Banded symmetric Toeplitz matrices: where linear algebra borrows from difference equations

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A Toeplitz matrix, named after the German mathematician Otto Toeplitz (1881-1940), is of the form $T = [t_{r-s}]_{r,s=0}^{n-1}$. (It's ok, and convenient for Toeplitz matrices, to number rows and columns from 0 to $n-1$.) A symmetric Toeplitz matrix is of the form $T_n = [t_{|r-s|}]_{r,s=0}^{n-1}$. For example,

$$T_5 = \begin{bmatrix} t_0 & t_1 & t_2 & t_3 & t_4 \\ t_1 & t_0 & t_1 & t_2 & t_3 \\ t_2 & t_1 & t_0 & t_1 & t_2 \\ t_3 & t_2 & t_1 & t_0 & t_1 \\ t_4 & t_3 & t_2 & t_1 & t_0 \end{bmatrix}$$

is a 5×5 symmetric Toeplitz matrix. We will assume that $t_0, t_1, \dots, t_{k-1}$ are all real numbers. From your linear algebra course you know that a symmetric matrix with real entries has real eigenvalues and is always diagonalizable; that is, T_n has real eigenvalues and n linearly independent eigenvectors.

A Toeplitz matrix is said to be banded if there is an integer $d < n - 1$ such that $t_\ell = 0$ if $\ell > d$. In this case, we say that T has bandwidth d. For example,

$$T_5 = \begin{bmatrix} t_0 & t_1 & t_2 & 0 & 0 \\ t_1 & t_0 & t_1 & t_2 & 0 \\ t_2 & t_1 & t_0 & t_1 & t_2 \\ 0 & t_2 & t_1 & t_0 & t_1 \\ 0 & 0 & t_2 & t_1 & t_0 \end{bmatrix}$$

is a 5×5 banded symmetric Toeplitz matrix with bandwidth 2.

The eigenvalue problem for very large (n can be in the thousands!) symmetric banded Toeplitz matrices pops up in many statistical problems. In your linear algebra course you learned to solve the eigenvalue problem for a matrix A by factoring its characteristic polynomial

$$p(\lambda) = \det(A - \lambda I).$$

Sorry, that's impossible for big matrices. In general there is no computationally useful way to obtain the characteristic polynomial of a large symmetric matrix (or any other large matrix). All methods for finding a single eigenvalue of an arbitrary $n \times n$ symmetric matrix carry a computational cost (it's called complexity) proportional to n^3. So, if you double the size of the matrix you make the problem of obtaining a single eigenvalue eight times more difficult. However, the situation is different for banded symmetric Toeplitz matrices.

Let's start with the simplest case: $d = 1$.

$$T_n = \begin{bmatrix} t_0 & t_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ t_1 & t_0 & t_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & t_1 & t_0 & t_1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & t_1 & t_0 & t_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & t_1 & t_0 \end{bmatrix}_{n\times n}$$

with $t_1 \neq 0$. This is a symmetric <u>tridiagonal</u> Toeplitz matrix. A vector

$$x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

is a λ-eigenvector of T_n if and only if

$$\begin{aligned} t_0x_0 + t_1x_1 &= \lambda x_0 \\ t_1x_{j-1} + t_1x_0 + t_1x_{j+1} &= \lambda x_j, \quad 1 \leq j \leq n-1, \\ t_1x_{n-1} + t_0x_n &= \lambda x_n \end{aligned}$$

which we can rewrite as

$$t_1x_{j-1} + (t_0 - \lambda)x_j + t_1x_{j+1} = 0, \quad 0 \leq j \leq n,$$

(a <u>homogeneous difference equation</u>) if we define $x_0 = x_{n+1} = 0$ (<u>boundary conditions</u>).

