EFFICIENT APPLICATION OF THE SCHAUDER—TYCHONOFF THEOREM TO FUNCTIONAL DIFFERENTIAL SYSTEMS

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We consider the $n \times n$ $(n \geq 1)$ system of functional differential equations

$$X' = FX, \ t > t_0. \tag{1}$$

For now we make no specific assumptions on the form of the functional F. For example, (1) may be a system of ordinary differential equations, an integro–differential system, a system with one or more deviating arguments, or a combination of these. To allow for the possibility that the values of $(FX)(t)$ for $t \geq t_0$ may depend on the values of $X(\tau)$ for some $\tau < t_0$ (as in the case of a delay equation, for example), we make the following definition.

Definition 1. *If $-\infty < t_0 < \infty$, then $\mathcal{C}_n(t_0)$ is the space of continuous n-vector functions $X = (x_1, \dots, x_n)$ on $(-\infty, \infty)$ which are constant on $(-\infty, t_0]$, with the topology induced by the following definition of convergence:*

$$X_j \to X \quad as \quad j \to \infty$$

if

$$\lim_{j\to\infty}\,[\sup_{-\infty<t\leq T}\|X_j(t)-X(t)\|\,]=0$$

for every T in $(-\infty,\infty)$. (Here $\|\cdot\|$ is any convenient vector norm.)

Notice that $\mathcal{C}_n(t_1)\subset\mathcal{C}_n(t_0)$ if $t_0\leq t_1$. We will say X is a *solution of (1) on* $[t_0,\infty)$ if $X\in\mathcal{C}_n(a)$ for some $a\leq x_0$ and X satisfies (1) for $t\geq t_0$ (derivative from the right at t_0). We are interested in giving conditions on the functional F which imply that (1) has a solution $\hat{X}$ such that $\lim_{t\to\infty}\hat{X}(t)=C$, where C is a given constant vector. A *global* result of this kind is a conclusion that such an $\hat{X}$ exists on a *given* interval $[t_0,\infty)$; a *local* (near ∞) result is one which implies that such a solution exists on $[t_0,\infty)$ provided that t_0 is sufficiently large.

The Schauder–Tychonoff theorem has proved to be a powerful tool for establishing existence theorems of the kind that interest us here. More precisely, the following special case of this theorem, which is essentially the form cited by Coppel (*Stability and Asymptotic Behavior of Differential Equations*) has yielded many useful results. (I have modified Coppel's formulation somewhat to adapt it to the situation considered in Definition 1.)

Lemma 1. *Let $\mathcal{S}$ be a closed convex subset of $C_n(t_0)$, and suppose that $\mathcal{T}$ is a transformation of $\mathcal{S}$ such that (a) $\mathcal{T}(S)\subset S$; (b) $\mathcal{T}$ is* *continuous (i.e., if* $\{X_j\} \subset S$ *and* $X_j \to X$*, then* $\mathcal{T}X_j \to \mathcal{T}X$*); and (c) the family of functions* $\mathcal{T}(\mathcal{S})$ *is uniformly bounded and equicontinuous on every compact subinterval of* $[t_0, \infty)$*. Then there is an* $\hat{X}$ *in* $\mathcal{S}$ *such that* $\mathcal{T}\hat{X} = \hat{X}$*.*

The following theorem illustrates one way in which this theorem can be applied to our problem. It is basically a conventional result, in that it is modelled after a technique commonly used to study the asymptotic theory of ordinary differential equations; however, the present formulation in terms of functional differential equations may be new.

Theorem 1. *Suppose that there are constants* a *and* M $(M > 0)$ *and a continuous function* $w\colon [a, \infty) \to (0, \infty)$ *such that* $FX \in C_n[a, \infty)$ *and* $\|(FX)(t)\| \leq w(t)$ *for* $t \geq a$ *whenever*

$$X \in \mathcal{C}_n(a) \text{ and } \|X(t)\| \leq M, \ t \geq a. \tag{2}$$

Suppose further that

$$\int_a^\infty w(s)\, ds < \infty,$$

and that $\lim_{j\to\infty}(FX_j)(t) = (FX)(t)$ *(pointwise) if each* X_j *satisfies (2) and* $X_j \to X$*. Let* C *be a given constant, with* $\|C\| \leq M$*. Then the system (1) has a solution* $\hat{X}$ *on some interval* $[t_0, \infty)$*, such that* $\lim_{t\to\infty}\hat{X}(t) = C$*.*

