Notes on the Real Symmetric Toeplitz Eigenvalue Problem

W. F. Trench

Following Andrew [1], we say that an \( n \)-vector \( x = [x_1 \ x_2 \ \cdots \ x_n]^T \) is symmetric if
\[
x_j = x_{n-j+1}, \quad 1 \leq j \leq n,
\]
or skew-symmetric if
\[
x_j = -x_{n-j+1}, \quad 1 \leq j \leq n.
\]
(Some authors call such vectors reciprocal and anti-reciprocal.)

The following theorems are special cases of results stated explicitly by Cantoni and Butler [2], but already implicit in Andrew [1]. (It’s important to cite Andrew in this way. Many authors – including Trench – have overlooked the full scope of Andrew’s results.) These theorems imply that if \( T \) is an RST matrix of order \( n \) then \( \mathbb{R}^n \) has an orthonormal basis consisting of \( \lceil n/2 \rceil \) symmetric and \( \lfloor n/2 \rfloor \) skew-symmetric eigenvectors of \( T \). They also yield efficient methods for computing eigenvalues and eigenvectors of real symmetric Toeplitz matrices.

In [3] I defined an eigenvalue \( \lambda \) of \( T \) to be even (odd) if \( T \) has a symmetric (skew-symmetric) \( \lambda \)-eigenvector. In the following theorems \( J_m \) is the \( m \times m \) matrix with ones on the secondary diagonal and zeros elsewhere. In the proofs we rely heavily on the relations
\[
JJ = I \quad \text{and} \quad JT = TJ
\]
if \( T \) is a symmetric Toeplitz matrix.

We can partition \( T_{2m} \) as
\[
T_{2m} = \begin{bmatrix} T_m & J_mH_m \\ H_mJ_m & T_m \end{bmatrix} \tag{1}
\]
where \( H_m = (t_{i+j-1})_{i,j=1}^m \).

We can partition \( T_{2m+1} \) as
\[
T_{2m+1} = \begin{bmatrix} T_m & J_mu_m & J_mG_m \\ u_m^t J_m & t_0 & u_m^t \\ G_mJ_m & u_m & T_m \end{bmatrix} \tag{2}
\]
where \( G_m = (t_{i+j})_{i,j=1}^m \) and
\[
u_m = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{bmatrix}.
\]
Theorem 1 Suppose that \( \mu \) is an eigenvalue of 
\[ A_m = T_m + H_m \]
with associated eigenvector \( x \). Then \( \mu \) is an even eigenvalue of \( T_{2m} \), with associated symmetric eigenvector
\[ p = \begin{bmatrix} J_m x \\ x \end{bmatrix}. \]

Proof. From (1),
\[ T_{2m}p = \begin{bmatrix} T_m & J_m H_m \\ H_m J_m & T_m \end{bmatrix} \begin{bmatrix} J_m x \\ x \end{bmatrix} = \begin{bmatrix} (T_m J_m + J_m H_m) x \\ (T_m + H_m) x \end{bmatrix} = \begin{bmatrix} J_m(T_m + H_m) x \\ (T_m + H_m) x \end{bmatrix} = \mu p. \]

Theorem 2 Suppose that \( \nu \) is an eigenvalue of 
\[ B_m = T_m - H_m \]
with associated eigenvector \( y \). Then \( \nu \) is an odd eigenvalue of \( T_{2m} \), with associated skew-symmetric eigenvector
\[ q = \begin{bmatrix} -J_m y \\ y \end{bmatrix}. \]

Proof. From (1),
\[ T_{2m}q = \begin{bmatrix} T_m & J_m H_m \\ H_m J_m & T_m \end{bmatrix} \begin{bmatrix} -J_m y \\ y \end{bmatrix} = \begin{bmatrix} (-T_m J_m + J_m H_m)y \\ (T_m - H_m)y \end{bmatrix} = \begin{bmatrix} -J_m(T_m - H_m)y \\ (T_m - H_m)y \end{bmatrix} = \nu p. \]

Theorem 3 Suppose that \( \mu \) is an eigenvalue of 
\[ C_m = \begin{bmatrix} t_0 & \sqrt{2}u_m^t \\ \sqrt{2}u_m & T_m + G_m \end{bmatrix}, \]
with eigenvector \( \begin{bmatrix} \alpha \\ x \end{bmatrix} \), where \( \alpha \) is a scalar. Then \( \mu \) is an even eigenvalue of \( T_{2m+1} \), with associated symmetric eigenvector
\[ p = \begin{bmatrix} J_m x \\ \alpha \sqrt{2} x \end{bmatrix}. \]
Proof. From (2),

\[
T_{2m+1} p = \begin{bmatrix} T_m & J_m u_m & J_m G_m \\ u_m^t J_m & t_0 & u_m^t \\ G_m J_m & u_m & T_m \end{bmatrix} \begin{bmatrix} J_m x \\ x \end{bmatrix} = \begin{bmatrix} T_m J_m + \alpha \sqrt{2} J_m u_m + J_m G_m x \\ u_m^t x + t_0 \alpha \sqrt{2} + u_m^t x \\ G_m x + \alpha \sqrt{2} u_m + T_m x \end{bmatrix} = \mu p.
\]

(Remember that \(T_m J_m = J_m T_m\).)

\[\square\]

**Theorem 4** Suppose that \(\nu\) is an eigenvalue of

\[D_m = T_m - G_m\]

with eigenvector \(y\). Then \(\nu\) is an odd eigenvalue of \(T_{2m+1}\), with associated skew-symmetric eigenvector

\[q = \begin{bmatrix} -J_m y \\ 0 \\ y \end{bmatrix}.
\]

Proof. From (2),

\[
T_{2m+1} q = \begin{bmatrix} T_m & J_m u_m & J_m G_m \\ u_m^t J_m & t_0 & u_m^t \\ G_m J_m & u_m & T_m \end{bmatrix} \begin{bmatrix} -J_m y \\ 0 \\ y \end{bmatrix} = \begin{bmatrix} -T_m J_m y + J_m G_m y \\ -u_m^t y + u_m^t y \\ -G_m y + T_m y \end{bmatrix} = \nu q.
\]

(Remember that \(T_m J_m = J_m T_m\).)

\[\square\]

Now assume that

\[t_r = \frac{1}{\pi} \int_0^\pi f(\theta) \, d\theta.
\]

Then

\[a_{rs} = \frac{1}{\pi} \int_0^\pi f(\theta) \cos(r-s) \theta + \cos(r + s - 1) \theta \, d\theta
\]

\[= \frac{2}{\pi} \int_0^\pi f(\theta) \cos(r - 1/2) \theta \cos(s - 1/2) \theta \, d\theta \quad r, s = 1, \ldots, n,
\]

\[b_{rs} = \frac{1}{\pi} \int_0^\pi f(\theta) \cos(r-s) \theta - \cos(r + s - 1) \theta \, d\theta
\]

\[= \frac{2}{\pi} \int_0^\pi f(\theta) \sin(r - 1/2) \theta \sin(s - 1/2) \theta \, d\theta \quad r, s = 1, \ldots, n,
\]
\[ c_{rs} = \frac{1}{\pi} \int_{0}^{\pi} f(\theta) [\cos(r - s)\theta + \cos(r + s)\theta] \, d\theta \]
\[ = \frac{2}{\pi} \int_{0}^{\pi} f(\theta) \cos r\theta \cos s\theta \, d\theta \quad r, s = 1, \ldots, n, \]

(the zeroth row and column of \( C \) are not included here), and

\[ d_{rs} = \frac{1}{\pi} \int_{0}^{\pi} f(\theta) [\cos(r - s)\theta - \cos(r + s)\theta] \, d\theta \]
\[ = \frac{2}{\pi} \int_{0}^{\pi} f(\theta) \sin r\theta \sin s\theta \, d\theta \quad r, s = 1, \ldots, n. \]

References