The characteristic polynomial of the difference equation

$$t_1 x_{j-1} + (t_0 - \lambda)x_j + t_1 x_{j+1} = 0 \qquad \text{(DE)}$$

is

$$p(z;\lambda) = t_1 + (t_0 - \lambda)z + t_1 z^2 = t_1(z - z_1(\lambda))(z - z_2(\lambda));$$

thus, $p(z_1(\lambda)) = p(z_2(\lambda)) = 0$. (We don't know $z_1(\lambda)$ and $z_2(\lambda)$ yet; be patient.) If we let

$$x_j = c_1 z_1^j(\lambda) + c_2 z_2^j(\lambda)$$

where c_1 and c_2 are arbitrary constants, then the left side of (DE) equals

$$c_1 z_1^{j-1} p(z_1(\lambda)) + c_2 z_2^{j-1} p(z_2(\lambda)) = 0$$

for any choice of c_1 and c_2. Now let's work on the boundary conditions. Since $x_0 = 0$ if and only if $c_2 = -c_1$,

$$x_j = c(z_1^j(\lambda) - z_2^j(\lambda)).$$

Now $x_{n+1} = 0$ if and only if $(z_1(\lambda)/z_2(\lambda))^{n+1} = 1$, which is true if and only if

$$z_1(\lambda) = \gamma_q \exp\left(\frac{q\pi i}{n+1}\right) \quad \text{and} \quad z_2(\lambda) = \gamma_q \exp\left(\frac{-q\pi i}{n+1}\right),$$

where $\exp(i\theta) = e^{i\theta} = cos\theta + i \sin\theta$, $q = 1, \ldots, n$ and γ_q is to be determined. (Letting $q = 0$ does not produce an eigenvector because if $z_1(\lambda) = z_2(\lambda)$) then $x_j = 0$ for all j).

Taking note that are q possibilities, the eigenvectors have the form

$$x_q = \begin{bmatrix} x_{0q} \\ x_{1q} \\ \vdots \\ x_{n-1,q} \end{bmatrix}$$

where

$$x_{jq} = c(z_1^j(\lambda_q) - z_2^j(\lambda_q)),$$

$$z_{1q}(\lambda) = \gamma_q \exp\left(\frac{q\pi i}{n+1}\right), \quad \text{and} \quad z_{2q}(\lambda) = \gamma_q \exp\left(\frac{-q\pi i}{n+1}\right),$$

so

$$x_{jq} = c\left(\exp\frac{jq\pi i}{n+1} - \exp\frac{jq\pi i}{n+1}\right) = 2ci\sin\frac{jq\pi}{n+1}.$$

Since c is arbitrary, it makes sense to let $c = 1/2\gamma_q i$. (Don't worry that maybe $\gamma_q = 0$; we'll see that it isn't.) Then

$$x_{jq} = \sin\frac{jq\pi}{n+1}, \quad 0 \le j \le n-1.$$

ALL SYMMETRIC TRIDIAGONAL TOEPLITZ MATRICES HAVE THE SAME EIGENVECTORS!

Now let's find λ_q, the eigenvalue associated with q.

$$t_1 + (t_0 - \lambda_q) + t_1 z^2 = t_1(z - z_1(\lambda))(z_2 - z_2(\lambda))$$

which equals

$$t_1 \left(z^2 - (z_1(\lambda) + z_2(\lambda))z + z_1(\lambda) z_2(\lambda) \right)$$

Since

$$z_1(\lambda) = \gamma_q \exp\left(\frac{q\pi i}{n+1}\right) \quad \text{and} \quad z_2(\lambda) = \gamma_q \exp\left(\frac{-q\pi i}{n+1}\right),$$

$$t_1 + (t_0 - \lambda_q) z + t_1 z^2 = t_1 \left(z^2 - 2\gamma_q z \cos\left(\frac{q\pi}{n+1}\right) + \gamma_q^2 \right).$$

Equating coefficients on the two sides yields $\gamma_q = 1$ and

$$\lambda_q = t_0 + 2t_1 \cos\left(\frac{q\pi}{n+1}\right), \quad 1 \le q \le n.$$