Proof. Choose $t_0 \geq a$ so that

$$\int_{t_0}^{\infty} r(t)\,dt < M - \|C\|.$$

Define

$$\mathcal{S} = \{X \in \mathcal{C}_n(t_0) \,\big|\, \|X(t)\| \leq M, t \geq t_0\}.$$

Then $\mathcal{S}$ is closed and convex. Define the transformation $Y = \mathcal{T}X$ by

$$Y(t) = \begin{cases} C - \int_t^{\infty} (FX)(S)\,ds, & t \geq t_0, \\ \\ C - \int_{t_0}^{\infty} F(X)(s)\,ds, & t < t_0. \end{cases} \qquad 1 \leq i \leq n. \tag{3}$$

If $X \in \mathcal{S}$, then

$$\|Y(t)\| \leq C + \int_{t_0}^{\infty} \|(FX)(s)\|\,ds$$

$$\leq C + \int_{t_0}^{\infty} w(s)\,ds \leq \|C\| + M - \|C\| = M;$$

therefore, $\mathcal{T}(\mathcal{S}) \subset \mathcal{S}$. Since $\mathcal{S}$ is a family of functions which are uniformly bounded on $[t_0, \infty)$, so is $\mathcal{T}(\mathcal{S})$. Differentiating (3) shows that $\|Y'(t)\| \leq w(t)$ if $t \geq t_0$ and $Y'(t) = 0$ if $t < t_0$. Therefore, $\mathcal{T}(\mathcal{S})$ is equicontinuous on every interval $(-\infty, T]$, by the mean value theorem. To see that $\mathcal{T}$ is continuous, let $\{X_j\} \subset \mathcal{S}$ and $X_j \to X$ as $j \to \infty$. Then

$$\|Y_j(t) - Y(t)\| \leq \int_{t_0}^{\infty} \|(FX_j)(s) - (FX)(s)\| \, ds. \tag{4}$$

Since the integrand here $\to 0$ as $t \to \infty$ and is dominated by $2w(s)$, our integrability assumption on w implies that the integral on the right of (4) $\to 0$ as $t \to \infty$, by Lebesgue's dominated convergence theorem. Hence, $Y_j \to Y$ as $t \to \infty$. Now the Schauder–Tychonoff theorem implies that $\mathcal{T}\hat{X} = \hat{X}$ for some $\hat{X}$ in $\mathcal{S}$; i. e.,

$$\hat{X}(t) = C - \int_t^{\infty} (F\hat{X})(s) \, ds.$$

Obviously, $\hat{X}'(t) = (F\hat{X})(t)$ for $t > t_0$, and $lim_{t\to\infty}\hat{X}(t) = C$.

Although useful results can be obtained from this thereom, it is clear that the integrability condition on the functional F is very strong, since it implies that the integrals

$$\int_t^{\infty} \|(FX)(s)\| ds, \; X \in \mathcal{S}, \tag{5}$$

all converge, and even uniformly for all X in $\mathcal{S}$ (i.e. $\int_t^\infty \|FX\| \, ds \leq \int_t^\infty w(s) \, ds$). It is quite possible to obtain useful results without requiring that the integrals (5) converge at all, so long as the integrals

$\int^{\infty}(FX)(s)\,ds$ $(X \in \mathcal{S})$ converge in the ordinary (i.e., perhaps conditional) sense, and satisfy a uniform estimate of the form

$$\|\int_t^{\infty}(FX)(s)\,ds\| \leq \rho(t),\ \ X \in \mathcal{S}, \tag{6}$$

for some function ρ such that $\lim_{t\to\infty}\rho(t) = 0$. Moreover, it is important to exploit not just the assumption that the integrals in (6) converge, but also their rate of convergence. *Whenever possible, we should integrate before taking absolute values.* This point is often missed.

