\beta,k) = 1$

Henceforth we assume that $(\alpha,k) = (\beta,k) = 1$ except where stated otherwise. For $0 \le s \le k-1$, we define $\gamma(s)$ to be the unique member of $\mathbb{Z}_k$ such that

$$\alpha\gamma(s) + \beta s \equiv \mu \pmod{k};$$

thus,

$$\gamma(s) \equiv \widehat{\alpha}(\mu - \beta s) \pmod{k}, \tag{14}$$

where $\widehat{\alpha}$ is the unique member of $\mathbb{Z}_k$ such that $\widehat{\alpha}\alpha \equiv 1 \pmod{k}$. Then γ is a permutation of $\mathbb{Z}_k$.

Theorem 4 *Let*

$$V_\gamma = \left[\begin{array}{cccc} P_{\gamma(0)} & P_{\gamma(1)} & \cdots & P_{\gamma(k-1)} \end{array}\right].$$

Then $A \in \mathcal{A}(R,S,\alpha,\beta,\mu)$ *if and only if*

$$A = V_\gamma \left(\bigoplus_{s=0}^{k-1} F_s\right) Q^* = \sum_{s=0}^{k-1} P_{\gamma(s)} F_s Q_s^*, \tag{15}$$

with

$$F_s = C_{\gamma(s),s} = P_{\gamma(s)}^* A Q_s \in \mathbb{C}^{c_{\gamma(s)} \times d_s}, \quad 0 \le s \le k-1. \tag{16}$$

Hence,

$$A^{\dagger} = Q \left(\bigoplus_{s=0}^{k-1} F_s^{\dagger}\right) V_\gamma^* = \sum_{s=0}^{k-1} Q_s F_s^{\dagger} P_{\gamma(s)}^*. \tag{17}$$

PROOF. Since (4) and (7) imply that

$$A \left[\begin{array}{cccc} Q_0 & Q_1 & \cdots & Q_{k-1} \end{array}\right] = \left[\begin{array}{cccc} P_0 & P_1 & \cdots & P_{k-1} \end{array}\right] C$$

and $C_{rs} = 0$ if $r \ne \gamma(s)$, it follows that $AQ_s = P_{\gamma(s)} C_{\gamma(s),s}$; hence (2) implies that $C_{\gamma(s),s} = P_{\gamma(s)}^* A Q_s$. Moreover,

$$\left[\begin{array}{cccc} P_0 & P_1 & \cdots & P_{k-1} \end{array}\right] C = V_\gamma \left(\bigoplus_{s=0}^{k-1} F_s\right).$$

This implies (15), which in turn implies (17) □

Since the following corollary deals with different values of μ, we temporarily define $\gamma(s,\mu) \equiv \widehat{\alpha}(\mu - \beta s) \pmod{k}$.

Corollary 1 *Any $A \in \mathbb{C}^{m\times n}$ can be written uniquely as*

$$A = \sum_{\mu=0}^{k-1} A^{(\mu)},$$

where $A^{(\mu)} \in \mathcal{A}(R, S, \alpha, \beta, \mu)$, $0 \le \mu \le k-1$. Specifically, if A is as in (7), *then $A^{(\mu)}$ is given uniquely by*

$$A^{(\mu)} = P\left(\left[C_{rs}^{(\mu)}\right]_{r,s=0}^{k-1}\right) Q^*,$$

where

$$C_{rs}^{(\mu)} = \begin{cases} 0 & \text{if } r \ne \gamma(s,\mu), \\ C_{\gamma(s,\mu),s} & \text{if } r = \gamma(s,\mu), \end{cases} \quad 0 \le s \le k-1.$$

Throughout this paper it is to be understood that, for fixed α, β, and μ, F_0, …, F_{k-1} are as in (16), where we have suppressed the dependence of F_s on α, β, and μ for simplicity of notation.

We say that $z \in \mathbb{C}^n$ is (S, s)-symmetric if $Sz = \zeta^s z$ and $w \in \mathbb{C}^m$ is (R, s)-symmetric if $Rw = \zeta^s w$. These definitions have their origins in Andrew's [2] definitions of symmetric and skew-symmetric vectors: $z \in \mathbb{C}^n$ is symmetric (skew-symmetric) if $Jx = x$ ($Jx = -x$), where $J = \left[\delta_{i,n-j+1}\right]_{i,j=1}^n$. (For other extensions of Andrew's definitions, see [12, 14, 15].)

Arbitrary $z \in \mathbb{C}^n$ and $w \in \mathbb{C}^m$ can be written uniquely as

$$z = \sum_{r=0}^{k-1} Q_r x_r \quad \text{and} \quad w = \sum_{s=0}^{k-1} P_s y_s, \tag{18}$$

with

$$x_r = Q_r^* z \in \mathbb{C}^{d_r} \quad \text{and} \quad y_r = P_r^* w \in \mathbb{C}^{c_r}, \quad 0 \le r \le k-1. \tag{19}$$

From (1) and (18),

$$Sz = \sum_{r=0}^{k-1} \zeta^r Q_r x_r.$$

Therefore, (2) implies that z is (S, s)-symmetric if and only if $z = Q_s x_s$ for some $x_s \in \mathbb{C}^{d_s}$. Similarly, w is (R, s)-symmetric if and only if $w = P_s y_s$ for some $y_s \in \mathbb{C}^{c_s}$.

Theorem 4 implies the following theorem.