Now suppose $d > 1$. To see where we're going, a nonzero vector

$$x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

is a λ-eigenvector of

$$T_5 = \begin{bmatrix} t_0 & t_1 & t_2 & 0 & 0 \\ t_1 & t_0 & t_1 & t_2 & 0 \\ t_2 & t_1 & t_0 & t_1 & t_2 \\ 0 & t_2 & t_1 & t_0 & t_1 \\ 0 & 0 & t_2 & t_1 & t_0 \end{bmatrix}$$

if and only if

$$\begin{bmatrix} t_0 & t_1 & t_2 & 0 & 0 \\ t_1 & t_0 & t_1 & t_2 & 0 \\ t_2 & t_1 & t_0 & t_1 & t_2 \\ 0 & t_2 & t_1 & t_0 & t_1 \\ 0 & 0 & t_2 & t_1 & t_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \lambda \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

or, equivalently,

$$\begin{aligned} t_0x_0 + t_1x_1 + t_2x_2 \quad &= \quad \lambda x_0 \\ t_1x_0 + t_0x_1 + t_1x_2 + t_2x_3 \quad &= \quad \lambda x_1 \\ t_2x_0 + t_1x_1 + t_2x_2 + t_3x_3 + t_4x_4 \quad &= \quad \lambda x_2 \\ t_2x_1 + t_1x_2 + t_0x_3 + t_1x_4 \quad &= \quad \lambda x_3 \\ t_2x_2 + t_1x_3 + t_0x_4 \quad &= \quad \lambda x_4, \end{aligned}$$

(repeated for clarity)

$$\begin{aligned}
t_0x_0 + t_1x_1 + t_2x_2 &= \lambda x_0 \\
t_1x_0 + t_0x_1 + t_1x_2 + t_2x_3 &= \lambda x_1 \\
t_2x_0 + t_1x_1 + t_2x_2 + t_3x_3 + t_4x_4 &= \lambda x_2 \\
t_2x_1 + t_1x_2 + t_0x_3 + t_1x_4 &= \lambda x_3 \\
t_2x_2 + t_1x_3 + t_0x_4 &= \lambda x_4,
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
t_2x_{-2} + t_1x_{-1} + t_0x_0 + t_1x_1 + t_2x_2 &= \lambda x_0 \\
t_2x_{-1} + t_1x_0 + t_0x_1 + t_1x_2 + t_2x_3 &= \lambda x_1 \\
t_2x_0 + t_1x_1 + t_0x_2 + t_1x_3 + t_2x_4 &= \lambda x_2 \\
t_2x_1 + t_1x_2 + t_0x_3 + t_1x_4 + t_2x_5 &= \lambda x_3 \\
t_2x_2 + t_1x_3 + t_0x_4 + t_1x_5 + t_2x_6 &= \lambda x_4
\end{aligned}$$

if we impose the boundary conditions

$$x_{-2} = x_{-1} = x_5 = x_6 = 0.$$

Better yet,

$$\sum_{\ell=-2}^{2} t_{|\ell|} x_{\ell+r} = \lambda x_r, \quad 0 \le r \le 4.$$

(repeated for clarity)

$$\sum_{\ell=-2}^{2} t_{|\ell|} x_{\ell+r} = \lambda x_r, \quad 0 \le r \le 4,$$

with boundary conditions

$$x_{-2} = x_{-1} = x_5 = x_6 = 0.$$

For the general case where $T_n = [t_{|r-s|}]_{r,s=0}^{n-1}$ with $t_\ell = 0$ if $\ell > d$, the eigenvalue problem can be written as

$$\sum_{\ell=-d}^{d} t_{|\ell|} x_{\ell+r} = \lambda x_r, \quad 0 \le r \le n-1, \qquad \text{(DE)}$$

subject to

$$x_r = 0, \quad -d \le r \le -1, \quad n \le r \le n+d-1. \qquad \text{(BC)}$$

Eqn. (DE) is a difference equation and the conditions in (BC) are called boundary conditions. Obviously, (DE) and (BC) both hold for any λ if $x_r = 0$ for $-d \le r \le n+d-1$. However, that's not interesting, since an eigenvector must be nonzero. Finding the values of λ for which (DE) has nonzero solutions that satisfy (BC) is a boundary value problem.