To illustrate this point, we will apply Theorem 1 to the linear functional system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(g_1(t)) \\ x_2(g_2(t)) \\ \vdots \\ x_n(g_n(t)) \end{bmatrix}. \tag{7}$$

Theorem 2. *Suppose that* $\{g_{ij}\}$ *and* $\{a_{ij}\}$ *are continuous on* $[a,\infty)$ *and* $\int_a^{\infty}|a_{ij}(t)|\,dt < \infty$ *for* $1 \leq i,j \leq n$. *Let* $C = (c_1, c_2, \dots, c_n)$ *be a given constant vector. Then the system (7) has a solution* $\hat{X}$ *such that* $\lim_{t\to\infty}\hat{X} = C$.

Proof. Here $X = (x_1, x_2, \ldots, x_n)$ and we use the vector norm

$$\|X\| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

The continuity of the $\{g_i\}$ and the $\{a_{ij}\}$ implies that $FX \in C_n[a, \infty)$ if $X \in \mathcal{C}_n(a)$ and that $\{FX_j\}$ converges pointwise to FX if $X_j \to X$. (In fact, $\{FX_j\}$ converges uniformly to FX on every interval $(-\infty, T]$, but this is not important.) The integrability condition can be expressed as

$$\int_a^\infty \|A(t)\|\, dt < \infty,$$

where A is the matrix in (7). If $\|X(t)\| \leq M$, then $\|(FX)(t)\| \leq w(t) = M\|A(t)\|$. Therefore, Theorem 1 implies the conclusion.

It is somewhat surprising that the conclusion of Theorem 1 does not depend in any way on the nature of the deviating arguments $\{g_i\}$. We have assumed nothing about then other than continuity; they may be advanced, retarded, or mixed, independently of each other. In particular, we have not assumed that $\lim_{t\to\infty} g_i(t) = \infty$.

Theorems 1 and 2 may be of interest, but in some situations their hypotheses are unnecessarily strong and their conclusions are relatively weak. The following standard corollary of Theorem 2 illustrates this.

Corollary 1. *Suppose that the* $\{a_{ij}\}$ *are as in Theorem 2, and let*

C be an arbitrary constant n–vector. Then the linear system $X'(t) = A(t)X(t)$ has a solution $\hat{X}$ such that $\hat{X} \to C$ as $t \to \infty$.

Example 1. Consider the system

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \frac{\sin t}{t^\alpha} \begin{bmatrix} a_1 t^{-1} & b_1 \\ a_2 t^{-2} & b_2 t^{-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \; t \geq a > 0, \qquad (8)$$

where $b_1, b_2 \neq 0$ and $\alpha > 0$. Since

$$\int^\infty t^{-\alpha} |\sin t| \, dt \begin{cases} = \infty \text{ if} & \alpha < 1, \\ < \infty \text{ if} & \alpha \geq 1, \end{cases}$$

Corollary 1 does not apply to this system if $0 \leq \alpha < 1$; if $\alpha \geq 1$, then Corollary 1 implies that if c_1 and c_2 are given constants, then (8) has a solution $\hat{X} = (\hat{x}_1, \hat{x}_2)$ such that

$$\lim_{t \to \infty} x_i(t) = c_i, \;\; i = 1, 2.$$

The corollary provides no estimate of the *order* of convergence here, but it is straightforward to show that if $\alpha > 1$, then

$$x_1(t) = c_1 + O(t^{-\alpha+1}) \text{ and } x_2(t) = c_2 + O(t^{-\alpha}).$$

However, a more efficient use of integrability conditions for problems like this will show later that the true situation is as follows:

Suppose that $\alpha > 0$. Then:

(i) If c_1 *is arbitrary and* $c_2 \neq 0$*, then (8) has a solution* $\hat{X}$ *such that*

$$x_1(t) = c_1 + O(t^{-\alpha}) \text{ and } x_2(t) = c_1 + O(t^{-\alpha-1}).$$

(ii) If c_1 *is arbitrary and* $c_2 = 0$*, then (8) has a solution* $\hat{X}$ *such that*

$$x_1(t) = c_1 + O(t^{-\alpha-1}) \text{ and } x_2(t) = O(t^{-\alpha-2}).$$

The following theorem makes more efficient use of the Schauder–Tychonoff theorem (Lemma 1).

