Linear differential systems with coefficient matrices that commute with a spectrally separated matrix function in

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Asymptotic preconditioning of linear homogeneous systems of differential equations, Linear Algebra Appl. 434 (2011), 1631-1637.

R is a real or complex $n \times n$ matrix-valued function on an interval $\mathscr{I}$ that can be written as

$$R = P\mathbf{D}P^{-1}$$

where $n_1 + \cdots + n_k = n$,

$$P = \left[\begin{array}{cccc} P_1 & P_2 & \cdots & P_k \end{array}\right],\ P_\ell \in \mathbb{C}_1^{n \times n_\ell}(\mathscr{I}),\ 1 \le \ell \le k,$$

$$\mathbf{D} = \mathrm{diag}(\mu_1 I_{n_1}, \mu_2 I_{n_2}, \ldots, \mu_k I_{n_k})$$

and $\mu_1(t)$, $\ldots$, $\mu_k(t)$ are distinct for every $t \in \mathscr{I}$. Thus, R is diagonalizable and has k distinct eigenvalues for all $t \in \mathscr{I}$, with constant multiplicities $n_1, \ldots, n_k$.

Our objective: To consider differential systems $x' = A(t)x$ where $A \in \mathbb{C}_0^{n \times n}(\mathscr{I})$ and $RA = AR$ on $\mathscr{I}$.

Later we'll add another condition on R. First, what are the consequences of the commutativity?

We can write

$$P^{-1} = \begin{bmatrix} \widehat{P}_1 \\ \widehat{P}_2 \\ \vdots \\ \widehat{P}_k \end{bmatrix} \quad \textit{where} \quad \widehat{P}_r P_s = \delta_{rs} I_{n_r}, \quad 1 \le r, s \le k.$$

Theorem 1 $RA = AR$ *on* $\mathcal{J}$ *if and only if*

$$A = P\,\mathrm{diag}(F_1, \ldots, F_k) P^{-1} = \sum_{\ell=1}^{k} P_\ell F_\ell \widehat{P}_\ell$$

with

$$F_\ell = \widehat{P}_\ell A P_\ell, \quad 1 \le \ell \le k.$$

PROOF. *We can always write*

$$A = PCP^{-1} = P \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1k} \\ C_{21} & C_{22} & \cdots & C_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ C_{k1} & C_{k2} & \cdots & C_{kk} \end{bmatrix} P^{-1}$$

with $C_{rs} \in \mathbb{C}^{n_r \times n_s}(\mathcal{J})$*. Just let* $C = P^{-1}AP$ *and partition* C *into blocks. (This is purely conceptual; we don't really have to do it.) Then*

$$\begin{aligned} RA &= (P\mathbf{D}P^{-1})(PCP^{-1}) = P\mathbf{D}CP^{-1} \\ &= P \begin{bmatrix} \mu_1 C_{11} & \mu_1 C_{12} & \cdots & \mu_1 C_{1k} \\ \mu_2 C_{21} & \mu_2 C_{22} & \cdots & \mu_2 C_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_k C_{k1} & \mu_k C_{k2} & \cdots & \mu_k C_{kk} \end{bmatrix} P^{-1} \end{aligned}$$

and

$$\begin{aligned} AR &= (PCP^{-1})(P\mathbf{D}P^{-1}) = PC\mathbf{D}P^{-1} \\ &= P \begin{bmatrix} \mu_1 C_{11} & \mu_2 C_{12} & \cdots & \mu_k C_{1k} \\ \mu_1 C_{21} & \mu_2 C_{22} & \cdots & \mu_k C_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_1 C_{k1} & \mu_2 C_{k2} & \cdots & \mu_k C_{kk} \end{bmatrix} P^{-1}. \end{aligned}$$

Hence $RA = AR$ if and only if

$$\mu_r C_{rs} = \mu_s C_{rs}, \quad 1 \le r, s \le k. \tag{1}$$

Since $\mu_r \neq \mu_s$ if $r \neq s$, (1) is equivalent to $C_{rs} = 0$ if $r \neq s$, $1 \le r, s \le k$. For better notation, write

$$C_{\ell\ell} = F_\ell \in \mathbb{C}^{n_\ell \times n_\ell} \quad \textit{and} \quad \mathbf{F} = \operatorname{diag}(F_1, F_2, \dots, F_k).$$

We have now shown that $RA = AR$ on $\mathcal{J}$ if and only if $P^{-1}AP = \mathbf{F}$ or, equivalently, $A = P\mathbf{F}P^{-1}$; i.e., R and A have the same block diagonal form with respect to P (for all $t \in \mathcal{J}$).