The characteristic polynomial of the difference equation

$$\sum_{\ell=-d}^{d} t_{|\ell|} x_{\ell+r} = \lambda x_r, \quad 0 \le r \le n-1, \qquad \text{(DE)}$$

is

$$P(z,\lambda) = \sum_{\ell=-d}^{d} t_{|\ell|} z^{\ell} - \lambda.$$

The zeros of $P(z,\lambda)$ are continuous functions of λ and, since $P(z,\lambda) = P(1/z,\lambda)$, they occur in reciprocal pairs

$$(z_1(\lambda), 1/z_1(\lambda)), \ldots, (z_d(\lambda), 1/z_d(\lambda)).$$

It can be shown (don't you hate that?) that these zeros are distinct except for at most finitely many "bad values" of λ. We'll assume that none of these bad values are actually eigenvalues of T_n. (This is a pretty safe bet.) Then (DE) holds if

$$x_r = \sum_{s=1}^{d} \left(a_s z_s^r(\lambda) + b_s z_s^{-r}(\lambda) \right), \quad -d \le r \le n+d-1,$$

where $a_1, \ldots, a_d$ and $b_1, \ldots, b_d$ are arbitrary constants.

PROOF. Recall that $P(z,\lambda) = \sum_{\ell=-d}^{d} t_{|\ell|} z^{\ell} - \lambda$. If

$$x_r = \sum_{s=1}^{d} \left(a_s z_s^r(\lambda) + b_s z_s^{-r}(\lambda) \right), \; -d \le r \le n+d-1, \text{ then}$$

$$\begin{aligned}
\sum_{\ell=-d}^{d} t_{|\ell|} x_{\ell+r} - \lambda x_r &= \sum_{\ell=-d}^{d} t_{|\ell|} \sum_{s=1}^{d} \left(a_s z_s^{\ell+r} + b_s z_s^{-\ell-r} \right) \\
&\quad -\lambda \sum_{s=1}^{d} \left(a_s z_s^r(\lambda) + b_s z_s^{-r}(\lambda) \right) \\
&= \sum_{s=1}^{d} a_s z_s^r(\lambda) \left(\sum_{\ell=-d}^{d} t_{|\ell|} z_s^{\ell}(\lambda) - \lambda \right) \\
&+ \sum_{s=1}^{d} b_s z_s^{-r}(\lambda) \left(\sum_{\ell=-d}^{d} t_{|\ell|} z_s^{-\ell}(\lambda) - \lambda \right)
\end{aligned}$$

$$= \sum_{s=1}^{d} \left(a_s z_s^r(\lambda) P(z_s(\lambda), \lambda) + b_s z_s^{-r}(\lambda) P(1/z_s(\lambda), \lambda) \right) = 0$$

(look at the top of this page) for all r. Note that $a_1, \ldots, a_d$ and $b_1, \ldots, b_d$ are completely arbitrary up to this point.

Now we must choose them to satisfy the boundary conditions

$$x_r = 0, \quad -d \le r \le -1, \quad n \le r \le n + d - 1; \quad \text{(BC)}$$

that is,

$$\sum_{s=1}^{d} \left(a_s z_s^r(\lambda) + b_s z_s^{-r}(\lambda) \right) = 0,$$

for $-d \le r \le -1$ and $n \le r \le n + d - 1$. For example, if $d = 2$, we must have

$$\begin{bmatrix} z_1^{-1}(\lambda) & z_2^{-1}(\lambda) & z_1(\lambda) & z_2(\lambda) \\ z_1^{-2}(\lambda) & z_2^{-2}(\lambda) & z_1^2(\lambda) & z_2^2(\lambda) \\ z_1^{n}(\lambda) & z_2^{n}(\lambda) & z_1^{-n}(\lambda) & z_2^{-n}(\lambda) \\ z_1^{n+1}(\lambda) & z_2^{n+1}(\lambda) & z_1^{-n-1}(\lambda) & z_2^{-n-1}(\lambda) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

