Inverses of lower triangular Toeplitz matrices

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Define $a_n = 0$ if $n < 0$ and assume that $a_0 \neq 0$. Let

$$T_N = (a_{r-s})_{r,s=1}^N, \quad N = 1, 2, \dots,$$

be the family of lower triangular Toeplitz matrices generated by the formal power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{1}$$

The family of inverses is given by

$$T_N^{-1} = (b_{r-s})_{r,s=1}^N, \quad N = 1, 2, \dots,$$

where

$$g(z) = \sum_{n=0}^{\infty} b_n x^n \tag{2}$$

is the formal reciprocal of $f(z)$; that is, $b_n = 0$ if $n < 0$, $b_0 = 1/a_0$ and

$$\sum_{r=0}^{n} a_r b_{n-r} = 0, \quad n > 0.$$

This formula can be used to compute $\{b_n\}_{n=1}^{\infty}$ recursively:

$$b_n = -\frac{1}{b_0} \sum_{r=1}^{n} a_r b_{n-r}.$$

This seems to me to be the most practical way to proceed.

I can think of two “explicit" expressions for b_n; however, although they may be of some theoretical interest, they don’t seem very useful.

FORMULA I.

For every $n \geq 1$,

$$T_{n+1} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Applying Cramer’s rule here yields

$$b_n = \frac{(-1)^n}{a_0^{n+1}} \begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{vmatrix}.$$

FORMULA II.

Temporarily assume that (1) has a positive radius of convergence. Then so does (2). Moreover,

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \text{and} \quad b_n = \frac{g^{(n)}(0)}{n!}. \tag{3}$$

According to the formula of Faa di Bruno (Ch.-J de La Vallee Poussin, *Cours d'analsye infinitesimale*, Vol. 1, 12th Ed., Libraire Universitaire Louvain, Gauthier-Villars, Paris, 1959) for the derivatives of a composite function,

$$\frac{d^n}{dx^n}h(f(x)) = \sum_{k=1}^{n} h^{(k)}(f(x)) \sum_k \frac{n!}{k_1! \cdots k_n!} \left(\frac{f'(x)}{1!}\right)^{k_1} \left(\frac{f''(x)}{2!}\right)^{k_2} \cdots \left(\frac{f^{(n)}(x)}{n!}\right)^{k_n},$$

where $\sum_k$ is over all partitions of k as a sum of nonnegative integers $k_1, k_2 \ldots, k_n$ such that

$$k_1 + k_2 + \cdots + k_n = k \quad \text{and} \quad k_1 + 2k_2 + \cdots n k_n = n.$$

We're interested in $g(x) = 1/f(x)$; therefore, we take $h(u) = 1/u$ in obtain

$$\frac{d^n}{dx^n}g(x) = \sum_{k=1}^{n} \frac{(-1)^k k!}{(f(x))^{k+1}} \sum_k \frac{n!}{k_1! \cdots k_n!} \left(\frac{f'(x)}{1!}\right)^{k_1} \left(\frac{f''(x)}{2!}\right)^{k_2} \cdots \left(\frac{f^{(n)}(x)}{n!}\right)^{k_n}.$$

Setting $x = 0$ here and recalling (3) yields

$$b_n = \sum_{k=1}^{n} \frac{(-1)^k k!}{a_0^{k+1}} \sum_k \frac{1}{k_1! \cdots k_n!} a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}.$$

Although we derived this formula assuming that (1) has a positive radius of convergence, it is a purely algebraic result that doesn't require this assumption.