Now we want a formula for $F_0, \dots, F_k$. Since $A = PFP^{-1}$, $AP = PF$; i,e,

$$\left[\begin{array}{cccc} AP_1 & AP_2 & \cdots & AP_k \end{array}\right] = \left[\begin{array}{cccc} P_1F_1 & P_2F_2 & \cdots & P_kF_k \end{array}\right],$$

so $AP_\ell = P_\ell F_\ell$, $1 \le \ell \le k$. Since $\widehat{P}_\ell P_\ell = I_{n_\ell}$, it follows that $F_\ell = \widehat{P}_\ell A P_\ell$, $1 \le \ell \le k$. □

Example: (Variable block circulant system) If P is constant we can use this result to simplify the solution of $x' = A(t)x$. For example, consider the block circulant matrix function

$$A = \left[\begin{array}{cccccc} A_0 & A_1 & A_2 & \cdots & A_{k-2} & A_{k-1} \\ A_{k-1} & A_0 & A_1 & \cdots & A_{k-3} & A_{k-2} \\ A_{k-2} & A_{k-1} & A_0 & \cdots & A_{k-4} & A_{k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A_1 & A_2 & A_3 & \cdots & A_{k-1} & A_0 \end{array}\right] \in \mathbb{C}_0^{kd\times kd}(\mathcal{J})$$

with $A_0, A_1, \dots, A_{k-1} \in \mathbb{C}_0^{d\times d}(\mathcal{J})$. Then $RA = AR$ if

$$R = \left[\begin{array}{cccccc} 0 & I_d & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_d & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_d \\ I_d & 0 & 0 & \cdots & 0 & 0 \end{array}\right]$$

(replace A_1 by I_d, all other A_ℓ's by $0_{d\times d}$).

Denote $\zeta = e^{2\pi i/k}$. *Then*

$$R = P\,\mathrm{diag}(I_d, \zeta I_d, \zeta^2 I_d, \ldots, \zeta^{k-1} I_d) P^{-1}$$

where $P = \begin{bmatrix} P_1 & P_2 & \cdots & P_k \end{bmatrix}$ *with*

$$P_\ell = \frac{1}{\sqrt{k}} \begin{bmatrix} I_d \\ \zeta^{\ell-1} I_d \\ \zeta^{2(\ell-1)} I_d \\ \vdots \\ \zeta^{(k-1)(\ell-1)} I_d \end{bmatrix}, \quad 1 \le \ell \le k.$$

Therefore Theorem 1 implies that

$$A = P\,\mathrm{diag}(F_1, \ldots, F_k) P^{-1} = \sum_{\ell=1}^{k} P_\ell F_\ell \widehat{P}_\ell$$

with

$$F_\ell = \widehat{P}_\ell A P_\ell = \frac{1}{k} \sum_{m=0}^{k-1} \zeta^{(\ell-1)m} A_m, \quad 1 \le \ell \le k,$$

(after some computation). Therefore, $x' = A(t)x$ *if and only if*

$$x = \sum_{\ell=1}^{k} P_\ell y_\ell \quad \textit{where} \quad y_\ell' = F_\ell(t) y, \quad 1 \le \ell \le k.$$