For clarity,

$$\begin{bmatrix} z_1^{-1}(\lambda) & z_2^{-1}(\lambda) & z_1(\lambda) & z_2(\lambda) \\ z_1^{-2}(\lambda) & z_2^{-2}(\lambda) & z_1^{2}(\lambda) & z_2^{2}(\lambda) \\ z_1^{n}(\lambda) & z_2^{n}(\lambda) & z_1^{-n}(\lambda) & z_2^{-n}(\lambda) \\ z_1^{n+1}(\lambda) & z_2^{n+1}(\lambda) & z_1^{-n-1}(\lambda) & z_2^{-n-1}(\lambda) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = 0.$$

Let

$$P(\lambda) = \begin{bmatrix} z_1^{-1}\lambda & z_2^{-1}(\lambda) \\ z_1^{-2}\lambda & z_2^{-2}(\lambda) \end{bmatrix}, \quad Q(\lambda) = \begin{bmatrix} z_1\lambda & z_2(\lambda) \\ z_1^{2}\lambda & z_2^{2}(\lambda) \end{bmatrix}$$

$$R_n(\lambda) = \begin{bmatrix} z_1^{n}\lambda & z_2^{n}(\lambda) \\ z_1^{n+1}\lambda & z_2^{n+1}(\lambda) \end{bmatrix},$$

$$S_n(\lambda) = \begin{bmatrix} z_1^{-n}\lambda & z_2^{-n}(\lambda) \\ z_2^{-n-1}\lambda & z_2^{-n-1}(\lambda) \end{bmatrix},$$

$$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Then the boundary conditions are satisfied if and only if

$$\begin{bmatrix} P(\lambda) & Q(\lambda) \\ R_n(\lambda) & S_n(\lambda) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0.$$

In general, let

$$P(\lambda) = [z_s^{-r}(\lambda)]_{r,s=1}^{d}, \quad Q(\lambda) = [z_s^{r}(\lambda)],$$

$$R_n(\lambda) = [z_s^{n+r-1}(\lambda)], \quad S_n(\lambda) = [z_s^{-n+r-1}(\lambda)]$$

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_d \end{bmatrix}.$$

Then the boundary conditions are satisfied if and only if

$$\begin{bmatrix} P(\lambda) & Q(\lambda) \\ R_n(\lambda) & S_n(\lambda) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0. \qquad \text{(S)}$$

Let

$$D_n(\lambda) = \begin{vmatrix} P(\lambda) & Q(\lambda) \\ R_n(\lambda) & S_n(\lambda) \end{vmatrix} \quad \text{(determinant)}.$$

An eigenvector of T_n must be a nonzero vector. Since (S) has only the trivial solution $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ if $D_n(\lambda) \neq 0$, it follows that λ is an eigenvalue of T_n if and only if $D_n(\lambda) = 0$. For ways to find the zeros of $D_n(\lambda)$, see my papers RP-44, 61, 63, and 78. Since $D_n(\lambda)$ doesn't become more complicated as n increases, the difficulty of finding individual eigenvalues of T_n is independent of n.

We can take this a little further. The eigenvectors of a symmetric Toeplitz matrix have a special property that I haven't mentioned. To identify this property, let J_n be the "flip matrix," which has 1's on its secondary diagonal and 0's elsewhere. For example,

$$J_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that

$$\begin{aligned} J_5^2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I_5. \end{aligned}$$

In general, $J_n^2 = I_n$; that is, J_n is its own inverse.