Theorem 3. *$C = (c_1, c_2, \ldots, c_n)$ be a given constant vector. Let* $\gamma_1, \ldots, \gamma_n$ *be continuous, positive and nonincreasing on* $[t_0, \infty)$ *and let* $M_1, \ldots, M_n$ *be positive constants. Let* $\mathcal{S}$ *be the set of functions* $X = (x_1, \ldots, x_n)$ *in* $C_n(t_0)$ *such that*

$$| x_i(t) - c_i | \leq M_i \gamma_i(t), \; t \geq t_0, \; 1 \leq i \leq n.$$

Suppose that F *satisfies the following assumptions:*

(i) $FX \in C_n[t_0, \infty)$ *if* $X \in \mathcal{S}$.

(ii) The family of functions $\mathcal{F} = \{FX \mid X \in \mathcal{S}\}$ *is uniformly bounded on each subinterval of* $[t_0, \infty)$.

(iii) If $\{X_j\} \subset \mathcal{S}$ *and* $X_j \to X$ *(uniform convergence on every*

interval $(-\infty, T]$ *), then*

$$\lim_{j\to\infty}(FX_j)(t) \;=\; (FX)(t) \text{ (pointwise)},\; t \geq t_0.$$

(iv) The integrals $\int^\infty (FX)(s)\,ds$ $(X \in \mathcal{S})$, *converge, perhaps conditionally, and there are nonincreasing functions* $\rho_1, \rho_2, \ldots, \rho_n$ *such that*

$$0 < \rho_i(t) \leq M_i\gamma_i(t),\; 1 \leq i \leq n, \tag{9}$$

$$\lim_{t\to\infty} \rho_i(t) = 0, 1 \leq i \leq n,$$

and, if $X \in \mathcal{S}$ *and* $t \geq t_0$,

$$\mid \int_t^\infty f_iX\ ds \mid \leq\ \rho_i(t), 1 \leq i \leq n. \tag{10}$$

Then (1) has a solution $\hat{X}$ *on* $[t_0, \infty)$ *such that*

$$\mid \hat{x}_i(t) - c_i \mid \leq\ \rho_i(t),\; t \geq t_0,\; 1 \leq i \leq n.$$

Proof. The transformation $Y = \mathcal{T}X$ can be written in terms of components as

$$y_i(t) = \begin{cases} c_i - \int_t^\infty (f_iX)(s)\,ds, & t \geq t_0, \\ \\ c_i - \int_{t_0}^\infty (f_iX)(s)\,ds, & t < t_0. \end{cases} \qquad 1 \leq i \leq n. \tag{11}$$

Therefore, from (9),(10) and (11),

$$|y_i(t) - c_i| \leq \rho_i(t) \leq M_i\gamma_i(t);$$

hence, $\mathcal{T}(\mathcal{S}) \subset \mathcal{S}$, and $\mathcal{T}(\mathcal{S})$ is uniformly bounded on $[t_0, \infty)$, since $\mathcal{S}$ is. Differentiating (11) shows that $y_i'(t) = (f_iX)(t)$ if $t \geq t_0$ and $y_i'(t) = 0$ if $t < t_0$; hence, the mean value theorem and assumption (iii) imply that the family $\mathcal{T}(\mathcal{S})$ is equicontinuous on every interval $(-\infty, T]$. The proof that $\mathcal{T}$ is continuous is somewhat more delicate than it was in Theorem 1, since the integrals in question may converge conditionally. Suppose that $\{X_j\} \subset \mathcal{S}$ and $X_j \to X = (x_1, x_2, \dots, x_n)$ as $j \to \infty$. Denote $X_j = (x_{1j}, x_{2j}, \dots, x_{nj})$; then

$$y_{ij}(t) - y_i(t) = \begin{cases} \int_t^\infty (f_iX_j - f_iX)\, ds, & t \geq t_0, \\ \int_{t_0}^\infty (f_iX_j - f_iX)\, ds, & t < t_0. \end{cases}$$

Let

$$H_{ij} = \sup_{-\infty<t<\infty} |y_{ij}(t) - y_i(t)|,\ 1 \leq i \leq n,\ j = 1, 2, \cdots.$$