Therefore, the only way to understand $x' = A(t)x$ (whatever we mean by this), is to understand the component systems $y'_\ell = F_\ell(t)y_\ell$, $1 \le \ell \le k$. For example, if $\mathcal{I} = [a, \infty)$ a standard theorem of Bôcher says that if

$$\int^\infty \|A(t)\|\,dt < \infty$$

then every nonzero solution of $x' = A(t)x$ approaches a nonzero limit as $t \to \infty$, or, equivalently, if $\xi \in \mathbb{C}^{kd}$ then $x' = A(t)x$ has a unique solution x such that

$$\lim_{t\to\infty} x(t) = \xi.$$

This theorem is not directly applicable to $x' = A(t)x$ if

$$\int^\infty \|F_\ell(t)\|\,dt = \infty$$

for some $\ell \in \{1, \ldots, k\}$, which implies that

$$\int^\infty \|A(t)\|\,dt = \infty.$$

However, if

$$\mathcal{S} = \left\{\ell \;\Big|\; \int^\infty \|F_\ell(t)\|\,dt < \infty\right\} \ne \emptyset$$

and $u_\ell \in \mathbb{C}^d$, $\ell \in \mathcal{S}$, *then applying the theorem separately to* $y_\ell' = F_\ell(t)y_\ell$, $\ell \in \mathcal{S}$, *shows that* $x' = A(t)x$ *has a unique solution*

$$x = \sum_{\ell \in \mathcal{S}} P_\ell y_\ell \quad \textit{such that} \quad \lim_{t \to \infty} y_\ell(t) = u_\ell. \quad \ell \in \mathcal{S},$$

To carry this over to the case where P *is a variable matrix, we need another assumption:*

$$P' = P \,\mathrm{diag}(U_1, U_2, \dots, U_k) \quad \textit{with} \quad U_\ell \in \mathbb{C}_0^{n_\ell \times n_\ell};$$

i.e.,

$$P_\ell' = P_\ell U_\ell \,\textit{with}\; U_\ell, \quad 1 \le \ell \le k.$$

Thus, for each t, $P_\ell'(t)$ *is in the column space of* $P(t)$ *(the* $\mu_k(t)$*-eigenspace of* $R(t)$*). In this case we say that* R *is* <u>*spectrally separated*</u>*. (I'm open to suggestions of better terminology.)*

To solve

$$x' = A(t)x + f(t), \quad x(t_0) = x_0,$$

write

$$x_0 = \sum_{\ell=1}^{k} P_\ell y_{0\ell} \ \textit{ with } \ y_{0\ell} = \widehat{P}_\ell x_0 \in \mathbb{C}^{n_\ell}$$

and

$$f = \sum_{\ell=1}^{k} P_\ell h_\ell \ \textit{ with } \ h_\ell = \widehat{P}_\ell f \in \mathbb{C}^{n_\ell}(\mathscr{I}),$$

and solve the k *initial value problems*

$$y_\ell = (F_\ell(t) - U_\ell(t))y_\ell + h_\ell(t), \ y_\ell(t_0) = y_{0\ell}, \ 1 \le \ell \le k.$$

Recall that $A = P\mathbf{F}P^{-1}$ *where* $\mathbf{F} = \operatorname{diag}(F_1, \dots, F_k)$. *Denote* $\mathbf{U} = \operatorname{diag}(U_1, \dots, U_k)$, *so* $P' = P\mathbf{U}$. *To solve* $x' = A(t)x$, *write*

$$x = Py = \begin{bmatrix} P_1 & P_2 & \cdots & P_k \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} \text{ with } y_\ell \in \mathbb{C}_0^{n_\ell}(\mathscr{I}).$$

Since

$$Ax = (P\mathbf{F}P^{-1})(Py) = PFy$$

and

$$x' = Py' + P'y = Py' + P\mathbf{U}y = P(y' + \mathbf{U}y)$$

it follows that $x' = A(t)x$ *if and only if*

$$y' + \mathbf{U}y = \mathbf{F}y \iff y' = (\mathbf{F} - \mathbf{U})y,$$

which is equivalent to

$$y_\ell' = (F_\ell(t) - U_\ell(t))y, \qquad 1 \le \ell \le k.$$

Recall that a fundamental matrix for $x' = A(t)x$ is an invertible $n \times n$ matrix function X such that $X' = A(t)X$.

Theorem 2 *If $RA = AR$ on $\mathscr{I}$ and $\mathbf{Y} = \bigoplus_{\ell=1}^{k} Y_\ell$ where $Y_1, Y_2, \dots, Y_k$ are fundamental matrices for the systems $y_\ell' = (F_\ell(t) - U_\ell(t))y_\ell$, $1 \le \ell \le k$, then $X = P\mathbf{Y}$ is a fundamental matrix for $x' = A(t)x$. Moreover, if $t_0 \in \mathscr{I}$ and $x_0 \in \mathbb{C}^n$ then the solution of the initial value problem $x' = A(t)x$, $x(t_0) = x_0$, is*

$$x(t) = \sum_{\ell=1}^{k} P_\ell(t)Y_\ell(t)Y_\ell^{-1}(t_0)y_{0\ell} \text{ where } y_{0\ell} = \widehat{P}_\ell(t_0)x_0,$$

$1 \le \ell \le k$. The general solution of $x' = A(t)x$ is

$$x(t) = \sum_{\ell=1}^{k} P_\ell(t)Y_\ell(t)c_\ell \text{ where } c_\ell \in \mathbb{C}^{n_\ell}, \quad 1 \le \ell \le k.$$

Theorem 3 *Suppose* $A \in \mathbb{C}^{n\times n}(\mathscr{I})$ *is* R*-symmetric,* $f \in \mathbb{C}^n(\mathscr{I})$*, and* $t_0 \in \mathscr{I}$*. Let* $Y_1, Y_2, \dots, Y_k$ *be fundamental matrices for the systems* $y'_\ell = (F_\ell(t) - U_\ell(t))y_\ell$*,* $1 \le \ell \le k$*. Then the solution of*

$$x' = A(t)x + f(t), \quad x(t_0) = x_0,$$

is

$$x(t) = \sum_{\ell=1}^{k} P_\ell(t)Y_\ell(t)\left(Y_\ell^{-1}(t_0)y_{0\ell} + \int_{t_0}^{t} Y_\ell^{-1}(\tau)h_\ell(\tau)\,d\tau\right),$$

where

$$y_{0\ell} = \widehat{P}_\ell(t_0)x_0 \quad \textit{and} \quad h_\ell = \widehat{P}_\ell f, \quad 1 \le \ell \le k.$$

Theorem 4 *Suppose* $A \in \mathbb{C}^{n\times n}(\mathcal{I})$*. Let*

$$\mathcal{S}_A = \left\{ x \in \mathbb{C}_1^{n\times n}(\mathcal{I}) \,\middle|\, x'(t) = A(t)x(t),\ t \in \mathcal{I} \right\}$$

(*solution set of* $x' = A(t)x$) *and*

$$\mathcal{E}_R = \bigcup_{\ell=1}^{k} \left\{ x \in \mathbb{C}_1^{n\times n}(\mathcal{I}) \,\middle|\, R(t)x(t) = \mu_\ell(t)x(t),\ t \in \mathcal{I} \right\}$$

(*union of the time-varying eigenspaces of* R). *Then* A *is* R*-symmetric if and only if* $\mathcal{S}_A$ *has a basis in* $\mathcal{E}_R$.

PROOF. *If* $RA = AR$ *on* $\mathcal{I}$ *then the general solution of* $x' = A(t)x$ *is* $x = \sum_{\ell=1}^{k} P_\ell y_\ell$*. Since* $RP_\ell = \mu_\ell P_\ell$, $1 \le \ell \le k$*. This implies necessity.*

For sufficiency, if $\mathcal{S}_A$ has a basis in $\mathcal{E}_R$ then $x' = Ax$ has a fundamental matrix of the form

$$X = P\mathbf{Y} = \left[\begin{array}{cccc} P_1 & P_2 & \cdots & P_k \end{array}\right] \mathrm{diag}(Y_1, Y_2, \dots, Y_k),$$

where

$$Y_\ell \ \textit{and} \ Y_\ell^{-1} \in \mathbb{C}_1^{n_\ell \times n_\ell}(\mathcal{J}), \quad 1 \le \ell \le k.$$

Therefore $AP\mathbf{Y} = (P\mathbf{Y})' = P'\mathbf{Y} + P\mathbf{Y}'$, so

$$\begin{aligned} A &= (P'\mathbf{Y} + P\mathbf{Y}')\mathbf{Y}^{-1}P^{-1} = P'P^{-1} + P(\mathbf{Y}'\mathbf{Y}^{-1})P^{-1} \\ &= P(P^{-1}P')P^{-1} + P(\mathbf{Y}'\mathbf{Y}^{-1})P^{-1} \\ &= P(\mathbf{U} + \mathbf{Y}'\mathbf{Y}^{-1})P^{-1} = P\mathbf{F}P^{-1} \end{aligned}$$

(since $P' = PU$), with

$$\mathbf{F} = \mathbf{U} + \mathbf{Y}'\mathbf{Y}^{-1} = \bigoplus_{\ell=0}^{k-1}(U_\ell + Y_\ell' Y_\ell^{-1}).$$

Hence $RA = AR$ on $\mathcal{J}$, by Theorem 1.

Closing comment on $x' = A(t)x$:

Suppose $\mathcal{I} = [a, \infty)$ and $RA = AR$ on $\mathcal{I}$. Since the general solution of $x' = A(t)x$ is of the form

$$y = \sum_{\ell=1}^{k} P_\ell y_\ell \quad \text{where} \quad y_\ell' = (F_\ell(t) - U_\ell(t))y_\ell,$$

it seems that the best (only?) way to study the asymptotic behavior of solutions of $x' = A(t)x$ is to study the separate behaviors of the components $y_1, \ldots, y_k$.

For example, Bôcher's theorem implies the following result.

Theorem *Suppose that $RA = AR$ on $\mathcal{I}$ and $\int^\infty \|F_\ell - U_\ell\| dt < \infty$ for all ℓ in a nonempty sunset $\mathcal{S}$ of $\{1, \ldots, k\}$. For each $\ell \in \mathcal{S}$ let $u_\ell \in \mathbb{C}^{n_\ell}$ be given. Then $x' = A(t)x$ has a unique solution $x = \sum_{\ell \in \mathcal{S}} P_\ell y_\ell$ such that $\lim_{t\to\infty} y_\ell(t) = u_\ell$, $\ell \in \mathcal{S}$.*

DISCRETE FORMULATION

Let $\mathbb{Z}_+$ *be the set of nonnegative integers and consider linear systems of difference equations*

$$x_{t+1} = (I + A_t)x_t, \quad t \in \mathbb{Z}_+, \quad x_0 = \xi,$$

where $I + A_t \in \mathbb{C}^{n\times n}$ *is invertible for all* $t \geq 0$*. Let*

$$\mathbb{P}_t = \left[\begin{array}{cccc} P_{1t} & P_{2t} & \cdots & P_{kt} \end{array} \right] \text{ with } \mathbb{P}_t^{-1} = \left[\begin{array}{c} \widehat{P}_{1t} \\ \widehat{P}_{2t} \\ \vdots \\ \widehat{P}_{kt} \end{array} \right],$$

where $P_{\ell t} \in \mathbb{C}^{n\times n_\ell}(\mathbb{Z}_+)$, $\widehat{P}_{\ell t} \in \mathbb{C}^{n_\ell\times n}(\mathbb{Z}_+)$,
and $\widehat{P}_{\ell t} P_{mt} = \delta_{\ell m} I_{n_\ell}$, $1 \leq \ell, m \leq k$, $t \in \mathbb{Z}_+$*. Let*

$$R_t = \mathbb{P}_t \operatorname{diag}(\mu_{1t} I_{n_1}, \ldots, \mu_{kt} I_{n_k})\mathbb{P}_t^{-1},$$

where $\mu_{1t}, \ldots, \mu_{kt}$ *are distinct for* $t \in \mathbb{Z}_+$*. Finally, let* $\mathbb{P}_{t+1} = \mathbb{P}_t(\mathbf{I} + \mathbf{U}_t)$*, where* $\mathbf{U}_t = \operatorname{diag}(U_{1t}, \ldots, U_{kt})$ *with* $U_{\ell t} \in \mathbb{C}^{n_\ell\times n_\ell}(\mathbb{Z}_+)$, $1 \leq \ell \leq k$*, and* $I + \mathbf{U}_t$ *invertible for all* $t \in \mathbb{Z}_+$*.*

Theorem 5 *$R_t A_t = A_t R_t$ for all $t \in \mathbb{Z}_+$ if and only if*

$$A_t = \mathbb{P}_t \mathbf{F}_t \mathbb{P}_t^{-1} = \sum_{\ell=0}^{k-1} P_{\ell t} F_{\ell t} \widehat{P}_{\ell t} \qquad (2)$$

with

$$F_{\ell t} = \widehat{P}_{\ell t} A_t P_{\ell t} \in \mathbb{C}^{n_\ell \times n_\ell}(\mathbb{Z}_+), \quad 1 \leq \ell \leq k, \quad t \in \mathbb{Z}_+.$$

Now suppose $RA = AR$ *on* $\mathcal{I}$ *and want to solve*

$$x_{t+1} = (I + A_t)x_t, \quad t > 0. \tag{3}$$

Write $x_t = P_t y_t = \sum_{\ell=1}^{k} P_{\ell t} y_{\ell t}$. *Then*

$$x_{t+1} = \sum_{\ell=1}^{k} P_{\ell,t+1} y_{\ell,t+1} = \sum_{\ell=1}^{k} P_{\ell t}(I_{n_\ell} + U_{\ell t}) y_{\ell,t+1}$$

and

$$\begin{aligned}(I + A_t)x_t &= \left(\sum_{\ell=1}^{k} P_{\ell t}(I_{n_\ell} + F_{\ell t})\widehat{P}_{\ell t}\right)\left(\sum_{m=1}^{k} P_{mt} y_{mt}\right) \\ &= \sum_{\ell=1}^{k} P_{\ell t}(I_{n_\ell} + F_{\ell t}) y_{\ell t}.\end{aligned}$$

Therefore (3) *holds if only if*

$$(I_{n_\ell}+U_{\ell t}) y_{\ell,t+1} = (I_{n_\ell}+F_{\ell t}) y_{\ell t}, \quad 1 \le \ell \le k, \quad t \in \mathbb{Z}_+,$$

or, equivalently,

$$y_{\ell,t+1} = (I_{n_\ell}+U_{\ell t})^{-1}(I_{n_\ell}+F_{\ell t}) y_{\ell t}, \quad 1 \le \ell \le k, \quad t \in \mathbb{Z}_+.$$

Theorem 6 *Suppose* $R_t A_t = A_t R_t$ *for all* $t \in \mathbb{Z}_+$ *and let*

$$Q_{\ell t} = \mathbb{P}_{\ell t} \prod_{j=1}^{t-1} (I_{n_\ell} + U_{\ell j})^{-1} (I_{n_\ell} + F_{\ell j}), \quad t \in \mathbb{Z}_+, \quad Q_{\ell 0} = I_{n_\ell},$$

$1 \le \ell \le k$ *Then*

$$X_t = \begin{bmatrix} Q_{1t} & Q_{2t} & \cdots & Q_{kt} \end{bmatrix} \quad t > 0, \quad X_0 = I$$

is a fundamental matrix for the system

$$x_{t+1} = (I + A_t) x_t, \quad t \in \mathbb{Z}_+.$$

The discrete analog of Bôcher's theorem can be adapted to prove the following theorem.

Theorem 7 *Suppose $R_t A_t = A_t R_t$ for all $t \in \mathbb{Z}_+$ and*

$$\sum_{t=0}^{\infty} \|(I_{n_\ell} + U_{\ell t})^{-1}(I + F_{\ell t}) - I_{n_\ell}\| < \infty$$

for all ℓ in a nonempty subset $\mathcal{S}$ of $\{1, \ldots, k\}$. For each $\ell \in \mathcal{S}$ let u_ℓ be a given vector in $\mathbb{C}^{n_\ell}$. Then the system $x_{t+1} = (I + A_t)x_t$ has a unique solution

$$x_t = \sum_{\ell \in \mathcal{S}} P_{\ell t} y_{\ell t} \quad \text{such that} \quad \lim_{t \to \infty} y_{\ell t} = u_\ell, \quad \ell \in \mathcal{S}.$$

AN ITERESTING QUESTION

Consider

$$x' = A(t)x, \quad t > t_0, \tag{4}$$

where $A \in \mathbb{C}^{n\times n}[t_0, \infty)$ *(continuous) but has no particular structure. A system like this is "nice" if it has linear asymptotic equilibrium (every nontrivial solution approaches a nonzero limit), which is true, for example, if* $\int^\infty \|A(t)\|\,dt < \infty$*. However, suppose that* $\int^\infty \|A(t)\|\,dt = \infty$*. In this case it seems reasonable to look for a continuously differentiable and invertible matrix* $P = P(t)$ *such that every nontrivial solution of* (1) *can be written as* $x = Py$*, where* y *approaches a nonzero limit as* $t \to \infty$*. In this case I'd like to say that* P *is a preconditioner for* (4)*.*

An easy sufficient (but not necessary) condition: Since $x' = Py' + P'y$ *and* $Ax = APy$*,* $x' = Ax$ *if and only if* $u' = P^{-1}(AP - P')$*. Hence* P *is a preconditioner for* (1) *if*

$$\int^\infty \|P^{-1}(AP - P')\|\,dt < \infty.$$

Some other definitions that can extended in this way:

Definition 1 Let $I = [a, \infty)$, $A \in \mathbb{C}_0^{n\times n}(\mathcal{I})$, and let $\mathcal{s}_a$ be the solution set of (4). Then:

(a) Eqn. (4) is stable if there is a constant M such that $|x(t)| \leq M|x(a)|$ for all $x \in \mathcal{S}_A$.

(b) Eqn. (4) is strictly stable if there is a constant M such that $\|x(t)\| \leq M\|x(\tau)\|$ for all $x \in \mathcal{S}_A$ and $t, \tau \geq a$.

(c) $x' = A(t)x$ is uniformly stable if there is a constant M such that $\|x(t)\| \leq \|x(\tau)\|$ for all $x \in \mathcal{S}_A$ and $t \geq \tau \geq a \in \mathcal{J}$.

(d) Eqn. (4) is uniformly asymptotically stable if there are constants M and $\nu > 0$ such that $\|x(t)\| \leq \|x(\tau)\|e^{-\nu(t-\tau)}$ for all $x \in \mathcal{S}_A$ and $t \geq \tau \geq a$.

(e) Eqn. (4) has linear asymptotic equilibrium if every nontrivial solution of $x' = A(t)x$ approaches a nonzero constant vector as $t \to \infty$.

Definitions **(c)** *and* **(d)** *can be combined in the following definition, which may be new. Let* ρ *be continuous and positive on* $\left\{(t, \tau \,\middle|\, t \geq \tau \geq a\right\}$ *and suppose that*

$$\rho(t,t) = 1 \text{ and } \rho(t,\tau) \leq \rho(t,s)\rho(s,\tau) \text{ if } t \geq s \geq \tau \geq a.$$

We say that (4) *is* ρ*-stable if there is a constant* M *such that*

$$\|x(t)\| \leq \frac{\|x(\tau)\|}{\rho(s,t)} \text{ for all } x \in \mathcal{S}_A \text{ and } t \geq \tau \geq a.$$

Definition 2 Suppose $P \in \mathbb{C}_1^{n\times n}(\mathcal{I})$. Then:

(a) Eqn. (4) is stable relative to P if there is a constant M such that

$$\|P^{-1}(t)x(t)\| \leq M\|P^{-1}(a)x(a)\|$$

(b) Eqn. (4) is strictly stable relative to P if there is a constant M such that

$$\|P^{-1}(t)x(t)\| \leq M\|P^{-1}(\tau)x(\tau)\| \quad \text{for all} \quad x \in \mathcal{S}_A$$

and $t, \tau \geq a$.

(c) Eqn. (4) is ρ-stable relative to P if there is a constant K such that

$$\|P^{-1}(t)x(t)\| \leq M\frac{\|P^{-1}(\tau)x(\tau)\|}{\rho(t,\tau)} \quad \text{for all} \quad x \in \mathcal{S}_A$$

and $t \geq \tau \geq a$.

(d) Eqn. (4) has linear asymptotic equilibrium relative to P if $\lim_{t\to\infty} P^{-1}(t)x(t)$ exists and is nonzero for every nontrivial $x \in \mathcal{S}_A$.