Multiplying a vector by J_n reverses ("flips") the components of the vector. For example, if

$$x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

then

$$J_5 x = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \\ x_0 \end{bmatrix}.$$

In general, if

$$x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{bmatrix} \quad \text{then} \quad J_n x = \begin{bmatrix} x_{n-1} \\ x_{n-2} \\ \vdots \\ x_1 \\ x_0 \end{bmatrix}$$

We say that a vector x is symmetric if $J_n x = x$, or skew-symmetric if $J_n x = -x$. Adding symmetric vectors produces a symmetric vector, and multiplying a symmetric vector by a real number produces a symmetric vector; hence, the symmetric vectors in $\mathbb{R}^n$ form a subspace of $\mathbb{R}^n$. Similarly, the skew-symmetric vectors form a subspace of $\mathbb{R}^n$. If $n = 2m$ then each of these subspaces has dimension m. If $n = 2m + 1$ then the subspace of symmetric vectors has dimension $m + 1$ and the subspace of skew symmetric vectors has dimension m. The zero vector is the only vector that is both symmetric and skew-symmetric.

For example, if $n = 4$ then

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the subspace of symmetric vectors, while

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

form a basis for the subspace of skew-symmetric vectors.

If $n = 5$ then

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

form a basis for the subspace of symmetric vectors, while

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

form a basis for the subspace of skew-symmetric vectors. In general, if $n = 2m$ then the subspace of symmetric vectors and the subspace of skew-symmetric vectors are both m-dimensional. If $n = 2m + 1$ then the subspace of symmetric vectors is $(m + 1)$-dimensional and the subspace of skew-symmetric vectors is m-dimensional.

Multiplying a matrix on the left by J_n reverses the rows of the matrix, so

$$\begin{aligned} J_5T_5 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t_0 & t_1 & t_2 & t_3 & t_4 \\ t_1 & t_0 & t_1 & t_2 & t_3 \\ t_2 & t_1 & t_0 & t_1 & t_2 \\ t_3 & t_2 & t_1 & t_0 & t_1 \\ t_4 & t_3 & t_2 & t_1 & t_0 \end{bmatrix} \\ &= \begin{bmatrix} t_4 & t_3 & t_2 & t_1 & t_0 \\ t_3 & t_2 & t_1 & t_0 & t_1 \\ t_2 & t_1 & t_0 & t_1 & t_2 \\ t_1 & t_0 & t_1 & t_2 & t_3 \\ t_0 & t_1 & t_2 & t_3 & t_4 \end{bmatrix} \end{aligned}$$

Multiplying a matrix on the right by J_n reverses the columns of the matrix, so

$$\begin{aligned} J_5 T_5 J_5 &= (J_5 T_5) J_5 \\ &= \begin{bmatrix} t_4 & t_3 & t_2 & t_1 & t_0 \\ t_3 & t_2 & t_1 & t_0 & t_1 \\ t_2 & t_1 & t_0 & t_1 & t_2 \\ t_1 & t_0 & t_1 & t_2 & t_3 \\ t_0 & t_1 & t_2 & t_3 & t_4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} t_0 & t_1 & t_2 & t_3 & t_4 \\ t_1 & t_0 & t_1 & t_2 & t_3 \\ t_2 & t_1 & t_0 & t_1 & t_2 \\ t_3 & t_2 & t_1 & t_0 & t_1 \\ t_4 & t_3 & t_2 & t_1 & t_0 \end{bmatrix} = T_5. \end{aligned}$$

In general, $J_n T_n J_n = T_n$. What does this tell us about the eigenvectors of T_n?