Theorem 5 *Suppose that $A \in \mathcal{A}(R, S, \alpha, \beta, \mu)$ and $F_s = \Omega_s \Sigma_s \Phi_s^*$ is a singular value decomposition of F_s, $0 \le s \le k-1$. Then*

$$A = \Omega\left(\bigoplus_{s=0}^{k-1} \Sigma_s\right)\Phi^*$$

with

$$\Omega = \begin{bmatrix} P_{\gamma(0)}\Omega_0 & P_{\gamma(1)}\Omega_1 & \cdots & P_{\gamma(k-1)}\Omega_{k-1} \end{bmatrix}$$

and

$$\Phi = \begin{bmatrix} Q_0\Phi_0 & Q_1\Phi_1 & \cdots & Q_{k-1}\Phi_{k-1} \end{bmatrix}$$

is a singular value decomposition of A. Thus, each singular value of F_s is a singular value of A associated with an $(R, \gamma(s))$-symmetric left singular vector and an (S, s)-symmetric right singular vector, $0 \le s \le k-1$.

Theorem 5 is related to [12, Theorems 11, 18], [13, Theorems 4.3, 5.3], and [15, Theorem3].

4 The least squares problem

Suppose that $G \in \mathbb{C}^{p\times q}$ and consider the least squares problem for G: If $u \in \mathbb{C}^p$, find $v \in \mathbb{C}^q$ such that

$$\|Gv - u\| = \min_{\xi\in\mathbb{C}^q} \|G\xi - u\|, \tag{20}$$

where $\|\cdot\|$ is the 2-norm. It is well known that this problem has a unique solution if and only if $\operatorname{rank}(G) = q$. In this case, $v = (G^*G)^{-1}G^*u$. In any case, the optimal solution of (20) is the unique n-vector v_0 of minimum norm that satisfies (20); thus, $v_0 = G^\dagger u$. The general solution of (20) is $v = v_0 + q$ with q in the null space of G, and

$$\|Gv - u\| = \|(GG^\dagger - I)u\|$$

for all such v.

We now consider the least squares problem for a matrix $A \in \mathcal{A}(R, S, \alpha, \beta, \mu)$. From (15) and (18),

$$Az - w = \sum_{s=0}^{k-1} P_{\gamma(s)}F_s x_s - \sum_{s=0}^{k-1} P_s y_s = \sum_{s=0}^{k-1} P_{\gamma(s)}(F_s x_s - y_{\gamma(s)}),$$

so (2) implies that

$$\|Az - w\|^2 = \sum_{s=0}^{k-1} \|F_s x_s - y_{\gamma(s)}\|^2.$$

This implies the following theorem.

Theorem 6 *Suppose that $A \in \mathcal{A}(R, S, \alpha, \beta, \mu)$. Let $w \in \mathbb{C}^m$ be given as in* (18). *Then $z \in \mathbb{C}^n$, written as in* (18), *satisfies*

$$\|Az - w\| = \min_{\xi\in\mathbb{C}^n} \|A\xi - w\| \tag{21}$$

if and only if

$$\|F_s x_s - y_{\gamma(s)}\| = \min_{\psi_s\in C^{d_s}} \|F_s\psi_s - y_{\gamma(s)}\|, \quad 0 \le s \le k-1,$$

with F_s as in (16). *Therefore* (21) *has a unique solution, given by*

$$z = \sum_{s=0}^{k-1} Q_s (F_s^* F_s)^{-1} F_s^* y_{\gamma(s)},$$

if and only $\operatorname{rank}(F_s) = d_s$, $0 \le s \le k-1$. *In any case, the optimal solution of* (21) *is*

$$z_0 = \sum_{s=0}^{k-1} Q_s F_s^\dagger y_{\gamma(s)}.$$

The general solution of (21) *is* $z = z_0 + \sum_{s=0}^{k-1} Q_s u_s$, *where* $F_s u_s = 0$, $0 \le s \le k-1$, *and*

$$\|Az - w\|^2 = \sum_{s=0}^{k-1} \|(F_s F_s^\dagger - I_{c_{\gamma(s)}}) y_{\gamma(s)}\|^2$$

for all such z.

5 The case where $m = n$ and $R = S$

In this section we assume that $m = n$, $S = R$, and $A \in \mathcal{A}(R, \alpha, \beta, u)$. Hence, (15) becomes

$$A = \sum_{s=0}^{k-1} P_{\gamma(s)} F_s P_s^* \tag{22}$$

and we can replace (18) and (19) by

$$z = \sum_{r=0}^{k-1} P_r x_r \quad \text{and} \quad w = \sum_{s=0}^{k-1} P_s y_s, \tag{23}$$

with

$$x_r = P_r^* z \in \mathbb{C}^{c_r} \quad \text{and} \quad y_r = P_r^* w \in \mathbb{C}^{c_r}, \quad 0 \le r \le k-1.$$

Let

$$\mathcal{S}_R = \bigcup_{s=0}^{k-1} \left\{ z \in \mathbb{C}^n \;\middle|\; Rz = \zeta^s z \right\}; \tag{24}$$

thus, $z \in \mathcal{S}_R$ if and only z is (R, s)-symmetric for some $s \in \mathbb{Z}_k$.