Then, if $t_1 \geq t_0$,

$$H_{ij} \leq \int_{t_0}^{t_1} |f_iX_j - f_iX|\, ds + \Big|\int_{t_1}^\infty f_iX_j\, ds\Big| + \Big|\int_{t_1}^\infty f_iX_j\, ds\Big|$$

$$\leq \int_{t_0}^{t_1} |f_iX_j - f_iX|\, ds + 2\rho_i(t_1),$$

from (10). Since the last integrand is uniformly bounded on $[t_0, t_1]$ for all j and $\to 0$ pointwise as $t \to \infty$, the last integral $\to 0$ as $t \to \infty$, by the bounded convergence theorem. Hence,

$$\overline{\lim}_{j\to\infty} H_{ij} \le 2\rho_i(t_1)$$

for every t_1. Since $\lim_{t_1\to\infty} \rho_i(t_1) = 0$, this implies that $\lim_{j\to\infty} H_{ij} = 0$ for $1 \le i \le n$; that is, $y_{ij}(t) \to y_i(t)$ uniformly on $(-\infty, \infty)$ as $j \to \infty$. Now Lemma 1 implies the conclusion.

One should not apply Theorem 3 by stating general integrability conditions and then seeking systems to which they apply. (A result of this type: *If A is a continuous $n \times n$ matrix on $[t_0, \infty)$ and C is a constant vector, then the system $X' = A(t)X$ has a solution $\hat{X}$ such that $\lim_{t\to\infty} \hat{X}(t) = c$.)* It is important to think in terms of a specific system $X' = FX$ and a specific "target vector" C, and to base the choice of $\gamma_1, \dots, \gamma_n$ on the integrability properties of FX for functions X "near" C in some appropriate sense. One way to approach this is to think of FX as

$$FX = FC + (FX - FC),$$

and use the integrability properties of FC to formulate an appropriate choice of $\gamma_1, \dots, \gamma_n$ which is consistent with the integrability properties of $FX - FC$. The following theorem is along these lines.

Theorem 4. *Let $\mathcal{S}$, $\gamma_1, \gamma_2, \dots, \gamma_n$, $M_1, M_2, \dots M_n$ and C be as in Theorem 3, and suppose that F satisfies assumptions (i) and (iii) on the set $\mathcal{S}$ of functions $X = (x_1, \dots, x_n)$ in $C_n(t_0)$ such that*

$$| x_i(t) - c_i | \leq \; M_i \gamma_i(t), \; t \geq t_0, \; 1 \leq i \leq n.$$

Suppose further that $\int^\infty FC\, dt$ converges (perhaps conditionally) and that

$$\sup_{t \geq t_0} (\gamma_1(t))^{-1} \Big| \int_t^\infty f_i C\, ds \Big| = A_i < \infty, \; i \leq \; i \leq n. \tag{12}$$

Suppose also that

$$|(f_i X)(t) - (f_i C)(t)| \leq M_i w_i(t), \; 1 \leq i \leq n, \; t \geq t_0, \tag{13}$$

for all X in $\mathcal{S}$, where

$$\sup_{t \geq t_0} (\gamma_i(t))^{-1} \int_t^\infty w_i\, ds = \theta_i < 1, \; 1 \leq i \leq n. \tag{14}$$

Finally, let

$$M_i \geq \frac{A_i}{1 - \theta_i}. \tag{15}$$

Then the conclusion of Theorem 3 holds.

Proof. Write

$$f_i X = f_i C + (f_i X - f_i C); \tag{16}$$

then

$$|f_i X| \leq |f_i C| + M w_i,$$

which implies assumption (ii) of Theorem 4 (that the family $\{FX \mid X \in \mathcal{S}\}$ is uniformly bounded on finite subintervals of $[t_0, \infty)$). To verify Assumption (iv), note that (16) implies the inequality

$$\Big|\int_t^\infty f_i X\, ds\Big| \leq \int_t^\infty |f_i X - f_i C|\, ds + \Big|\int_t^\infty f_i C\, ds\Big|;$$

therefore, (13) implies that

$$\Big|\int_t^\infty f_i X\, ds\Big| \leq \rho_i(t) =_{\text{df}} M_i \int_t^\infty w_i\, ds + \sup_{\tau \geq t}\Big|\int_\tau^\infty f_i C\, ds\Big|.$$

From (12) and (14),

$$\rho_i(t) \leq (M_i \theta_i + A_i)\gamma_i(t),$$

and now (15) implies that $\rho_i \leq M_i \gamma_i$. Hence, Theorem 3 implies the conclusion.

