# FUNCTIONS DEFINED BY IMPROPER INTEGRALS

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# 1 Foreword

This is a revised version of Section 7.5 of my *Advanced Calculus* (Harper & Row, 1978). It is a supplement to my textbook *Introduction to Real Analysis*, which is referenced several times here. You should review Section 3.4 (Improper Integrals) of that book before reading this document.

# 2 Introduction

In Section 7.2 (pp. 462–484) we considered functions of the form

$$F(y) = \int_a^b f(x, y)\, dx, \quad c \le y \le d.$$

We saw that if $f$ is continuous on $[a, b] \times [c, d]$, then $F$ is continuous on $[c, d]$ (Exercise 7.2.3, p. 481) and that we can reverse the order of integration in

$$\int_c^d F(y)\, dy = \int_c^d \left( \int_a^b f(x, y)\, dx \right) dy$$

to evaluate it as

$$\int_c^d F(y)\, dy = \int_a^b \left( \int_c^d f(x, y)\, dy \right) dx$$

(Corollary 7.2.3, p. 466).

Here is another important property of $F$.

**Theorem 1** *If $f$ and $f_y$ are continuous on $[a, b] \times [c, d]$, then*

$$F(y) = \int_a^b f(x, y)\, dx, \quad c \le y \le d, \tag{1}$$

*is continuously differentiable on $[c, d]$ and $F'(y)$ can be obtained by differentiating (1) under the integral sign with respect to $y$; that is,*

$$F'(y) = \int_a^b f_y(x, y)\, dx, \quad c \le y \le d. \tag{2}$$

*Here $F'(a)$ and $f_y(x, a)$ are derivatives from the right and $F'(b)$ and $f_y(x, b)$ are derivatives from the left.*

**Proof** If $y$ and $y + \Delta y$ are in $[c, d]$ and $\Delta y \ne 0$, then

$$\frac{F(y + \Delta y) - F(y)}{\Delta y} = \int_a^b \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}\, dx. \tag{3}$$

From the mean value theorem (Theorem 2.3.11, p. 83), if $x \in [a, b]$ and $y, y + \Delta y \in [c, d]$, there is a $y(x)$ between $y$ and $y + \Delta y$ such that

$$f(x, y + \Delta y) - f(x, y) = f_y(x, y)\Delta y = f_y(x, y(x))\Delta y + (f_y(x, y(x)) - f_y(x, y))\Delta y.$$

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From this and (3),

$$\left| \frac{F(y + \Delta y) - F(y)}{\Delta y} - \int_a^b f_y(x, y)\, dx \right| \leq \int_a^b |f_y(x, y(x)) - f_y(x, y)|\, dx. \quad (4)$$

Now suppose $\epsilon > 0$. Since $f_y$ is uniformly continuous on the compact set $[a, b] \times [c, d]$ (Corollary 5.2.14, p. 314) and $y(x)$ is between $y$ and $y + \Delta y$, there is a $\delta > 0$ such that if $|\Delta| < \delta$ then

$$|f_y(x, y) - f_y(x, y(x))| < \epsilon, \quad (x, y) \in [a, b] \times [c, d].$$

This and (4) imply that

$$\left| \frac{F(y + \Delta y - F(y))}{\Delta y} - \int_a^b f_y(x, y)\, dx \right| < \epsilon(b - a)$$

if $y$ and $y + \Delta y$ are in $[c, d]$ and $0 < |\Delta y| < \delta$. This implies (2). Since the integral in (2) is continuous on $[c, d]$ (Exercise 7.2.3, p. 481, with $f$ replaced by $f_y$), $F'$ is continuous on $[c, d]$. $\qquad \square$

**Example 1** Since

$$f(x, y) = \cos xy \quad \text{and} \quad f_y(x, y) = -x \sin xy$$

are continuous for all $(x, y)$, Theorem 1 implies that if

$$F(y) = \int_0^\pi \cos xy\, dx, \quad -\infty < y < \infty, \quad (5)$$

then

$$F'(y) = -\int_0^\pi x \sin xy\, dx, \quad -\infty < y < \infty. \quad (6)$$

(In applying Theorem 1 for a specific value of $y$, we take $R = [0, \pi] \times [-\rho, \rho]$, where $\rho > |y|$.) This provides a convenient way to evaluate the integral in (6): integrating the right side of (5) with respect to $x$ yields

$$F(y) = \left. \frac{\sin xy}{y} \right|_{x=0}^\pi = \frac{\sin \pi y}{y}, \quad y \neq 0.$$

Differentiating this and using (6) yields

$$\int_0^\pi x \sin xy\, dx = \frac{\sin \pi y}{y^2} - \frac{\pi \cos \pi y}{y}, \quad y \neq 0.$$

To verify this, use integration by parts. $\qquad \blacksquare$

We will study the continuity, differentiability, and integrability of

$$F(y) = \int_a^b f(x, y)\, dx, \quad y \in S,$$

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where $S$ is an interval or a union of intervals, and $F$ is a convergent improper integral for each $y \in S$. If the domain of $f$ is $[a, b) \times S$ where $-\infty < a < b \leq \infty$, we say that $F$ is *pointwise convergent on $S$* or simply *convergent on $S$*, and write

$$\int_a^b f(x, y)\, dx = \lim_{r \to b-} \int_a^r f(x, y)\, dx \tag{7}$$

if, for each $y \in S$ and every $\epsilon > 0$, there is an $r = r_0(y)$ (which also depends on $\epsilon$) such that

$$\left| F(y) - \int_a^r f(x, y)\, dx \right| = \left| \int_r^b f(x, y)\, dx \right| < \epsilon, \quad r_0(y) \leq y < b. \tag{8}$$

If the domain of $f$ is $(a, b] \times S$ where $-\infty \leq a < b < \infty$, we replace (7) by

$$\int_a^b f(x, y)\, dx = \lim_{r \to a+} \int_r^b f(x, y)\, dx$$

and (8) by

$$\left| F(y) - \int_r^b f(x, y)\, dx \right| = \left| \int_a^r f(x, y)\, dx \right| < \epsilon, \quad a < r \leq r_0(y).$$

In general, pointwise convergence of $F$ for all $y \in S$ does not imply that $F$ is continuous or integrable on $[c, d]$, and the additional assumptions that $f_y$ is continuous and $\int_a^b f_y(x, y)\, dx$ converges do not imply (2).