Theorem 7 *If A is singular, then the null space of A has a basis in $\mathcal{S}_R$.*

PROOF. Let $\mathcal{N}(A)$ be the nullspace of A. From (2), (22), and (23), $z \in \mathcal{N}(A)$ if and only if $F_s x_s = 0$, $0 \le s \le k-1$. Recall that $F_s \in \mathbb{C}^{c_{\gamma(s)} \times c_s}$, $0 \le s \le k-1$. Let $\mathcal{U} = \left\{ s \in \mathbb{Z}_k \;\middle|\; \operatorname{rank}(F_s) < c_s \right\}$. Since A is singular, $\mathcal{U} \ne \emptyset$. If $s \in \mathcal{U}$ and

$\{x_s^{(1)}, x_s^{(2)}, \cdots, x_s^{(m_s)}\}$ is a basis for the null space of F_s, then $P_s x_s^{(1)}$, $P_s x_s^{(2)}$, $\dots$, $P_s x_s^{(m_s)}$ are linearly independent (R, s)-symmetric vectors in $\mathcal{N}(A)$, and

$$\bigcup_{s\in\mathcal{U}} \{P_s x_s^{(1)}, P_s x_s^{(2)}, \cdots, P_s x_s^{(m_s)}\}$$

is a basis for $\mathcal{N}(A)$. $\square$

Now suppose that γ has m orbits $\mathcal{O}_0, \dots, \mathcal{O}_{m-1}$. If $m = 1$, then γ is a k-cycle and $\mathbb{Z}_k = \left\{\gamma^j(0) \,\middle|\, 0 \le j \le k-1\right\}$. In any case, there are unique integers $0 = s_0 < \cdots < s_{m-1}$ such that

$$\mathbb{Z}_k = \bigcup_{\ell=0}^{m-1} \mathcal{O}_\ell, \quad \text{where} \quad \mathcal{O}_\ell = \left\{\gamma^j(s_\ell) \,\middle|\, 0 \le j \le k_\ell - 1\right\}$$

and $k_0 + \cdots + k_{m-1} = k$. Now define

$$\Gamma_\ell = \sum_{j=0}^{k_\ell-1} P_{\gamma^{j+1}(s_\ell)} F_{\gamma^j(s_\ell)} P^*_{\gamma^j(s_\ell)}, \tag{25}$$

$$z_\ell = \sum_{j=0}^{k_\ell-1} P_{\gamma^j(s_\ell)} x_{\gamma^j(s_\ell)}, \quad \text{and} \quad w_\ell = \sum_{j=0}^{k_\ell-1} P_{\gamma^j(s_\ell)} y_{\gamma^j(s_\ell)}. \tag{26}$$

Then (15) and (18) can be written as

$$A = \sum_{\ell=0}^{m-1} \Gamma_\ell, \quad z = \sum_{\ell=0}^{m-1} z_\ell, \quad \text{and} \quad w = \sum_{\ell=0}^{m-1} w_\ell.$$

This, (2), (25), and (26) imply that $Az = w$ if and only if

$$\Gamma_\ell z_\ell = w_\ell, \quad 0 \le \ell \le m-1.$$

However, $\Gamma_\ell z_\ell = w_\ell$ if and only if

$$\sum_{j=0}^{k_\ell-1} P_{\gamma^{j+1}(s_\ell)} F_{\gamma^j(s_\ell)} x_{\gamma^j(s_\ell)} = \sum_{j=0}^{k_\ell-1} P_{\gamma^j(s_\ell)} y_{\gamma^j(s_\ell)} = \sum_{j=0}^{k_\ell-1} P_{\gamma^{j+1}(s_\ell)} y_{\gamma^{j+1}(s_\ell)},$$

which is equivalent to

$$F_{\gamma^j(s_\ell)} x_{\gamma^j(s_\ell)} = y_{\gamma^{j+1}(s_\ell)}, \quad 0 \le j \le k_\ell - 1. \tag{27}$$

This system can be written as

$$F_{s_\ell} x_{s_\ell} = y_{s_\ell} \ \text{ if } k_\ell = 1, \quad \begin{bmatrix} 0 & F_{\gamma(s_\ell)} \\ F_{s_\ell} & 0 \end{bmatrix} \begin{bmatrix} x_{s_\ell} \\ x_{\gamma(s_\ell)} \end{bmatrix} = \begin{bmatrix} y_{s_\ell} \\ y_{\gamma(s_\ell)} \end{bmatrix} \text{ if } k_\ell = 2,$$

or

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & F_{\gamma^{k_\ell-1}(s_\ell)} \\ F_{s_\ell} & 0 & \cdots & 0 & 0 \\ 0 & F_{\gamma(s_\ell)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & F_{\gamma^{k_\ell-2}(s_\ell)} & 0 \end{bmatrix} \begin{bmatrix} x_{s_\ell} \\ x_{\gamma(s_\ell)} \\ x_{\gamma^2(s_\ell)} \\ \vdots \\ x_{\gamma^{k_\ell-1}(s_\ell)} \end{bmatrix} = \begin{bmatrix} y_{s_\ell} \\ y_{\gamma(s_\ell)} \\ y_{\gamma^2(s_\ell)} \\ \vdots \\ y_{\gamma^{k_\ell-1}(s_\ell)} \end{bmatrix}$$

if $k_\ell > 2$. In any case, let us abbreviate this system as $H_\ell\phi_\ell = \psi_\ell$. Then we have proved the following theorem.