Remark1. If for some i the integral $\sigma_i(t) = \int_t^\infty w_i\, ds$ is small (as $t \to \infty$) compared to $\int_t^\infty f_i C\, ds$, then we have the more precise estimate

$$x_i(t) = c_i - \int_t^\infty f_i C\, ds + O(w_i(t)).$$

We now apply Theorem 4 to the linear system

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

Theorem 5. *Suppose that* $\{a_{ij}\}$ *are continuous on* $[a, \infty)$ *and* $\int_a^\infty a_{ij}(t)\,dt$ *converges (perhaps conditionally) for* $1 \le i, j \le n$. *Let* $C = (c_1, c_2, \ldots, c_n)$ *be a given constant vector, and suppose that* $\gamma_1, \gamma_2, \ldots, \gamma_n$ *are nonincreasing positive functions on* $[a, \infty)$ *such that*

$$\int_t^\infty f_i C\,ds = O(\gamma_i(t)),\ 1 \le i \le n, \tag{17}$$

and define

$$w_i(t) = \sum_{j=1}^n |a_{ij}(t)|\gamma_j(t).$$

Suppose further that

$$\overline{\lim}(\gamma_i(t))^{-1} \int_t^\infty w_i(s)\,ds = \tilde{\theta}_i < 1,\ 1 \le i \le n. \tag{18}$$

Then the system $X' = AX$ *has a solution* $\hat{X}$ *such that*

$$\hat{x}_i(t) = c_i + O(\gamma_i(t)),$$

for $1 \le i \le n$; *moreover, if* $\tilde{\theta}_i = 0$, *then this can be replaced by more precise estimate*

$$\hat{x}_i(t) = c_i + \int_t^\infty f_i C\,ds + O\Big(\int_t^\infty w_i\,ds\Big).$$

Proof. Here we have

$$(f_i X)(t) = \sum_{j=1}^n a_{ij}(t) x_j(t),$$

$$(f_iC)(t) = \sum_{j=1}^{n} a_{ij}(t)c_j,$$

and

$$|(f_iX)(t) - (f_iC)(t)| \leq \sum_{j=1}^{n} |a_{ij}(t)||x_j - c_j|.$$

Choose $\theta_1, \theta_2, \ldots \theta_n$ so that $\tilde{\theta}_i < \theta_i < 1 \; 1 \leq i \leq n$, and then choose t_0 so large that

$$(\gamma_i(t))^{-1} \int_t^\infty w_i(s)\, ds = \theta_i < 1,\ 1 \leq i \leq n,\ t \geq t_0;$$

this is possible, because of (18). From (17), there are finite numbers $A_1, A_2, \ldots, A_n$ such that

$$\sup_{t \geq t_0} (\gamma_1(t))^{-1} \Big| \int_t^\infty f_iC\, ds \Big| = A_i,\ i \leq\ i \leq n.$$

Now choose $M_i \geq A_1/(1 - \theta_i),\ 1 \leq i \leq n$, and invoke Theorem 4. This completes the proof.

We emphasize that it is in general a bad tactic to choose $\gamma_1, \ldots, \gamma_n$ to be unnecessarily large, and the appropriate choice may depend upon the target vector C, as the following example will show. These points are often ignored in standard results in this area. Unfortunately, there appears to be no algorithm for deciding on the optimum choice for $\gamma_1, \gamma_2, \ldots, \gamma_n$. The obvious way to start is to estimate the functions $\phi_i(t) = \sup_{\tau \geq t} \left| \int_t^\infty f_i C \, ds \right| \; 1 \leq i \leq n$. Since we have assumed that

$$\int_{f_i C}^\infty = dsO(\gamma_i(t)) \; 1 \leq i \leq n$$

, it is clear that we must choose γ_i so that

The sharpest results can be obtained, often under the weakest hypotheses, by choosing $\gamma_i, \ldots, \gamma_n$ consistent with these requirements, and so that $\gamma_i(t)$ approaches zero as rapidly as possible as $t \to \infty$ Since $\gamma_1, \ldots, \gamma_n$ determine $\mathcal{S}$ and therefore also $w_1, \ldots, w_n$ this "smallest" choice of $\gamma_1, \ldots, \gamma_n$ will usually require the mildest integrability conditions. Sometimes it is possible to simply take $\gamma_i = k_i \phi_i$. However, this does not always work, since the conditions on $\gamma_1, \gamma_2, \ldots, \gamma_n$ are interrelated; for example, recall the condition from Theorem 5.