**Example 2** The function

$$f(x, y) = ye^{-|y|x}$$

is continuous on $[0, \infty) \times (-\infty, \infty)$ and

$$F(y) = \int_0^\infty f(x, y)\, dx = \int_0^\infty ye^{-|y|x}\, dx$$

converges for all $y$, with

$$F(y) = \begin{cases} -1 & y < 0, \\ 0 & y = 0, \\ 1 & y > 0; \end{cases}$$

therefore, $F$ is discontinuous at $y = 0$.

**Example 3** The function

$$f(x, y) = y^3 e^{-y^2 x}$$

is continuous on $[0, \infty) \times (-\infty, \infty)$. Let

$$F(y) = \int_0^\infty f(x, y)\, dx = \int_0^\infty y^3 e^{-y^2 x}\, dx = y, \quad -\infty < y < \infty.$$

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Then
$$F'(y) = 1, \quad -\infty < y < \infty.$$

However,

$$\int_0^\infty \frac{\partial}{\partial y}(y^3 e^{-y^2 x}) \, dx = \int_0^\infty (3y^2 - 2y^4 x)e^{-y^2 x} \, dx = \begin{cases} 1, & y \neq 0, \\ 0, & y = 0, \end{cases}$$

so

$$F'(y) \neq \int_0^\infty \frac{\partial f(x, y)}{\partial y} \, dx \quad \text{if} \quad y = 0.$$

# 3  Preparation

We begin with two useful convergence criteria for improper integrals that do not involve a parameter. Consistent with the definition on p. 152, we say that $f$ is locally integrable on an interval $I$ if it is integrable on every finite closed subinterval of $I$.

**Theorem 2 ( <span style="color:blue">Cauchy</span> Criterion for Convergence of an Improper Integral I)** *Suppose $g$ is locally integrable on $[a, b)$ and denote*

$$G(r) = \int_a^r g(x) \, dx, \quad a \leq r < b.$$

*Then the improper integral $\int_a^b g(x) \, dx$ converges if and only if, for each $\epsilon > 0$, there is an $r_0 \in [a, b)$ such that*

$$|G(r) - G(r_1)| < \epsilon, \quad r_0 \leq r, r_1 < b. \tag{9}$$

**Proof**  For necessity, suppose $\int_a^b g(x) \, dx = L$. By definition, this means that for each $\epsilon > 0$ there is an $r_0 \in [a, b)$ such that

$$|G(r) - L| < \frac{\epsilon}{2} \quad \text{and} \quad |G(r_1) - L| < \frac{\epsilon}{2}, \quad r_0 \leq r, r_1 < b.$$

Therefore

$$\begin{aligned} |G(r) - G(r_1)| &= |(G(r) - L) - (G(r_1) - L)| \\ &\leq |G(r) - L| + |G(r_1) - L| < \epsilon, \quad r_0 \leq r, r_1 < b. \end{aligned}$$

For sufficiency, (9) implies that

$$|G(r)| = |G(r_1) + (G(r) - G(r_1))| < |G(r_1)| + |G(r) - G(r_1)| \leq |G(r_1)| + \epsilon,$$

$r_0 \leq r \leq r_1 < b$. Since $G$ is also bounded on the compact set $[a, r_0]$ (Theorem 5.2.11, p. 313), $G$ is bounded on $[a, b)$. Therefore the monotonic functions

$$\overline{G}(r) = \sup\left\{ G(r_1) \,\middle|\, r \leq r_1 < b \right\} \quad \text{and} \quad \underline{G}(r) = \inf\left\{ G(r_1) \,\middle|\, r \leq r_1 < b \right\}$$

are well defined on $[a, b)$, and

$$\lim_{r \to b-} \overline{G}(r) = \overline{L} \quad \text{and} \quad \lim_{r \to b-} \underline{G}(r) = \underline{L}$$

both exist and are finite (Theorem 2.1.11, p. 47). From (9),

$$
\begin{aligned}
|G(r) - G(r_1)| &= |(G(r) - G(r_0)) - (G(r_1) - G(r_0))| \\
&\leq |G(r) - G(r_0)| + |G(r_1) - G(r_0)| < 2\epsilon,
\end{aligned}
$$

so

$$\overline{G}(r) - \underline{G}(r) \leq 2\epsilon, \quad r_0 \leq r, r_1 < b.$$

Since $\epsilon$ is an arbitrary positive number, this implies that

$$\lim_{r \to b-} (\overline{G}(r) - \underline{G}(r)) = 0,$$

so $\overline{L} = \underline{L}$. Let $L = \overline{L} = \underline{L}$. Since

$$\underline{G}(r) \leq G(r) \leq \overline{G}(r),$$

it follows that $\lim_{r \to b-} G(r) = L$. $\qquad\qquad\square$

We leave the proof of the following theorem to you (Exercise 2).

**Theorem 3 (Cauchy Criterion for Convergence of an Improper Integral II)** *Suppose $g$ is locally integrable on $(a, b]$ and denote*

$$G(r) = \int_r^b g(x)\, dx, \quad a \leq r < b.$$

*Then the improper integral $\int_a^b g(x)\, dx$ converges if and only if, for each $\epsilon > 0$, there is an $r_0 \in (a, b]$ such that*

$$|G(r) - G(r_1)| < \epsilon, \quad a < r, r_1 \leq r_0.$$

To see why we associate Theorems 2 and 3 with Cauchy, compare them with Theorem 4.3.5 (p. 204)

## 4 Uniform convergence of improper integrals

Henceforth we deal with functions $f = f(x, y)$ with domains $I \times S$, where $S$ is an interval or a union of intervals and $I$ is of one of the following forms:

- $[a, b)$ with $-\infty < a < b \leq \infty$;

- $(a, b]$ with $-\infty \leq a < b < \infty$;

- $(a, b)$ with $-\infty \leq a \leq b \leq \infty$.

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In all cases it is to be understood that $f$ is locally integrable with respect to $x$ on $I$. When we say that the improper integral $\int_a^b f(x, y)\, dx$ has a stated property "on S" we mean that it has the property for every $y \in S$.

**Definition 1** *If the improper integral*

$$\int_a^b f(x, y)\, dx = \lim_{r \to b-} \int_a^r f(x, y)\, dx \tag{10}$$

*converges on $S$, it is said to converge uniformly (or be uniformly convergent) on $S$ if, for each $\epsilon > 0$, there is an $r_0 \in [a, b)$ such that*

$$\left| \int_a^b f(x, y)\, dx - \int_a^r f(x, y)\, dx \right| < \epsilon, \quad y \in S, \quad r_0 \le r < b,$$

*or, equivalently,*

$$\left| \int_r^b f(x, y)\, dx \right| < \epsilon, \quad y \in S, \quad r_0 \le r < b. \tag{11}$$

The crucial difference between pointwise and uniform convergence is that $r_0(y)$ in (8) may depend upon the particular value of $y$, while the $r_0$ in (11) does not: one choice must work for all $y \in S$. Thus, uniform convergence implies pointwise convergence, but pointwise convergence does not imply uniform convergence.

**Theorem 4** (**Cauchy Criterion for Uniform Convergence I**) *The improper integral in (10) converges uniformly on $S$ if and only if, for each $\epsilon > 0$, there is an $r_0 \in [a, b)$ such that*

$$\left| \int_r^{r_1} f(x, y)\, dx \right| < \epsilon, \quad y \in S, \quad r_0 \le r, r_1 < b. \tag{12}$$

**Proof**  Suppose $\int_a^b f(x, y)\, dx$ converges uniformly on $S$ and $\epsilon > 0$. From Definition 1, there is an $r_0 \in [a, b)$ such that

$$\left| \int_r^b f(x, y)\, dx \right| < \frac{\epsilon}{2} \text{ and } \left| \int_{r_1}^b f(x, y)\, dx \right| < \frac{\epsilon}{2}, \quad y \in S, \quad r_0 \le r, r_1 < b. \tag{13}$$

Since

$$\int_r^{r_1} f(x, y)\, dx = \int_r^b f(x, y)\, dx - \int_{r_1}^b f(x, y)\, dx,$$

(13) and the triangle inequality imply (12).

For the converse, denote

$$F(y) = \int_a^r f(x, y)\, dx.$$

Since (12) implies that

$$|F(r, y) - F(r_1, y)| < \epsilon, \quad y \in S, \quad r_0 \leq r, r_1 < b, \tag{14}$$

Theorem 2 with $G(r) = F(r, y)$ ($y$ fixed but arbitrary in $S$) implies that $\int_a^b f(x, y)\, dx$ converges pointwise for $y \in S$. Therefore, if $\epsilon > 0$ then, for each $y \in S$, there is an $r_0(y) \in [a, b)$ such that

$$\left| \int_r^b f(x, y)\, dx \right| < \epsilon, \quad y \in S, \quad r_0(y) \leq r < b. \tag{15}$$

For each $y \in S$, choose $r_1(y) \geq \max[r_0(y), r_0]$. (Recall (14)). Then

$$\int_r^b f(x, y)\, dx = \int_r^{r_1(y)} f(x, y)\, dx + \int_{r_1(y)}^b f(x, y)\, dx,$$

so (12), (15), and the triangle inequality imply that

$$\left| \int_r^b f(x, y)\, dx \right| < 2\epsilon, \quad y \in S, \quad r_0 \leq r < b.$$

$\square$

In practice, we don't explicitly exhibit $r_0$ for each given $\epsilon$. It suffices to obtain estimates that clearly imply its existence.

**Example 4** For the improper integral of Example 2,

$$\left| \int_r^\infty f(x, y)\, dx \right| = \int_r^\infty |y| e^{-|y|x} = e^{-r|y|}, \quad y \neq 0.$$

If $|y| \geq \rho$, then

$$\left| \int_r^\infty f(x, y)\, dx \right| \leq e^{-r\rho},$$

so $\int_0^\infty f(x, y)\, dx$ converges uniformly on $(-\infty, \rho] \cup [\rho, \infty)$ if $\rho > 0$; however, it does not converge uniformly on any neighborhood of $y = 0$, since, for any $r > 0$, $e^{-r|y|} > \frac{1}{2}$ if $|y|$ is sufficiently small.

**Definition 2** *If the improper integral*

$$\int_a^b f(x, y)\, dx = \lim_{r \to a+} \int_r^b f(x, y)\, dx$$

*converges on $S$, it is said to converge uniformly (or be uniformly convergent) on $S$ if, for each $\epsilon > 0$, there is an $r_0 \in (a, b]$ such that*

$$\left| \int_a^b f(x, y)\, dx - \int_r^b f(x, y)\, dx \right| < \epsilon, \quad y \in S, \quad a < r \leq r_0,$$

*or, equivalently,*

$$\left| \int_a^r f(x, y)\, dx \right| < \epsilon, \quad y \in S, \quad a < r \leq r_0.$$

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We leave proof of the following theorem to you (Exercise 3).

**Theorem 5** (**Cauchy Criterion for Uniform Convergence II**) *The improper integral*

$$\int_a^b f(x,y)\,dx = \lim_{r \to a+} \int_r^b f(x,y)\,dx$$

*converges uniformly on $S$ if and only if, for each $\epsilon > 0$, there is an $r_0 \in (a,b]$ such that*

$$\left| \int_{r_1}^r f(x,y)\,dx \right| < \epsilon, \quad y \in S, \quad a < r, r_1 \le r_0.$$

We need one more definition, as follows.

**Definition 3** *Let $f = f(x,y)$ be defined on $(a,b) \times S$, where $-\infty \le a < b \le \infty$. Suppose $f$ is locally integrable on $(a,b)$ for all $y \in S$ and let $c$ be an arbitrary point in $(a,b)$. Then $\int_a^b f(x,y)\,dx$ is said to converge uniformly on $S$ if $\int_a^c f(x,y)\,dx$ and $\int_c^b f(x,y)\,dx$ both converge uniformly on $S$.*

We leave it to you (Exercise 4) to show that this definition is independent of $c$; that is, if $\int_a^c f(x,y)\,dx$ and $\int_c^b f(x,y)\,dx$ both converge uniformly on $S$ for some $c \in (a,b)$, then they both converge uniformly on $S$ for every $c \in (a,b)$.

We also leave it you (Exercise 5) to show that if $f$ is bounded on $[a,b] \times [c,d]$ and $\int_a^b f(x,y)\,dx$ exists as a proper integral for each $y \in [c,d]$, then it converges uniformly on $[c,d]$ according to all three Definitions 1–3.

**Example 5** Consider the improper integral

$$F(y) = \int_0^\infty x^{-1/2} e^{-xy}\,dx,$$

which diverges if $y \le 0$ (verify). Definition 3 applies if $y > 0$, so we consider the improper integrals

$$F_1(y) = \int_0^1 x^{-1/2} e^{-xy}\,dx \quad \text{and} \quad F_2(y) = \int_1^\infty x^{-1/2} e^{-xy}\,dx$$

separately. Moreover, we could just as well define

$$F_1(y) = \int_0^c x^{-1/2} e^{-xy}\,dx \quad \text{and} \quad F_2(y) = \int_c^\infty x^{-1/2} e^{-xy}\,dx, \tag{16}$$

where $c$ is any positive number.

Definition 2 applies to $F_1$. If $0 < r_1 < r$ and $y \ge 0$, then

$$\left| \int_r^{r_1} x^{-1/2} e^{-xy}\,dx \right| < \int_{r_1}^r x^{-1/2}\,dx < 2r^{1/2},$$

so $F_1(y)$ converges for uniformly on $[0, \infty)$.

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Definition 1 applies to $F_2$. Since

$$\left| \int_r^{r_1} x^{-1/2} e^{-xy}\, dx \right| < r^{-1/2} \int_r^\infty e^{-xy}\, dx = \frac{e^{-ry}}{y r^{1/2}},$$

$F_2(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$. It does not converge uniformly on $(0, \rho)$, since the change of variable $u = xy$ yields

$$\int_r^{r_1} x^{-1/2} e^{-xy}\, dx = y^{-1/2} \int_{ry}^{r_1 y} u^{-1/2} e^{-u}\, du,$$

which, for any fixed $r > 0$, can be made arbitrarily large by taking $y$ sufficiently small and $r = 1/y$. Therefore we conclude that $F(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$.

Note that that the constant $c$ in (16) plays no role in this argument.

**Example 6** Suppose we take

$$\int_0^\infty \frac{\sin u}{u}\, du = \frac{\pi}{2} \tag{17}$$

as given (Exercise 31(**b**)). Substituting $u = xy$ with $y > 0$ yields

$$\int_0^\infty \frac{\sin xy}{x}\, dx = \frac{\pi}{2}, \quad y > 0. \tag{18}$$

What about uniform convergence? Since $(\sin xy)/x$ is continuous at $x = 0$, Definition 1 and Theorem 4 apply here. If $0 < r < r_1$ and $y > 0$, then

$$\int_r^{r_1} \frac{\sin xy}{x}\, dx = -\frac{1}{y} \left( \frac{\cos xy}{x} \Big|_r^{r_1} + \int_r^{r_1} \frac{\cos xy}{x^2}\, dx \right), \quad \text{so} \quad \left| \int_r^{r_1} \frac{\sin xy}{x}\, dx \right| < \frac{3}{ry}.$$

Therefore (18) converges uniformly on $[\rho, \infty)$ if $\rho > 0$. On the other hand, from (17), there is a $\delta > 0$ such that

$$\int_{u_0}^\infty \frac{\sin u}{u}\, du > \frac{\pi}{4}, \quad 0 \le u_0 < \delta.$$

This and (18) imply that

$$\int_r^\infty \frac{\sin xy}{x}\, dx = \int_{yr}^\infty \frac{\sin u}{u}\, du > \frac{\pi}{4}$$

for any $r > 0$ if $0 < y < \delta/r$. Hence, (18) does not converge uniformly on any interval $(0, \rho]$ with $\rho > 0$.

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# 5 Absolutely Uniformly Convergent Improper Integrals

**Definition 4** (**Absolute Uniform Convergence I**) *The improper integral*

$$\int_a^b f(x, y)\, dx = \lim_{r \to b-} \int_a^r f(x, y)\, dx$$

*is said to converge absolutely uniformly on $S$ if the improper integral*

$$\int_a^b |f(x, y)|\, dx = \lim_{r \to b-} \int_a^r |f(x, y)|\, dx$$

*converges uniformly on $S$; that is, if, for each $\epsilon > 0$, there is an $r_0 \in [a, b)$ such that*

$$\left| \int_a^b |f(x, y)|\, dx - \int_a^r |f(x, y)|\, dx \right| < \epsilon, \quad y \in S, \quad r_0 < r < b.$$

To see that this definition makes sense, recall that if $f$ is locally integrable on $[a, b)$ for all $y$ in $S$, then so is $|f|$ (Theorem 3.4.9, p. 161). Theorem 4 with $f$ replaced by $|f|$ implies that $\int_a^b f(x, y)\, dx$ converges absolutely uniformly on $S$ if and only if, for each $\epsilon > 0$, there is an $r_0 \in [a, b)$ such that

$$\int_r^{r_1} |f(x, y)|\, dx < \epsilon, \quad y \in S, \quad r_0 \le r < r_1 < b.$$

Since

$$\left| \int_r^{r_1} f(x, y)\, dx \right| \le \int_r^{r_1} |f(x, y)|\, dx,$$

Theorem 4 implies that if $\int_a^b f(x, y)\, dx$ converges absolutely uniformly on $S$ then it converges uniformly on $S$.

**Theorem 6** ( **Weierstrass**'s **Test for Absolute Uniform Convergence I**) *Suppose $M = M(x)$ is nonnegative on $[a, b)$, $\int_a^b M(x)\, dx < \infty$, and*

$$|f(x, y)| \le M(x), \quad y \in S, \quad a \le x < b. \tag{19}$$

*Then $\int_a^b f(x, y)\, dx$ converges absolutely uniformly on $S$.*

**Proof** Denote $\int_a^b M(x)\, dx = L < \infty$. By definition, for each $\epsilon > 0$ there is an $r_0 \in [a, b)$ such that

$$L - \epsilon < \int_a^r M(x)\, dx \le L, \quad r_0 < r < b.$$

Therefore, if $r_0 < r \le r_1$, then

$$0 \le \int_r^{r_1} M(x)\, dx = \left( \int_a^{r_1} M(x)\, dx - L \right) - \left( \int_a^r M(x)\, dx - L \right) < \epsilon$$

This and (19) imply that

$$\int_r^{r_1} |f(x, y)|\, dx \leq \int_r^{r_1} M(x)\, dx < \epsilon, \quad y \in S, \quad a \leq r_0 < r < r_1 < b.$$

Now Theorem 4 implies the stated conclusion. $\qquad \square$

**Example 7** Suppose $g = g(x, y)$ is locally integrable on $[0, \infty)$ for all $y \in S$ and, for some $a_0 \geq 0$, there are constants $K$ and $p_0$ such that

$$|g(x, y)| \leq K e^{p_0 x}, \quad y \in S, \quad x \geq a_0.$$

If $p > p_0$ and $r \geq a_0$, then

$$\int_r^\infty e^{-px} |g(x, y)|\, dx \;=\; \int_r^\infty e^{-(p-p_0)x} e^{-p_0 x} |g(x, y)|\, dx$$

$$\leq \; K \int_r^\infty e^{-(p-p_0)x}\, dx = \frac{K e^{-(p-p_0)r}}{p - p_0},$$

so $\int_0^\infty e^{-px} g(x, y)\, dx$ converges absolutely on $S$. For example, since

$$|x^\alpha \sin xy| < e^{p_0 x} \quad \text{and} \quad |x^\alpha \cos xy| < e^{p_0 x}$$

for $x$ sufficiently large if $p_0 > 0$, Theorem 4 implies that $\int_0^\infty e^{-px} x^\alpha \sin xy\, dx$ and $\int_0^\infty e^{-px} x^\alpha \cos xy\, dx$ converge absolutely uniformly on $(-\infty, \infty)$ if $p > 0$ and $\alpha \geq 0$. As a matter of fact, $\int_0^\infty e^{-px} x^\alpha \sin xy\, dx$ converges absolutely on $(-\infty, \infty)$ if $p > 0$ and $\alpha > -1$. (Why?)

**Definition 5** (**Absolute Uniform Convergence II**) *The improper integral*

$$\int_a^b f(x, y)\, dx = \lim_{r \to a+} \int_r^b f(x, y)\, dx$$

*is said to converge absolutely uniformly on $S$ if the improper integral*

$$\int_a^b |f(x, y)|\, dx = \lim_{r \to a+} \int_r^b |f(x, y)|\, dx$$

*converges uniformly on $S$; that is, if, for each $\epsilon > 0$, there is an $r_0 \in (a, b]$ such that*

$$\left| \int_a^b |f(x, y)|\, dx - \int_r^b |f(x, y)|\, dx \right| < \epsilon, \quad y \in S, \quad a < r < r_0 \leq b.$$

We leave it to you (Exercise 7) to prove the following theorem.

**Theorem 7** (**Weierstrass's Test for Absolute Uniform Convergence II**) *Suppose $M = M(x)$ is nonnegative on $(a, b]$, $\int_a^b M(x)\, dx < \infty$, and*

$$|f(x, y)| \leq M(x), \quad y \in S, \quad x \in (a, b].$$

*Then $\int_a^b f(x, y)\, dx$ converges absolutely uniformly on $S$.*

**Example 8** If $g = g(x, y)$ is locally integrable on $(0, 1]$ for all $y \in S$ and

$$|g(x, y)| \leq Ax^{-\beta}, \quad 0 < x \leq x_0,$$

for each $y \in S$, then

$$\int_0^1 x^\alpha g(x, y) \, dx$$

converges absolutely uniformly on $S$ if $\alpha > \beta - 1$. To see this, note that if $0 < r < r_1 \leq x_0$, then

$$\int_{r_1}^r x^\alpha |g(x, y)| \, dx \leq A \int_{r_1}^r x^{\alpha-\beta} \, dx = \left. \frac{Ax^{\alpha-\beta+1}}{\alpha - \beta + 1} \right|_{r_1}^r < \frac{Ar^{\alpha-\beta+1}}{\alpha - \beta + 1}.$$

Applying this with $\beta = 0$ shows that

$$F(y) = \int_0^1 x^\alpha \cos xy \, dx$$

converges absolutely uniformly on $(-\infty, \infty)$ if $\alpha > -1$ and

$$G(y) = \int_0^1 x^\alpha \sin xy \, dx$$

converges absolutely uniformly on $(-\infty, \infty)$ if $\alpha > -2$.

By recalling Theorem 4.4.15 (p. 246), you can see why we associate Theorems 6 and 7 with Weierstrass.

## 6  Dirichlet's Tests

Weierstrass's test is useful and important, but it has a basic shortcoming: it applies only to absolutely uniformly convergent improper integrals. The next theorem applies in some cases where $\int_a^b f(x, y) \, dx$ converges uniformly on $S$, but $\int_a^b |f(x, y)| \, dx$ does not.

**Theorem 8 (Dirichlet's Test for Uniform Convergence I)** *If $g$, $g_x$, and $h$ are continuous on $[a, b) \times S$, then*

$$\int_a^b g(x, y)h(x, y) \, dx$$

*converges uniformly on $S$ if the following conditions are satisfied*:

**(a)** $\displaystyle \lim_{x \to b-} \left\{ \sup_{y \in S} |g(x, y)| \right\} = 0;$

**(b)** *There is a constant $M$ such that*

$$\sup_{y \in S} \left| \int_a^x h(u, y) \, du \right| < M, \quad a \leq x < b;$$

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**(c)** $\int_a^b |g_x(x, y)| \, dx$ *converges uniformly on* $S$.

**Proof** If

$$H(x, y) = \int_a^x h(u, y) \, du, \tag{20}$$

then integration by parts yields

$$
\begin{aligned}
\int_r^{r_1} g(x, y) h(x, y) \, dx &= \int_r^{r_1} g(x, y) H_x(x, y) \, dx \\
&= g(r_1, y) H(r_1, y) - g(r, y) H(r, y) \\
&\quad - \int_r^{r_1} g_x(x, y) H(x, y) \, dx.
\end{aligned} \tag{21}
$$

Since assumption **(b)** and (20) imply that $|H(x, y)| \le M$, $(x, y) \in (a, b] \times S$, Eqn. (21) implies that

$$\left| \int_r^{r_1} g(x, y) h(x, y) \, dx \right| < M \left( 2 \sup_{x \ge r} |g(x, y)| + \int_r^{r_1} |g_x(x, y)| \, dx \right) \tag{22}$$

on $[r, r_1] \times S$.

Now suppose $\epsilon > 0$. From assumption **(a)**, there is an $r_0 \in [a, b)$ such that $|g(x, y)| < \epsilon$ on $S$ if $r_0 \le x < b$. From assumption **(c)** and Theorem 6, there is an $s_0 \in [a, b)$ such that

$$\int_r^{r_1} |g_x(x, y)| \, dx < \epsilon, \quad y \in S, \quad s_0 < r < r_1 < b.$$

Therefore (22) implies that

$$\left| \int_r^{r_1} g(x, y) h(x, y) \right| < 3M\epsilon, \quad y \in S, \quad \max(r_0, s_0) < r < r_1 < b.$$

Now Theorem 4 implies the stated conclusion. $\qquad \square$

The statement of this theorem is complicated, but applying it isn't; just look for a factorization $f = gh$, where $h$ has a bounded antderivative on $[a, b)$ and $g$ is "small" near $b$. Then integrate by parts and hope that something nice happens. A similar comment applies to Theorem 9, which follows.

**Example 9** Let

$$I(y) = \int_0^\infty \frac{\cos xy}{x + y} \, dx, \quad y > 0.$$

The obvious inequality

$$\left| \frac{\cos xy}{x + y} \right| \le \frac{1}{x + y}$$

is useless here, since

$$\int_0^\infty \frac{dx}{x + y} = \infty.$$

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However, integration by parts yields

$$\int_r^{r_1} \frac{\cos xy}{x+y}\,dx = \left.\frac{\sin xy}{y(x+y)}\right|_r^{r_1} + \int_r^{r_1} \frac{\sin xy}{y(x+y)^2}\,dx$$

$$= \frac{\sin r_1 y}{y(r_1+y)} - \frac{\sin ry}{y(r+y)} + \int_r^{r_1} \frac{\sin xy}{y(x+y)^2}\,dx.$$

Therefore, if $0 < r < r_1$, then

$$\left|\int_r^{r_1} \frac{\cos xy}{x+y}\,dx\right| < \frac{1}{y}\left(\frac{2}{r+y} + \int_r^{\infty} \frac{1}{(x+y)^2}\right) \le \frac{3}{y(r+y)^2} \le \frac{3}{\rho(r+\rho)}$$

if $y \ge \rho > 0$. Now Theorem 4 implies that $I(y)$ converges uniformly on $[\rho, \infty)$ if $\rho > 0$.

We leave the proof of the following theorem to you (Exercise 10).

**Theorem 9** (**Dirichlet's Test for Uniform Convergence II**) *If $g$, $g_x$, and $h$ are continuous on $(a, b] \times S$, then*

$$\int_a^b g(x, y)h(x, y)\,dx$$

*converges uniformly on $S$ if the following conditions are satisfied:*

**(a)** $\displaystyle\lim_{x \to a+}\left\{\sup_{y \in S} |g(x, y)|\right\} = 0;$

**(b)** *There is a constant $M$ such that*

$$\sup_{y \in S}\left|\int_x^b h(u, y)\,du\right| \le M, \quad a < x \le b;$$

**(c)** $\int_a^b |g_x(x, y)|\,dx$ *converges uniformly on $S$.*

By recalling Theorems 3.4.10 (p. 163), 4.3.20 (p. 217), and 4.4.16 (p. 248), you can see why we associate Theorems 8 and 9 with Dirichlet.

# 7 Consequences of uniform convergence

**Theorem 10** *If $f = f(x, y)$ is continuous on either $[a, b) \times [c, d]$ or $(a, b] \times [c, d]$ and*

$$F(y) = \int_a^b f(x, y)\,dx \tag{23}$$

*converges uniformly on $[c, d]$, then $F$ is continuous on $[c, d]$. Moreover,*

$$\int_c^d \left(\int_a^b f(x, y)\,dx\right) dy = \int_a^b \left(\int_c^d f(x, y)\,dy\right) dx. \tag{24}$$

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**Proof**   We will assume that $f$ is continuous on $(a, b] \times [c, d]$. You can consider the other case (Exercise 14).

We will first show that $F$ in (23) is continuous on $[c, d]$. Since $F$ converges uniformly on $[c, d]$, Definition 1 (specifically, (11)) implies that if $\epsilon > 0$, there is an $r \in [a, b)$ such that

$$\left| \int_r^b f(x, y) \, dx \right| < \epsilon, \quad c \leq y \leq d.$$

Therefore, if $c \leq y, y_0 \leq d]$, then

$$
\begin{aligned}
|F(y) - F(y_0)| \; &= \; \left| \int_a^b f(x, y) \, dx - \int_a^b f(x, y_0) \, dx \right| \\
&\leq \; \left| \int_a^r [f(x, y) - f(x, y_0)] \, dx \right| + \left| \int_r^b f(x, y) \, dx \right| \\
&\quad + \left| \int_r^b f(x, y_0) \, dx \right|,
\end{aligned}
$$

so

$$|F(y) - F(y_0)| \leq \int_a^r |f(x, y) - f(x, y_0)| \, dx + 2\epsilon. \tag{25}$$

Since $f$ is uniformly continuous on the compact set $[a, r] \times [c, d]$ (Corollary 5.2.14, p. 314), there is a $\delta > 0$ such that

$$|f(x, y) - f(x, y_0)| < \epsilon$$

if $(x, y)$ and $(x, y_0)$ are in $[a, r] \times [c, d]$ and $|y - y_0| < \delta$. This and (25) imply that

$$|F(y) - F(y_0)| < (r - a)\epsilon + 2\epsilon < (b - a + 2)\epsilon$$

if $y$ and $y_0$ are in $[c, d]$ and $|y - y_0| < \delta$. Therefore $F$ is continuous on $[c, d]$, so the integral on left side of (24) exists. Denote

$$I = \int_c^d \left( \int_a^b f(x, y) \, dx \right) dy. \tag{26}$$

We will show that the improper integral on the right side of (24) converges to $I$. To this end, denote

$$I(r) = \int_a^r \left( \int_c^d f(x, y) \, dy \right) dx.$$

Since we can reverse the order of integration of the continuous function $f$ over the rectangle $[a, r] \times [c, d]$ (Corollary 7.2.2, p. 466),

$$I(r) = \int_c^d \left( \int_a^r f(x, y) \, dx \right) dy.$$

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From this and (26),

$$I - I(r) = \int_c^d \left( \int_r^b f(x, y) \, dx \right) dy.$$

Now suppose $\epsilon > 0$. Since $\int_a^b f(x, y) \, dx$ converges uniformly on $[c, d]$, there is an $r_0 \in (a, b]$ such that

$$\left| \int_r^b f(x, y) \, dx \right| < \epsilon, \quad r_0 < r < b,$$

so $|I - I(r)| < (d - c)\epsilon$ if $r_0 < r < b$. Hence,

$$\lim_{r \to b-} \int_a^r \left( \int_c^d f(x, y) \, dy \right) dx = \int_c^d \left( \int_a^b f(x, y) \, dx \right) dy,$$

which completes the proof of (24). $\square$

**Example 10** It is straightforward to verify that

$$\int_0^\infty e^{-xy} \, dx = \frac{1}{y}, \quad y > 0,$$

and the convergence is uniform on $[\rho, \infty)$ if $\rho > 0$. Therefore Theorem 10 implies that if $0 < y_1 < y_2$, then

$$\int_{y_1}^{y_2} \frac{dy}{y} = \int_{y_1}^{y_2} \left( \int_0^\infty e^{-xy} \, dx \right) dy = \int_0^\infty \left( \int_{y_1}^{y_2} e^{-xy} \, dy \right) dy$$

$$= \int_0^\infty \frac{e^{-xy_1} - e^{-xy_2}}{x} \, dx.$$

Since

$$\int_{y_1}^{y_2} \frac{dy}{y} = \log \frac{y_2}{y_1}, \quad y_2 \geq y_1 > 0,$$

it follows that

$$\int_0^\infty \frac{e^{-xy_1} - e^{-xy_2}}{x} \, dx = \log \frac{y_2}{y_1}, \quad y_2 \geq y_1 > 0.$$

**Example 11** From Example 6,

$$\int_0^\infty \frac{\sin xy}{x} \, dx = \frac{\pi}{2}, \quad y > 0,$$

and the convergence is uniform on $[\rho, \infty)$ if $\rho > 0$. Therefore, Theorem 10 implies that if $0 < y_1 < y_2$, then

$$\frac{\pi}{2}(y_2 - y_1) = \int_{y_1}^{y_2} \left( \int_0^\infty \frac{\sin xy}{x} \, dx \right) dy = \int_0^\infty \left( \int_{y_1}^{y_2} \frac{\sin xy}{x} \, dy \right) dx$$

$$= \int_0^\infty \frac{\cos xy_1 - \cos xy_2}{x^2} \, dx. \tag{27}$$

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The last integral converges uniformly on $(-\infty, \infty)$ (Exercise 10(**h**)), and is therefore continuous with respect to $y_1$ on $(-\infty, \infty)$, by Theorem 10; in particular, we can let $y_1 \to 0+$ in (27) and replace $y_2$ by $y$ to obtain

$$\int_0^\infty \frac{1 - \cos xy}{x^2}\,dx = \frac{\pi y}{2}, \quad y \geq 0.$$

The next theorem is analogous to Theorem 4.4.20 (p. 252).

**Theorem 11** *Let $f$ and $f_y$ be continuous on either $[a, b) \times [c, d]$ or $(a, b] \times [c, d]$. Suppose that the improper integral*

$$F(y) = \int_a^b f(x, y)\,dx$$

*converges for some $y_0 \in [c, d]$ and*

$$G(y) = \int_a^b f_y(x, y)\,dx$$

*converges uniformly on $[c, d]$. Then $F$ converges uniformly on $[c, d]$ and is given explicitly by*

$$F(y) = F(y_0) + \int_{y_0}^y G(t)\,dt, \quad c \leq y \leq d.$$

*Moreover, $F$ is continuously differentiable on $[c, d]$; specifically,*

$$F'(y) = G(y), \quad c \leq y \leq d, \tag{28}$$

*where $F'(c)$ and $f_y(x, c)$ are derivatives from the right, and $F'(d)$ and $f_y(x, d)$ are derivatives from the left.*

**Proof** We will assume that $f$ and $f_y$ are continuous on $[a, b) \times [c, d]$. You can consider the other case (Exercise 15).

Let

$$F_r(y) = \int_a^r f(x, y)\,dx, \quad a \leq r < b, \quad c \leq y \leq d.$$

Since $f$ and $f_y$ are continuous on $[a, r] \times [c, d]$, Theorem 1 implies that

$$F_r'(y) = \int_a^r f_y(x, y)\,dx, \quad c \leq y \leq d.$$

Then

$$
\begin{aligned}
F_r(y) &= F_r(y_0) + \int_{y_0}^y \left( \int_a^r f_y(x, t)\,dx \right) dt \\
&= F(y_0) + \int_{y_0}^y G(t)\,dt \\
&\quad + (F_r(y_0) - F(y_0)) - \int_{y_0}^y \left( \int_r^b f_y(x, t)\,dx \right) dt, \quad c \leq y \leq d.
\end{aligned}
$$

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Therefore,

$$\left| F_r(y) - F(y_0) - \int_{y_0}^{y} G(t)\, dt \right| \leq |F_r(y_0) - F(y_0)|$$

$$+ \left| \int_{y_0}^{y} \int_{r}^{b} f_y(x,t)\, dx \right|\, dt. \qquad (29)$$

Now suppose $\epsilon > 0$. Since we have assumed that $\lim_{r \to b-} F_r(y_0) = F(y_0)$ exists, there is an $r_0$ in $(a, b)$ such that

$$|F_r(y_0) - F(y_0)| < \epsilon, \quad r_0 < r < b.$$

Since we have assumed that $G(y)$ converges for $y \in [c, d]$, there is an $r_1 \in [a, b)$ such that

$$\left| \int_{r}^{b} f_y(x,t)\, dx \right| < \epsilon, \quad t \in [c, d], \quad r_1 \leq r < b.$$

Therefore, (29) yields

$$\left| F_r(y) - F(y_0) - \int_{y_0}^{y} G(t)\, dt \right| < \epsilon(1 + |y - y_0|) \leq \epsilon(1 + d - c)$$

if $\max(r_0, r_1) \leq r < b$ and $t \in [c, d]$. Therefore $F(y)$ converges uniformly on $[c, d]$ and

$$F(y) = F(y_0) + \int_{y_0}^{y} G(t)\, dt, \quad c \leq y \leq d.$$

Since $G$ is continuous on $[c, d]$ by Theorem 10, (28) follows from differentiating this (Theorem 3.3.11, p. 141). $\qquad \square$

**Example 12** Let

$$I(y) = \int_{0}^{\infty} e^{-yx^2}\, dx, \quad y > 0.$$

Since

$$\int_{0}^{r} e^{-yx^2}\, dx = \frac{1}{\sqrt{y}} \int_{0}^{r\sqrt{y}} e^{-t^2}\, dt,$$

it follows that

$$I(y) = \frac{1}{\sqrt{y}} \int_{0}^{\infty} e^{-t^2}\, dt,$$

and the convergence is uniform on $[\rho, \infty)$ if $\rho > 0$ (Exercise 8(**i**)). To evaluate the last integral, denote $J(\rho) = \int_{0}^{\rho} e^{-t^2}\, dt$; then

$$J^2(\rho) = \left( \int_{0}^{\rho} e^{-u^2}\, du \right) \left( \int_{0}^{\rho} e^{-v^2}\, dv \right) = \int_{0}^{\rho} \int_{0}^{\rho} e^{-(u^2 + v^2)}\, du\, dv.$$

Transforming to polar coordinates $r = r\cos\theta$, $v = r\sin\theta$ yields

$$J^2(\rho) = \int_{0}^{\pi/2} \int_{0}^{\rho} re^{-r^2}\, dr\, d\theta = \frac{\pi(1 - e^{-\rho^2})}{4}, \quad \text{so} \quad J(\rho) = \frac{\sqrt{\pi(1 - e^{-\rho^2})}}{2}.$$

Therefore

$$\int_0^\infty e^{-t^2}\,dt = \lim_{\rho\to\infty} J(\rho) = \frac{\sqrt{\pi}}{2} \quad \text{and} \quad \int_0^\infty e^{-yx^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{y}}, \quad y > 0.$$

Differentiating this $n$ times with respect to $y$ yields

$$\int_0^\infty x^{2n} e^{-yx^2}\,dx = \frac{1\cdot 3\cdots(2n-1)\sqrt{\pi}}{2^n y^{n+1/2}} \quad y > 0, \quad n = 1, 2, 3, \ldots,$$

where Theorem 11 justifies the differentiation for every $n$, since all these integrals converge uniformly on $[\rho, \infty)$ if $\rho > 0$ (Exercise 8(i)).

Some advice for applying this theorem: Be sure to check first that $F(y_0) = \int_a^b f(x, y_0)\,dx$ converges for at least one value of $y$. If so, differentiate $\int_a^b f(x, y)\,dx$ formally to obtain $\int_a^b f_y(x, y)\,dx$. Then $F'(y) = \int_a^b f_y(x, y)\,dx$ if $y$ is in some interval on which this improper integral converges uniformly.

# 8 Applications to Laplace transforms

The *Laplace transform* of a function $f$ locally integrable on $[0, \infty)$ is

$$F(s) = \int_0^\infty e^{-sx} f(x)\,dx$$

for all $s$ such that integral converges. Laplace transforms are widely applied in mathematics, particularly in solving differential equations.

We leave it to you to prove the following theorem (Exercise 26).

**Theorem 12** *Suppose $f$ is locally integrable on $[0, \infty)$ and $|f(x)| \le M e^{s_0 x}$ for sufficiently large $x$. Then the Laplace transform of $F$ converges uniformly on $[s_1, \infty)$ if $s_1 > s_0$.*

**Theorem 13** *If $f$ is continuous on $[0, \infty)$ and $H(x) = \int_0^\infty e^{-s_0 u} f(u)\,du$ is bounded on $[0, \infty)$, then the Laplace transform of $f$ converges uniformly on $[s_1, \infty)$ if $s_1 > s_0$.*

**Proof** If $0 \le r \le r_1$,

$$\int_r^{r_1} e^{-sx} f(x)\,dx = \int_r^{r_1} e^{-(s-s_0)x} e^{-s_0 x} f(x)\,dt = \int_r^{r_1} e^{-(s-s_0)t} H'(x)\,dt.$$

Integration by parts yields

$$\int_r^{r_1} e^{-sx} f(x)\,dt = e^{-(s-s_0)x} H(x)\Big|_r^{r_1} + (s - s_0)\int_r^{r_1} e^{-(s-s_0)x} H(x)\,dx.$$

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Therefore, if $|H(x)| \leq M$, then

$$\left| \int_r^{r_1} e^{-sx} f(x)\, dx \right| \leq M \left| e^{-(s-s_0)r_1} + e^{-(s-s_0)r} + (s-s_0) \int_r^{r_1} e^{-(s-s_0)x}\, dx \right|$$

$$\leq 3Me^{-(s-s_0)r} \leq 3Me^{-(s_1-s_0)r}, \quad s \geq s_1.$$

Now Theorem 4 implies that $F(s)$ converges uniformly on $[s_1, \infty)$.

The following theorem draws a considerably stonger conclusion from the same assumptions.

**Theorem 14** *If $f$ is continuous on $[0, \infty)$ and*

$$H(x) = \int_0^x e^{-s_0 u} f(u)\, du$$

*is bounded on $[0, \infty)$, then the Laplace transform of $f$ is infinitely differentiable on $(s_0, \infty)$, with*

$$F^{(n)}(s) = (-1)^n \int_0^\infty e^{-sx} x^n f(x)\, dx; \qquad (30)$$

*that is, the n-th derivative of the Laplace transform of $f(x)$ is the Laplace transform of $(-1)^n x^n f(x)$.*

**Proof** First we will show that the integrals

$$I_n(s) = \int_0^\infty e^{-sx} x^n f(x)\, dx, \quad n = 0, 1, 2, \ldots$$

all converge uniformly on $[s_1, \infty)$ if $s_1 > s_0$. If $0 < r < r_1$, then

$$\int_r^{r_1} e^{-sx} x^n f(x)\, dx = \int_r^{r_1} e^{-(s-s_0)x} e^{-s_0 x} x^n f(x)\, dx = \int_r^{r_1} e^{-(s-s_0)x} x^n H'(x)\, dx.$$

Integrating by parts yields

$$\int_r^{r_1} e^{-sx} x^n f(x)\, dx = r_1^n e^{-(s-s_0)r_1} H(r) - r^n e^{-(s-s_0)r} H(r)$$

$$- \int_r^{r_1} H(x) \left( e^{-(s-s_0)x} x^n \right)'\, dx,$$

where $'$ indicates differentiation with respect to $x$. Therefore, if $|H(x)| \leq M \leq \infty$ on $[0, \infty)$, then

$$\left| \int_r^{r_1} e^{-sx} x^n f(x)\, dx \right| \leq M \left( e^{-(s-s_0)r} r^n + e^{-(s-s_0)r} r^n + \int_r^\infty |(e^{-(s-s_0)x}) x^n)'|\, dx \right).$$

Therefore, since $e^{-(s-s_0)r} r^n$ decreases monotonically on $(n, \infty)$ if $s > s_0$ (check!),

$$\left| \int_r^{r_1} e^{-sx} x^n f(x)\, dx \right| < 3Me^{-(s-s_0)r} r^n, \quad n < r < r_1,$$

so Theorem 4 implies that $I_n(s)$ converges uniformly $[s_1, \infty)$ if $s_1 > s_0$. Now Theorem 11 implies that $F_{n+1} = -F_n'$, and an easy induction proof yields (30) (Exercise 25). $\square$

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**Example 13** Here we apply Theorem 12 with $f(x) = \cos ax$ ($a \neq 0$) and $s_0 = 0$. