Notes on the Real Symmetric Toeplitz Eigenvalue Problem

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Following Andrew [1], we say that an n–vector $x = [x_1\ x_2\ \cdots\ x_n]^T$ is *symmetric* if

$$x_j = x_{n-j+1}, \quad 1 \le j \le n,$$

or *skew–symmetric* if

$$x_j = -x_{n-j+1}, \quad 1 \le j \le n.$$

(Some authors call such vectors *reciprocal* and *anti-reciprocal*.)

The following theorems are special cases of results stated explicitly by Cantoni and Butler [2], but already implicit in Andrew [1]. (It's important to cite Andrew in this way. Many authors – including Trench – have overlooked the full scope of Andrew's results.) These theorems imply that if T is an RST matrix of order n then R^n has an orthonormal basis consisting of $\lceil n/2 \rceil$ symmetric and $\lfloor n/2 \rfloor$ skew–symmetric eigenvectors of T. They also yield effficient methods for computing eigenvalues and eigenvectors of real symmetric Toeplitz matrices.

In [3] I defined an eigenvalue λ of T to be *even* (*odd*) if T has a symmetric (skew–symmetric) λ–eigenvector. In the following theorems J_m is the $m \times m$ matrix with ones on the secondary diagonal and zeros elsewhere. In the proofs we rely heavily on the relations

$$JJ = I \quad \text{and} \quad JT = TJ$$

if T is a symmetric Toeplitz matrix.

We can partition T_{2m} as

$$T_{2m} = \begin{bmatrix} T_m & J_m H_m \\ H_m J_m & T_m \end{bmatrix} \tag{1}$$

where $H_m = (t_{i+j-1})_{i,j=1}^m$.

We can partition T_{2m+1} as

$$T_{2m+1} = \begin{bmatrix} T_m & J_m u_m & J_m G_m \\ u_m^t J_m & t_0 & u_m^t \\ G_m J_m & u_m & T_m \end{bmatrix} \tag{2}$$

where $G_m = (t_{i+j})_{i,j=1}^m$ and

$$u_m = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{bmatrix}.$$

Theorem 1 *Suppose that μ is an eigenvalue of*

$$A_m = T_m + H_m$$

with associated eigenvector x. Then μ is an even eigenvalue of T_{2m}, with associated symmetric eigenvector

$$p = \begin{bmatrix} J_m x \\ x \end{bmatrix}.$$

PROOF. From (1),

$$\begin{aligned} T_{2m}p &= \begin{bmatrix} T_m & J_m H_m \\ H_m J_m & T_m \end{bmatrix} \begin{bmatrix} J_m x \\ x \end{bmatrix} \\ &= \begin{bmatrix} (T_m J_m + J_m H_m)x \\ (T_m + H_m)x \end{bmatrix} = \begin{bmatrix} J_m (T_m + H_m)x \\ (T_m + H_m)x \end{bmatrix} = \mu p. \end{aligned}$$

□

Theorem 2 *Suppose that ν is an eigenvalue of*

$$B_m = T_m - H_m$$

with associated eigenvector y. Then ν is an odd eigenvalue of T_{2m}, with associated skew-symmetric eigenvector

$$q = \begin{bmatrix} -J_m y \\ y \end{bmatrix}.$$

PROOF. From (1),

$$\begin{aligned} T_{2m}q &= \begin{bmatrix} T_m & J_m H_m \\ H_m J_m & T_m \end{bmatrix} \begin{bmatrix} -J_m y \\ y \end{bmatrix} \\ &= \begin{bmatrix} (-T_m J_m + J_m H_m)y \\ (T_m - H_m)y \end{bmatrix} = \begin{bmatrix} -J_m (T_m - H_m)y \\ (T_m - H_m)y \end{bmatrix} = \nu p. \end{aligned}$$

□

Theorem 3 *Suppose that μ is an eigenvalue of*

$$C_m = \begin{bmatrix} t_0 & \sqrt{2}u_m^t \\ \sqrt{2}u_m & T_m + G_m \end{bmatrix},$$

with eigenvector $\begin{bmatrix} \alpha \\ x \end{bmatrix}$, where α is a scalar. Then μ is an even eigenvalue of T_{2m+1}, with associated symmetric eigenvector

$$p = \begin{bmatrix} J_m x \\ \alpha\sqrt{2} \\ x \end{bmatrix}.$$

PROOF. From (2),

$$
\begin{aligned}
T_{2m+1}p &= \begin{bmatrix} T_m & J_m u_m & J_m G_m \\ u_m^t J_m & t_0 & u_m^t \\ G_m J_m & u_m & T_m \end{bmatrix} \begin{bmatrix} J_m x \\ \alpha\sqrt{2} \\ x \end{bmatrix} \\
&= \begin{bmatrix} T_m J_m x + \alpha\sqrt{2} J_m u_m + J_m G_m x \\ u_m^t x + t_0 \alpha\sqrt{2} + u_m^t x \\ G_m x + \alpha\sqrt{2} u_m + T_m x \end{bmatrix} = \mu p.
\end{aligned}
$$

(Remember that $T_m J_m = J_m T_m$.) □

Theorem 4 *Suppose that* ν *is an eigenvalue of*

$$D_m = T_m - G_m$$

with eigenvector y*. Then* ν *is an odd eigenvalue of* T_{2m+1}*, with associated skew-symmetric eigenvector*

$$q = \begin{bmatrix} -J_m y \\ 0 \\ y \end{bmatrix}.$$

PROOF. From (2),

$$
\begin{aligned}
T_{2m+1}q &= \begin{bmatrix} T_m & J_m u_m & J_m G_m \\ u_m^t J_m & t_0 & u_m^t \\ G_m J_m & u_m & T_m \end{bmatrix} \begin{bmatrix} -J_m y \\ 0 \\ y \end{bmatrix} \\
&= \begin{bmatrix} -T_m J_m y + J_m G_m y \\ -u_m^t y + u_m^t y \\ -G_m y + T_m y \end{bmatrix} = \nu q.
\end{aligned}
$$

(Remember that $T_m J_m = J_m T_m$.) □

Now assume that

$$t_r = \frac{1}{\pi}\int_0^\pi f(\theta)\, d\theta.$$

Then

$$
\begin{aligned}
a_{rs} &= \frac{1}{\pi}\int_0^\pi f(\theta)[\cos(r-s)\theta + \cos(r+s-1)\theta]\, d\theta \\
&= \frac{2}{\pi}\int_0^\pi f(\theta)\cos(r-1/2)\theta\cos(s-1/2)\theta\, d\theta \quad r,s = 1,\dots,n,
\end{aligned}
$$

$$
\begin{aligned}
b_{rs} &= \frac{1}{\pi}\int_0^\pi f(\theta)[\cos(r-s)\theta - \cos(r+s-1)\theta]\, d\theta \\
&= \frac{2}{\pi}\int_0^\pi f(\theta)\sin(r-1/2)\theta\sin(s-1/2)\theta\, d\theta \quad r,s = 1,\dots,n,
\end{aligned}
$$

$$
\begin{aligned}
c_{rs} &= \frac{1}{\pi}\int_0^{\pi} f(\theta)[\cos(r-s)\theta+\cos(r+s)\theta]\,d\theta \\
&= \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos r\theta\cos s\theta\,d\theta \quad r,s=1,\ldots,n,
\end{aligned}
$$

(the zeroth row and column of C are not included here), and

$$
\begin{aligned}
d_{rs} &= \frac{1}{\pi}\int_0^{\pi} f(\theta)[\cos(r-s)\theta-\cos(r+s)\theta]\,d\theta \\
&= \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin r\theta\sin s\theta\,d\theta \quad r,s=1,\ldots,n.
\end{aligned}
$$

References

[1] A. L. Andrew, *Eigenvectors of certain matrices*, Linear Algebra Appl., 7 (1973), pp. 151–162.

[2] A. Cantoni and F. Butler, *Eigenvalues and eigenvectors of symmetric centrosymmetric matrices*, Linear Algebra Appl., 13 (1976), pp. 275–288.

[3] W. F. Trench, *Spectral evolution of a one-parameter extension of a real symmetric Toeplitz matrix*, SIAM J. Matrix Anal. Appl. 11 (1990), 601-611.