$$\overline{\lim}(\gamma_i(t))^{-1} \int_t^\infty \sum_{j=1}^n |a_{ij}(s)| \gamma_j(s) \, ds = \tilde{\theta}_i < 1, \;\; 1 \leq i \leq n.$$

Example 1 (Continuation). As mentioned above, a linear system $X' = A(t)X$ (with A continuous on $[0, \infty)$) has a solution $\hat{X}$ satisfying an arbitrary final condition $\lim_{t \to \infty} \hat{x}(t) = C$ if

$$\int^\infty \parallel A(t) \parallel dt < \infty.$$

This result can be obtained from Theorem 5 by simply taking $\gamma_i = 1\ 1 \leq i \leq n$.The system

$$\begin{bmatrix} x_1^{'} \\ x_2^{'} \end{bmatrix} = t^{-\alpha}\sin t \begin{bmatrix} a_1 t^{-1} & b_1 \\ a_2 t^{-2} & b_2 t^{-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ t > 0, \qquad (18)$$

with $b_1 \neq 0$ does not satisfy this inrtegrability condition if $\alpha \leq 1$; moreover, even if $\alpha > 1$, the standard theorem merely implies that if c_1 and c_2 are given constants, then (18) has a solution $(\hat{x}_1, \hat{x}_2)$ such that $\lim_{t\to\infty} \hat{x}_i(t) = c_i\ (i = 1, 2)$. However, Corollary 1 (with $Q = 0$) and Remark 1 imply that if $\alpha > 0$ and (c_1, c_2) is arbitrary, then (18) has a solution $(\hat{x}_1, \hat{x}_2)$ such that

$$\hat{x}_1(t) = c_1(1 - a_1 S_{\alpha+1}(t)) - b_1 c_2 S_\alpha(t) + 0(t^{-2\alpha})$$

and

$$\hat{x}_2(t) = -a_2 c_1 S_{\alpha+2}(t) + c_2(1 - b_2 S_{\alpha+1}(t)) + 0(t^{-2\alpha-1}),$$

where

$$S_\beta(t) = \int_t^\infty s^{-\beta} \sin s\ ds = 0(t^{-\beta}),\ \beta > 0.$$

This conclusion is obtained by letting $\gamma_1(t) = t^{-\alpha}$ and $\gamma_2(t) = t^{-\alpha-1}$. A sharper result is available if $c_2 = 0$; i.e., for every constant c_1, (18) has a solution $(\hat{x}_1, \hat{x}_2)$ such that

$$\hat{x}_1(t) = c_1(1 - a_1 S_{\alpha+1}(t)) + 0(t^{-2\alpha-1}),$$

and

$$\hat{x}_2(t) = -a_2 c_1 S_{\alpha+2}(t) + 0(t^{-2\alpha-2}).$$

This is obtained by letting $\gamma_1(t) = t^{-\alpha-1}$ and $\gamma_2(t) = t^{-\alpha-2}$.

We will now obtain a global result for the nonlinear integral equation

$$x'(t) = g(t)(x(t))^\alpha + \int_0^t P(t,\tau)(x(\tau))^\beta \, d\tau, \;\; t > 0, \tag{19}$$

where $g \in C[0,\infty)$ and P is continuous on $[0,\infty) \times [0,\infty)$.

