A Riemann integral proof of a generalized Riemann lemma

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According to the well-known Riemann lemma [1, pp. 431-2], if $f \in \mathrm{BV}[a,b]$ then

$$\int_a^b f(x)\cos\lambda x\,dx = O(1/\lambda) \tag{1}$$

and

$$\int_a^b f(x)\sin\lambda x\,dx = O(1/\lambda).$$

This is an important result if one is interested in Fourier series, and the proof is easy if one knows about the Riemann-Stieltjes integral; for example,

$$\int_a^b f(x)\cos\lambda x\,dx = \frac{1}{\lambda}\left[f(x)\sin\lambda x\Big|_a^b - \int_a^b \sin\lambda x\,df(x)\right],$$

which implies (1), since f and $\sin\lambda x$ are bounded on $[a,b]$ and $\int_a^b |df(x)| < \infty$. However, most students encountering Fourier series for the first time are not familiar with the Riemann-Stieltjes integral and do not know that a function of bounded variation is almost everywhere differentiable (or even what that means). For these students we offer the following proof of a generalized Riemann lemma.

THEOREM 1. *If $f \in BV[a,b]$ and g is continuous and has a bounded antiderivative G on $(-\infty,\infty)$ then*

$$\int_a^b f(x)g(\lambda x)\,dx = O(1/\lambda).$$

Proof. Let $P:\ a = x_0 < x_1 < \cdots < x_n = b$ be an arbitrary partition of $[a,b]$ and suppose that $\lambda > 0$. By the mean value theorem, for $j = 1, 2, \dots, n$ there is a $c_j \in (x_{j-1}, x_j)$ such that

$$\frac{G(\lambda x_j) - G(\lambda x_{j-1})}{x_j - x_{j-1}} = \lambda g(\lambda c_j). \tag{2}$$

Consider the Riemann sum

$$S_P = \sum_{j=1}^n f(c_j)g(\lambda c_j)(x_j - x_{j-1}).$$

Because of (2),

$$S_P = \frac{1}{\lambda} \sum_{j=1}^{n} f(c_j) \left(G(\lambda x_j) - G(\lambda x_{j-1}) \right),$$

and summation by parts yields

$$S_P = \frac{1}{\lambda} \left[f(c_n)G(\lambda b) - f(c_1)G(\lambda a) + \sum_{j=1}^{n-1} \left(f(c_j) - f(c_{j+1}) \right) G(\lambda x_j) \right].$$

Therefore

$$|S_P| \le \frac{M(2K+V)}{\lambda},$$

where M is an upper bound for $|G|$ on $(-\infty, \infty)$, K is an upper bound for $|f|$ on $[a, b]$, and V is the total variation of f on $[a, b]$. Since P is an arbitrary partition of $[a, b]$, this implies that

$$\left| \int_a^b f(x) g(\lambda x)\, dx \right| \le \frac{M(2K+V)}{\lambda}.$$

This completes the proof.

References

[1] H. Jeffreys and B. S. Jeffreys, *Methods of Mathematical Physics*, 3rd ed., Cambridge University Press, 1956.