A Riemann integral proof of a generalized Riemann lemma

According to the well-known Riemann lemma [1, pp. 431-2], if \( f \in BV[a, b] \) then
\[
\int_a^b f(x) \cos \lambda x \, dx = O(1/\lambda)
\]
and
\[
\int_a^b f(x) \sin \lambda x \, dx = O(1/\lambda).
\]
This is an important result if one is interested in Fourier series, and the proof is easy if one knows about the Riemann-Stieltjes integral; for example,
\[
\int_a^b f(x) \cos \lambda x \, dx = \frac{1}{\lambda} \left[ f(x) \sin \lambda x \Big|_a^b - \int_a^b \sin \lambda x \, df(x) \right],
\]
which implies (1), since \( f \) and \( \sin \lambda x \) are bounded on \([a, b]\) and \( \int_a^b |df(x)| < \infty \). However, most students encountering Fourier series for the first time are not familiar with the Riemann-Stieltjes integral and do not know that a function of bounded variation is almost everywhere differentiable (or even what that means). For these students we offer the following proof of a generalized Riemann lemma.

**Theorem 1.** If \( f \in BV[a, b] \) and \( g \) is continuous and has a bounded antiderivative \( G \) on \((-\infty, \infty)\) then
\[
\int_a^b f(x) g(\lambda x) \, dx = O(1/\lambda).
\]

**Proof.** Let \( P : a = x_0 < x_1 < \cdots < x_n = b \) be an arbitrary partition of \([a, b]\) and suppose that \( \lambda > 0 \). By the mean value theorem, for \( j = 1, 2, \ldots, n \) there is a \( c_j \in (x_{j-1}, x_j) \) such that
\[
\frac{G(\lambda x_j) - G(\lambda x_{j-1})}{x_j - x_{j-1}} = \lambda g(\lambda c_j).
\]
Consider the Riemann sum
\[
S_P = \sum_{j=1}^n f(c_j) g(\lambda c_j) (x_j - x_{j-1}).
\]
Because of (2),
\[ S_P = \frac{1}{\lambda} \sum_{j=1}^{n} f(c_j) \left( G(\lambda x_j) - G(\lambda x_{j-1}) \right), \]
and summation by parts yields
\[ S_P = \frac{1}{\lambda} \left[ f(c_n)G(\lambda b) - f(c_1)G(\lambda a) + \sum_{j=1}^{n-1} (f(c_j) - f(c_{j+1})) G(\lambda x_j) \right]. \]
Therefore
\[ |S_P| \leq \frac{M(2K + V)}{\lambda}, \]
where \( M \) is an upper bound for \( |G| \) on \((-\infty, \infty)\), \( K \) is an upper bound for \( |f| \) on \([a, b]\), and \( V \) is the total variation of \( f \) on \([a, b]\). Since \( P \) is an arbitrary partition of \([a, b]\), this implies that
\[ \left| \int_{a}^{b} f(x)g(\lambda x) \, dx \right| \leq \frac{M(2K + V)}{\lambda}. \]
This completes the proof.

**References**