First, suppose λ is an eigenvalue of T_n with multiplicity one, so the associated eigenspace is one-dimensional, and suppose that $T_n x = \lambda x$ with $x \neq 0$. Then $J_n T_n x = \lambda J_n x$. Since $J_n J_n = I_n$, it follows that $(J_n T_n J_n)(J_n x) = \lambda J_n x$. Therefore $T_n(J_n x) = \lambda(J_n x)$, because $J_n T_n J_n = T_n$. Since the λ-eigenspace of T_n is one-dimensonal, it follows that $J_n x = cx$ for some constant c. Therefore, since $\|J_n x\| = \|x\|$ (that is, x and $J_n x$ have the same length), it follows that $c = \pm 1$; that is, x is either symmetric or skew-symmetric. The situation is more complicated if λ is a repeated eigenvalue of T_n with multiplicity k. However, it can be shown that if $k = 2\ell$ then the λ-eigenspace of T_n has a basis consisting of ℓ symmetric and ℓ skew-symmetric vectors, while if $k = 2\ell + 1$ then the λ-eigenspace has a basis consistng of either ℓ symmetric and $\ell + 1$ skew-symmetric eigenvectors or $\ell + 1$ skew-symmetric and ℓ-symmetric eigenvectors. In any case, if $n = 2k$ then T_n has k symmetric linearly independent eigenvectors and k linearly independent skew-symmetric eigenvectors, while if $n = 2k + 1$ then T_n has $k + 1$ linearly independent symmetric eigenvectors and k linearly independent skew-symmetric vectors.

Recall that the components of the eigenvectors of T_n satisfy

$$\sum_{\ell=-d}^{d} t_{|\ell|} x_{\ell+r} = \lambda x_r, \quad 0 \le r \le n-1, \qquad \text{(DE)}$$

subject to

$$x_r = 0, \quad -d \le r \le -1, \quad n \le r \le n+d-1, \quad \text{(BC)}$$

and are therefore of the form

$$x_r = \sum_{s=1}^{d} \left(a_s z_s^r(\lambda) + b_s z_s^{-r}(\lambda) \right), \quad -d \le r \le n+d-1. \qquad \text{(A)}$$

However, we now know that we can assume at the outset that the eigenvectors of T_n are either symmetric, which means that $x_{n-r+1} = x_r$, or skew-symmetric, which means that $x_{n-r+1} = -x_r$. So let's build this into (A) at the start!

For clarity

$$x_r = \sum_{s=1}^{d} \left(a_s z_s^r(\lambda) + b_s z_s^{-r}(\lambda)\right), \quad -d \le r \le n+d-1.$$

To get a symmetric eigenvector, let

$$b_s = a_s z_s^{n+1}(\lambda) \quad \text{so} \quad x_r = \sum_{s=1}^{d} a_s \left(z_s^r(\lambda) + z_s^{n-r+1}(\lambda)\right).$$

Since $n-(n-r+1)-1 = r$, $x_{n-r+1} = x_r$. As for the boundary conditions

$$x_r = 0, \quad -d \le r \le -1, \quad n \le r \le n+d-1, \quad \text{(BC)}$$

it is enough to require that $x_r = 0$ for $-d \le r \le -1$, since this and the equality $x_{n-r+1} = x_r$ implies that $x_r = 0$ for $n \le r \le n+d-1$.

Therefore, λ is an eigenvalue with an associated symmetric eigenvector if and only if

$$\det\left(\left[z_s^{-r}(\lambda) + z_s^{n+r+1}(\lambda)\right]_{r,s=1}^{d}\right) = 0.$$

For clarity

$$x_r = \sum_{s=1}^{d} \left(a_s z_s^r(\lambda) + b_s z_s^{-r}(\lambda) \right), \quad -d \leq r \leq n + d - 1.$$

To get a skew-symmetric eigenvector, let

$$b_s = -a_s z_s^{n+1}(\lambda) \quad \text{so} \quad x_r = \sum_{s=1}^{d} a_s \left(z_s^r(\lambda) - z_s^{n-r+1}(\lambda) \right).$$

Therefore, λ is an eigenvalue with an associated skew-symmetric eigenvector if and only if

$$\det \left(\left[z_s^{-r}(\lambda) - z_s^{n+r+1}(\lambda) \right]_{r,s=1}^{d} \right) = 0.$$