Theorem 6. *Suppose that*

$$\int_t^\infty g(s)\, ds = O(\gamma(t)), \tag{20}$$

$$\int_t^\infty |g(s)|\gamma(s)\, ds = O(\gamma(t)), \tag{21}$$

$$\int_t^\infty \int_0^s P(s,\tau)\, d\tau\, ds = O(\gamma(t)), \tag{22}$$

and

$$\int_t^\infty \int_0^s |P(s,\tau)|\gamma(\tau)\,d\tau\,ds = O(\gamma(t)), \qquad (23)$$

where γ *is positive and nonincreasing on* $[0,\infty)$, *and* $\lim_{t\to\infty}\gamma(t) = 0$. *Suppose also that* $0 < \theta < 1$. *Then there is a constant* $c_0 > 0$ *such that (19) has a solution* $\hat{x}$ *on* $[0,\infty)$ *which satisfies the following conditions:*

$$|\hat{x}(t) - c| \le \theta c \ (t \ge 0) \ , \hat{x}(t) = c + O(\gamma(t)).$$

provided that either

(a) $\alpha, \beta > 1$ *and* $0 < c < c_0$*; or*

(b) $\alpha, \beta < 1$ *and* $c > c_0$.

(Notice that (20) and (21) do not imply that $\int^\infty |g(s)|\,ds < \infty$, nor do (22) and (23) imply that $\int^\infty \int_0^s |P(s,\tau)|\,d\tau\,ds < \infty$.)

Proof. For convenience, normalize γ so that $\gamma(0) = 1$. Here

$$(Fx)(t) = g(t)(x(t))^\alpha + \int_0^t P(t,\tau)(x(\tau))^\beta\,d\tau.$$

and

$$(Fc)(t) = \int_t^\infty g(s)\,ds c^\alpha + c^\beta \int_0^t P(t,\tau)\,d\tau.$$

If $c > 0$, let

$$\mathcal{S} = \{x \in C[0,\infty) \,\Big|\, |x(t) - c| \le \theta c\gamma(t),\ t \ge 0\}.$$

Then, if $x \in \mathcal{S}$, $|x(t) - c| \leq \theta c$ $(t \geq 0)$, since γ is nonincreasing. Obviously, F satisfies assumptions (i), (ii), and (iii) of Theorem 3. Now,

$$\int_t^\infty Fx\, ds = \int_t^\infty Fc\, ds + \int_t^\infty (Fx - Fc)\, ds$$

$$= c^\alpha \int_t^\infty g(s)\, ds + c^\beta \int_t^\infty \int_0^s P(s,\tau)\, d\tau\, ds$$

$$+ \int_t^\infty g(s)[(x(s))^\alpha - c^\alpha]\, ds \int_t^\infty \int_0^s P(s,\tau)[(x(s))^\beta - c^\beta]\, ds.$$

By the mean value theorem,

$$|x_\alpha - c_\alpha| \leq K(\alpha) =_{df} |\alpha|[(1 \pm \theta)c]^{\alpha-1}|x - c|$$

if $|x - c| \leq \theta c$ (with "+" if $\alpha > 1$, "$-$" if $\alpha < 1$. Since $|x(t) - c| \leq \theta c\gamma(t)$

if $x \in \mathcal{S}$, this means that

$$\left|\int_t^\infty Fx\,ds\right| \leq c^\alpha\left|\int_t^\infty g(s)\,ds\right|+c^\beta\left|\int_t^\infty \int_0^s P(s,\tau)\,d\tau\,ds\right|$$

$$Kc^\alpha \int_t^\infty |g(s)|\gamma(s)\,ds + Kc^\beta \int_t^\infty \int_0^s |P(s,\tau)|\gamma(\tau)\,d\tau\,ds,$$

where K is a constant which does not depend on α or β. Since all four integrals on the right are $O(\gamma(t))$, this means that

$$(\gamma(t))^{-1}\left|\int_t^\infty Fx\,ds\right| \leq Ac^\alpha + Bc^\beta,\ x \in \mathcal{S},\ t \geq 0,$$

where A and B are constants which do not depend on α or β. Since our requirement is that

$$(\gamma(t))^{-1}\left|\int_t^\infty Fx\,ds\right| \leq \theta c,\ x \in ,\mathcal{S},\ t \geq 0,$$

we have only to choose c so that

$$Ac^{\alpha-1} + Bc^{\beta-1} \leq \theta.$$

This is true for c sufficiently large if $\alpha, \beta < 1$, or for c sufficiently small if $\alpha, \beta > 1$.