Theorem 8 *If $w = \sum_{s=0}^{k-1} P_s y_s$, then the system $Az = w$ has a solution $z = \sum_{s=0}^{k-1} P_s x_s$ if and only if the systems $H_\ell\phi_\ell = \psi_\ell$, $0 \le \ell \le m-1$, all have solutions. Morever, if ϕ_ℓ is a λ-eigenvector of H_ℓ, then $z_\ell = \sum_{j=0}^{k_\ell-1} P_{\gamma^j(s_\ell)} x_{\gamma^j(s_\ell)}$ is a λ-eigenvector of A.*

Theorem 9 *Suppose $k_\ell > 1$. If $\lambda \neq 0$, then ϕ_ℓ is a λ-eigenvector of H_ℓ if and only if $x_{s_\ell} \neq 0$ and*

$$F_{\gamma^{k_\ell-1}(s_\ell)} \cdots F_{\gamma(s_\ell)} F_{s_\ell} x_{s_\ell} = \lambda^{k_\ell} x_{s_\ell}. \tag{28}$$

In this case,

$$x_{\gamma^{j+1}(s_\ell)} = \frac{1}{\lambda} F_{\gamma^j(s_\ell)} x_{\gamma^j(s_\ell)}, \quad 0 \le j \le k_\ell - 2, \tag{29}$$

and $x_{s_\ell}, \dots, x_{\gamma^{k_\ell-1}(s_\ell)}$ are all nonzero.

PROOF. We note from (16) with $d_s = c_s$ that $F_{\gamma^{k_\ell-1}(s_\ell)} \cdots F_{\gamma(s_\ell)} F_{s_\ell} x_{s_\ell} \in \mathbb{C}^{c_{s_\ell} \times c_{s_\ell}}$. (Recall that $\gamma^k(s_\ell) = s_\ell$.) From (27), $H_\ell\phi_\ell = \lambda\phi_\ell$ if and only

$$x_{\gamma^{j+1}(s_\ell)} = \frac{1}{\lambda} F_{\gamma^j(s_\ell)} x_{\gamma^j(s_\ell)} \tag{30}$$

for all j, because of the periodicity of $\gamma^j(s_\ell)$ with respect to j. Hence, if $x_{\gamma^{j_0}(s_\ell)} = 0$ for some j_0, then $x_{\gamma^j(s_\ell)} = 0$ for all j. Therefore, $x_{s_\ell} \neq 0$ if ϕ_ℓ is a λ-eigenvector of H_ℓ. Applying (30) for $0 \le j \le k_\ell - 1$ and noting that $x_{\gamma^{k_\ell}(s_\ell)} = x_{s_\ell}$ yields (28). □

Corollary 2 *If $k_\ell > 1$, $\zeta_\ell = e^{2\pi i/k_\ell}$, and*

$$\phi_\ell^{(0)} = \begin{bmatrix} x_{s_\ell} \\ x_{\gamma(s_\ell)} \\ x_{\gamma^2(s_\ell)} \\ \vdots \\ x_{\gamma^{k_\ell-1}(s_\ell)} \end{bmatrix}$$

is a λ-eigenvector of H_ℓ with $\lambda \neq 0$, then

$$\phi_\ell^{(r)} = \begin{bmatrix} x_{s_\ell} \\ \zeta_\ell^{-r} x_{\gamma(s_\ell)} \\ \zeta_\ell^{-2r} x_{\gamma^2(s_\ell)} \\ \vdots \\ \zeta_\ell^{-(k_\ell-1)r} x_{\gamma^{k_\ell-1}(s_\ell)} \end{bmatrix}$$

is $\lambda\zeta_\ell^r$-eigenvector of H_ℓ, $0 \le r \le k_\ell - 1$.

PROOF. Replacing λ by $\zeta^r\lambda$ in (28) and (29) leaves the former unchanged. This implies the conclusion. □

The results in this section take a particularly simple form if $n = k$, so that $c_s = d_s = 1$, $0 \le s \le k-1$. In this case, let $\{p_1, \dots, p_k\} \subset \mathbb{C}^k$ be an orthonormal set such that $Rp_s = \zeta^s p_s$, $0 \le s \le k-1$. Theorems 8, 9, and Corollary 2 with $P_s = p_s$ and $F_s = p^*_{\gamma(s)} A p_s$ imply the following theorem, which is related to [1, Lemma 4.3].

Theorem 10 *Suppose that $n = k$. If $k_\ell = 1$, then $\lambda = p^*_{s_\ell} A p_{s_\ell}$ is an eigenvalue of H_ℓ with associated eigenvector p_{s_ℓ}. If $k_\ell > 1$, let*

$$\Delta_\ell = \prod_{t=0}^{k_{\ell-1}} p^*_{\gamma^{t+1}(s_\ell)} A p_{\gamma^t(s_\ell)}.$$

If $\Delta_\ell \ne 0$, let $\lambda_\ell = \Delta_\ell^{1/k_\ell}$ and define

$$x_{s_\ell} = 1 \quad \textit{and} \quad x_{\gamma^{j+1}(s_\ell)} = \lambda_\ell^{-j-1} \prod_{t=0}^{j} p^*_{\gamma^{t+1}(s_\ell)} A p_{\gamma^t(s_\ell)}, \quad 0 \le j \le k_\ell - 2.$$

Then $\lambda_\ell \zeta_\ell^r$ is an eigenvalue of A with associated eigenvector

$$z_{\ell r} = \sum_{j=0}^{k_{\ell-1}} \zeta_\ell^{-rj} x_{\gamma^j(s_\ell)} p_{\gamma^j(s_\ell)}, \quad 0 \le r \le k_\ell - 1.$$

Any nonzero eigenvalue of A must be of the form just defined for some $\ell \in \{0, \dots, m-1\}$. A is singular if and only if the set $\mathcal{M} = \{s \mid p_s^* A p_s = 0\}$ is nonempty, in which case $\{p_s \mid s \in \mathcal{M}\}$ is a basis for $\mathcal{N}(A)$.

6 R-symmetric matrices

In this section we consider the special case where $m = n$, $S = R$, $\mu = 0$, $\alpha = 1$, and $\beta = k-1$. Since $R^{k-1} = R^{-1} = R^*$, $\mathcal{A}(R, 1, k-1, 0)$ is the set of matrices $A \in \mathbb{C}^{n\times n}$ such that $RAR^* = A$. We will say that such a matrix is R-symmetric. This is related to a definition in [12].

Our assumptions imply that $\gamma(s) = s$, $0 \le s \le k-1$ (see (14)), so Theorem 4 implies that A is R-symmetric if and only if

$$A = P\left(\bigoplus F_s\right)P^* = \sum_{s=0}^{k-1} P_s F_s P_s^* \tag{31}$$

with

$$F_s = P_s^* A P_s \in \mathbb{C}^{c_s \times c_s}, \quad 0 \le s \le k-1.$$

The next two theorems are immediate consequences of (31).

Theorem 11 *If A is R-symmetric, then λ is an eigenvalue of A if and only if λ is an eigenvalue of one or more of the matrices $F_0, F_1, \dots, F_{k-1}$. Assuming this to be true, let*

$$S_A(\lambda) = \left\{ s \in \mathbb{Z}_k \;\middle|\; \lambda \text{ is an eigenvalue of } F_s \right\}.$$

If $s \in S_A(\lambda)$ and $\{x_s^{(1)}, x_s^{(2)}, \cdots, x_s^{(m_s)}\}$ is a basis for $\mathcal{E}_{A_s}(\lambda)$, then $P_s x_s^{(1)}, P_s x_s^{(2)}, \dots, P_s x_s^{(m_s)}$ are linearly independent (R, s)-symmetric λ-eigenvectors of A. Moreover,

$$\bigcup_{s \in \mathcal{S}_A(\lambda)} \{P_s x_s^{(1)}, P_s x_s^{(2)}, \cdots, P_s x_s^{(m_s)}\}$$

is a basis for $\mathcal{E}_A(\lambda)$. Finally, A is diagonalizable if and only if $F_0, F_1, \dots, F_{k-1}$ are all diagonalizable. In this case, A has c_s linearly independent (R, s)-symmetric eigenvectors, $0 \le s \le k-1$.

Theorem 12 *If A is R-symmetric, then A is normal if and only if F_s is normal, $0 \le s \le k-1$. In this case, if $F_s = \Omega_s D_s \Omega_s^*$ is a spectral representation of A_s, $0 \le s \le k-1$, then*

$$A = \Omega \left(\bigoplus_{s=0}^{k-1} D_s \right) \Omega^*$$

with

$$\Omega = \left[\begin{array}{cccc} P_0\Omega_0 & P_1\Omega_1 & \cdots & P_{k-1}\Omega_{k-1} \end{array} \right]$$

is a spectral representation of A. Hence, A has c_s linearly independent (R, s)-symmetric eigenvectors, $0 \le s \le k-1$.

The next theorem is a generalization of Andrew's theorem [2, Theorem 2]. For other generalizations of Andrew's theorem, see [12, 14, 15].

Theorem 13

(i) *If A is R-symmetric and λ is an eigenvalue of A, then $\mathcal{E}_A(\lambda)$ has a basis in $\mathcal{S}_R$* (*recall* (24)).

(ii) *If A has n linearly independent eigenvectors in $\mathcal{S}_R$, then A is R-symmetric.*

PROOF. (i) Theorem 11.

(ii) Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A with associated linearly independent eigenvectors $z_1, \dots, z_n$ in $\mathcal{S}_R$. It suffices to show that $RAR^* z_j = Az_j$, $1 \le j \le n$. This is true, since if $Az_j = \lambda_j z_j$ and $Rz_j = \zeta^s z_j$, then

$$RAR^* z_j = \zeta^{-s} RAz_j = \zeta^{-s}\lambda_j Rz_j = \zeta^{-s}\zeta^s \lambda_j z_j = Az_j. \quad \square$$

7 Generalized block circulants

Henceforth ρ is a k-cyclic permutation of $\mathbb{Z}_k$ and σ is the permutation of $\mathbb{Z}_k$ such that

$$\rho^{\sigma(s)}(0) = s, \quad 0 \le s \le k-1. \tag{32}$$

For example, if $k = 7$ and $\rho = (0, 5, 6, 2, 3, 4, 1)$, then

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 6 & 3 & 4 & 5 & 1 & 2 \end{pmatrix}.$$

Let

$$\nu(r,s) = \rho^{\sigma(s)+\beta\sigma(r)}(0) = \rho^{\beta\sigma(r)}(s), \quad 0 \le r, s \le k-1. \tag{33}$$

We study matrices of the form

$$A = [\zeta^{\mu_1\sigma(s)+\mu_2\sigma(r)} A_{\nu(r,s)}]_{r,s=0}^{k-1}, \quad \text{where} \quad A_0, \dots, A_{k-1} \in \mathbb{C}^{d_1 \times d_2}. \tag{34}$$

For example, if $\rho = (0, 1, \dots, k-1)$, then $\sigma(s) = s$, $0 \le s \le k-1$, so

$$A = [\zeta^{s\mu_1 + r\mu_2} A_{(s+\beta r)(\mathrm{mod}\, k)}]_{r,s=0}^{k-1}.$$

Hence, if $\mu_1 = \mu_2 = 0$, then A is a block β-anticirculant if $\beta > 0$, or a block $|\beta|$-circulant if $\beta < 0$. (Note that we do not assume here that the blocks are square).

We will need the following lemma.

Lemma 1 *Let*

$$E = \begin{bmatrix} e_{\rho^{-1}(0)} & e_{\rho^{-1}(1)} & \cdots & e_{\rho^{-1}(k-1)} \end{bmatrix}, \tag{35}$$

where $\begin{bmatrix} e_0 & e_1 & \cdots & e_{k-1} \end{bmatrix} = I_k$. *Then* $E = UDU^*$, *where*

$$D = \mathrm{diag}(1, \zeta, \dots, \zeta^{k-1})$$

and

$$U = \begin{bmatrix} u_0 & u_1 & \cdots & u_{k-1} \end{bmatrix} = \frac{1}{\sqrt{k}} \left[\zeta^{s\sigma(r)}\right]_{r,s=0}^{k-1}.$$

PROOF. If q is an arbitrary integer, then

$$\sigma(\rho^q(r)) \equiv \sigma(r) + q \pmod{k}, \quad 0 \le r \le k-1, \tag{36}$$

since (32) implies that

$$\rho^{\sigma(\rho^q(r))}(0) = \rho^q(r) = \rho^q(\rho^{\sigma(r)}(0)) = \rho^{\sigma(r)+q}(0).$$

Therefore,

$$EU = \frac{1}{\sqrt{k}} \left[\zeta^{s\sigma(\rho(r))}\right]_{r,s=0}^{k-1} = \frac{1}{\sqrt{k}} \left[\zeta^{s(\sigma(r)+1)}\right]_{r,s=0}^{k-1} = UD,$$

where (36) with $q = 1$ implies the second equality. Since $UU^* = I_k$, it follows that $E = UDU^*$. $\square$

The following two theorem do not require that $(\beta, k) = 1$.

Theorem 14 *Let*

$$R = E \otimes I_{d_1} \quad \text{and} \quad S = E \otimes I_{d_2}. \tag{37}$$

Let $H = [H_{rs}]_{r,s=0}^{k-1}$ *, where* $H_{rs} \in \mathbb{C}^{d_1 \times d_2}$, $0 \le r, s \le k-1$. *Then*

$$RHS^{\beta} = \zeta^{\mu_2 - \beta\mu_1} H, \tag{38}$$

if and only if

$$H_{rs} = \zeta^{\mu_1\sigma(s) + \mu_2\sigma(r)} A_{\nu(r,s)}, \quad 0 \le r, s \le k-1, \tag{39}$$

where $A_0, \dots, A_{k-1} \in \mathbb{C}^{d_1 \times d_2}$. *In this case,*

$$A_s = \zeta^{-\mu_1\sigma(s)} H_{0s}, \quad 0 \le s \le k-1. \tag{40}$$

PROOF. Let P and Q be as in (4), with

$$P_s = u_s \otimes I_{d_1}, \text{ and } Q_s = u_s \otimes I_{d_2}, \quad 0 \le s \le k-1.$$

Then (1) holds, which implies (5) and (6). From (35) and (37), it is straightforward to verify that

$$RHS^{\beta} = \left[H_{\rho(r),\rho^{-\beta}(s)}\right]_{r,s=0}^{k-1}. \tag{41}$$

If (39) holds, then

$$RHS^{\beta} = \left[\zeta^{\mu_1\sigma(\rho^{-\beta}(s)) + \mu_2\sigma(\rho(r))} A_{\nu(\rho(r),\rho^{-\beta}(s))}\right]_{r,s=0}^{k-1}. \tag{42}$$

However, from (36),

$$\mu_1\sigma(\rho^{-\beta}(s)) + \mu_2\sigma(\rho(r)) \equiv \mu_1\sigma(s) + \mu_2\sigma(r) - \beta\mu_1 + \mu_2, \quad (\text{mod } k). \tag{43}$$

and

$$\sigma(\rho^{-\beta}(s)) + \beta\sigma(\rho(r)) \equiv \sigma(s) + \beta\sigma(r) \quad (\text{mod } k). \tag{44}$$

Now (39), (42), (43), and (44) imply (38).

Conversely, suppose that (38) holds. Then (41) implies that

$$H_{\rho(r),\rho^{-\beta}(s)} = \zeta^{\mu_2 - \beta\mu_1} H_{rs}, \quad 0 \le r, s \le k-1. \tag{45}$$

We will show by induction on r that

$$H_{\rho^r(0),s} = \zeta^{\mu_1\sigma(s) + r\mu_2} A_{\rho^{r\beta}(s)}, \quad 0 \le s \le k-1, \tag{46}$$

with $A_0, \dots, A_s$ as in (40); thus, (46) holds for $r = 0$. Now suppose $r \ge 0$ and (46) holds. Replacing r by $\rho^r(0)$ and s by $\rho^{\beta}(s)$ in (45) yields

$$H_{\rho^{r+1}(0),s} = \zeta^{\mu_2 - \beta\mu_1} H_{\rho^r(0),\rho^{\beta}(s)}.$$

Therefore, from (46) with s replaced by $\rho^{\beta}(s)$,

$$H_{\rho^{r+1}(0),s} = \zeta^{\mu_2 - \beta\mu_1 + \mu_1(\sigma(\rho^{\beta}(s)) + r\mu_2)} A_{\rho^{(r+1)\beta}(s)} = \zeta^{\mu_1\sigma(s) + (r+1)\mu_2} A_{\rho^{(r+1)\beta}(s)},$$

where the last equality is a consequence of (36). This completes the induction, so (46) holds for $0 \le r \le k-1$. Replacing r by $\sigma(r)$ in (46) and recalling (32) and (33) yields (39). $\square$

Theorem 15 *If*

$$A = \left[\zeta^{\mu_1\sigma(s)+\mu_2\sigma(r)} A_{\rho^{\beta\sigma(r)}(s)}\right]_{r,s=0}^{k-1}$$

and

$$B = \left[\zeta^{\nu_1\sigma(s)+\nu_2\sigma(r)} B_{\rho^{\delta\sigma(r)}(s)}\right]_{r,s=0}^{k-1},$$

where $A_0, \dots, A_{k-1}, B_0, \dots, B_{k-1} \in \mathbb{C}^{d\times d}$, *then*

$$AB = \left[\zeta^{\nu_1\sigma(s)+\tau\sigma(r)} C_{\rho^{-\beta\delta\sigma(r)}(s)}\right], \tag{47}$$

where

$$\tau = \mu_2 - \beta\mu_1 - \beta\nu_2$$

and

$$C_s = \sum_{j=0}^{k-1} \zeta^{(\mu_1+\nu_2)\sigma(j)} A_j B_{\rho^{\delta\sigma(j)}(s)}, \quad 0 \le s \le k-1. \tag{48}$$

PROOF. We apply Theorem 14 with $d_1 = d_2 = d$, so that $R = S$ (see (37)). Theorem 14 implies that

$$\text{(i)}\ \ RA = \zeta^{\mu_2-\beta\mu_1} AR^{-\beta} \quad \text{and} \quad \text{(ii)}\ RB = \zeta^{\nu_2-\delta\nu_1} BR^{-\delta}. \tag{49}$$

From (ii) and induction,

$$R^{-\beta}B = R^{k-\beta}B = \zeta^{(k-\beta)(\nu_2-\delta\nu_1)} R^{-(k-\beta)\delta} = \zeta^{-\beta(\nu_2-\delta\nu_1)} BR^{\beta\delta}.$$

From this and (49)(i),

$$RAB = \zeta^{\mu_2-\beta\mu_1} AR^{-\beta}B = \zeta^{\mu_2-\beta\mu_1-\beta(\nu_2-\delta\nu_1)} ABR^{\beta\delta}$$

Now Theorem 14 with β, μ_1, and μ_2 replaced by $k-\beta\delta$, ν_1, and τ implies (47). It is straightforward to verify (48), since (40) with appropriate substitutions implies that $C_s = \zeta^{-\nu_1\sigma(s)}(AB)_{0s}$.

Theorem 15 generalizes [1, Theorem 3.1]; namely, that the product of a g-circulant and an h-circulant is a gh-circulant. However, [1] does not specify the entries in the product, as in (48).

Theorem 16 *Suppose that A is as in* (34) *and* $(\beta, k) = 1$. *Define*

$$\gamma(s) \equiv \mu_2 - \beta(\mu_1 + s) \pmod{k}. \tag{50}$$

Then

$$A = \sum_{s=0}^{k-1} P_{\gamma(s)} F_s Q_s^* \tag{51}$$

with

$$P_s = u_s \otimes I_{d_1}, \quad Q_s = u_s \otimes I_{d_2}, \quad u_s = \frac{1}{\sqrt{k}} \begin{bmatrix} 1 \\ \zeta^{s\sigma(1)} \\ \zeta^{s\sigma(2)} \\ \vdots \\ \zeta^{s\sigma(k-1)} \end{bmatrix}, \tag{52}$$

and

$$F_s = \sum_{m=0}^{k-1} \zeta^{(\mu_1+s)\sigma(m)} A_m, \quad 0 \le s \le k-1, \tag{53}$$

independent of β and μ_2. Conversely, if A is as in (51) *with given $F_0, \dots, F_{k-1} \in \mathbb{C}^{d_1 \times d_2}$, then A is as in* (34) *with*

$$A_m = \frac{1}{k} \sum_{s=0}^{k-1} \zeta^{-(\mu_1+s)\sigma(m)} F_s, \quad 0 \le s \le k-1. \tag{54}$$

PROOF. If A is as (34), then Theorem 14 implies the assumptions of Theorem 1 with $\alpha = 1$ and $\mu = \mu_2 - \beta\mu_1$. If in addition $(\beta, k) = 1$, then Theorem 4 implies (51), where, from (16), (34), and (52),

$$F_s = P^*_{\gamma(s)} A Q_s = \frac{1}{k} \sum_{\ell,m=0}^{k-1} \zeta^{(\mu_2-\gamma(s))\sigma(\ell)+(\mu_1+s)\sigma(m)} A_{\nu(\ell,m)}, \quad 0 \le s \le k-1.$$

However, from (50),

$$\mu_2 - \gamma(s) \equiv \beta(\mu_1 + s) \pmod{k},$$

so

$$(\mu_2 - \gamma(s))\sigma(\ell) + (\mu_1 + s)\sigma(m) \equiv \xi(\ell, m) \pmod{k}.$$

where

$$\xi(\ell, m) = (\mu_1 + s)(\beta\sigma(\ell) + \sigma(m)). \tag{55}$$

Therefore,

$$F_s = \frac{1}{k} \sum_{\ell,m=0}^{k-1} \zeta^{\xi(\ell,m)} A_{\nu(\ell,m)}. \tag{56}$$

We want to rearrange the terms of this double sum to collect the coefficients of $A_0, \dots, A_{k-1}$. Our strategy for accomplishing this is motivated by the congruence

$$\sigma(\rho^{\beta\ell}(m)) + \beta(\sigma(\rho^{-\ell}(0)) \equiv (\sigma(m) + \beta\ell) + \beta(\sigma(0) - \ell) \equiv \sigma(m) \pmod{k},$$

(recall (36) and note that $\sigma(0) = 0$, from (32)) which, from (33) and (55), implies that

$$\nu(\rho^{-\ell}(0), \rho^{\beta\ell}(m)) \equiv m \pmod{k} \tag{57}$$

and

$$\xi(\rho^{-\ell}(0), \rho^{\beta\ell}(m)) \equiv (\mu_1 + s)\sigma(m) \pmod{k}. \tag{58}$$

Replacing ℓ by $\rho^{-\ell}(0)$ in (56) yields

$$F_s = \frac{1}{k} \sum_{\ell=0}^{k-1} \left(\sum_{m=0}^{k-1} \zeta^{\xi(\rho^{-\ell}(0),m)} A_{\nu(\rho^{-\ell}(0),m)} \right).$$

For each ℓ we now replace m by $\rho^{\beta\ell}(m)$ in the sum in parentheses to obtain

$$F_s = \frac{1}{k} \sum_{\ell,m=0}^{k-1} \zeta^{\xi(\rho^{-\ell}(0),\rho^{\beta\ell}(m))} A_{\nu(\rho^{-\ell}(0),\rho^{\beta\ell}(m))}.$$

Hence, (57) and (58) imply (53). Since (53) and (54) are equivalent, the converse also holds. $\square$

Theorem 17 *If A is as in* (34), *then* $(A^\dagger)^* \in \mathcal{A}(R, S, 1, \beta, \mu_2 - \beta\mu_1)$. *If in addition* $(\beta, k) = 1$, *then*

$$A^\dagger = \left[\zeta^{-\mu_1\sigma(r) - \mu_2\sigma(s)} D_{\nu(s,r)} \right]_{r,s=0}^{k-1},$$

where

$$D_m = \frac{1}{k} \sum_{s=0}^{k-1} \zeta^{(\mu_1+s)\sigma(m)} F_s^\dagger, \quad 0 \le m \le k-1, \tag{59}$$

and F_s is as in (53).

PROOF. Theorems 3 and 14 imply the first assertion. Now suppose $(\beta, k) = 1$. Temporarily, denote

$$D = \left[\zeta^{-\mu_1\sigma(r) - \mu_2\sigma(s)} D_{\nu(s,r)} \right]_{r,s=0}^{k-1}.$$

Since

$$D^* = \left[\zeta^{\mu_1\sigma(s) + \mu_2\sigma(r)} D^*_{\nu(r,s)} \right]_{r,s=0}^{k-1},$$

the argument used to obtain (38) shows that $RD^*S^\beta = \zeta^{\mu_2 - \beta\mu_1} D^*$. Hence, Theorem 4 with A replaced by D^* implies that

$$D^* = \sum_{s=0}^{k-1} P_{\gamma(s)} G_s Q_s^* \tag{60}$$

with

$$G_s = P^*_{\gamma(s)} D^*_s Q_s, \quad 0 \le s \le k-1.$$

By the argument used to obtain (53),

$$G_s = \sum_{m=0}^{k-1} \zeta^{(\mu_1+s)\sigma(m)} D^*_m, \quad 0 \le s \le k-1. \tag{61}$$

However, (59) is equivalent to

$$F_s^\dagger = \sum_{m=0}^{k-1} \zeta^{-(\mu_1+s)\sigma(m)} D_m, \quad 0 \le s \le k-1.$$

Comparing this with (61) shows that $G_s = (F_s^\dagger)^*$. This and (60) imply that

$$D = \sum_{s=0}^{k-1} Q_s F_s^\dagger P_{\gamma(s)}^*,$$

so (17) implies that $D = A^\dagger$. $\square$

If $m = n$, $S = R$, and $d_1 = d_2$, the results of Sections 5 and 6 can be applied to analyze the spectral properties of A in (34).

We close with the following theorem, which generalizes the well known formulas for the eigenvalues and eigenvectors of the standard circulant matrix $A = [a_{(s-r)(\mathrm{mod}\,k)}]_{r,s=0}^{k-1}$.

Theorem 18 *If $a_0, \dots, a_{k-1} \in \mathbb{C}$, then the eigenvalues and associated eigenvectors of $A = [a_{\rho^{-\sigma(r)}(s)}]_{r,s=0}^{k-1}$ are*

$$f_s = \sum_{m=0}^{k-1} a_m \zeta^{s\sigma(m)} \quad \textit{and} \quad u_s = \frac{1}{\sqrt{k}} \begin{bmatrix} 1 \\ \zeta^{s\sigma(1)} \\ \zeta^{s\sigma(2)} \\ \vdots \\ \zeta^{s\sigma(k-1)} \end{bmatrix}, \quad 0 \le s \le k-1. \tag{62}$$

Proof. A is of the form (34) with $\mu_1 = \mu_2 = 0$, $\beta = -1$, and $A_s = a_s$, $0 \le s \le k-1$. Hence $\gamma(s) = s$ (see (50)) and $P_s = Q_s = u_s$, from (52) with $d_1 = d_2 = 1$. Hence, from (51), $A = \sum_{s=0}^{k-1} f_s u_s u_s^*$ with f_s as in (62) (see (53)). $\square$

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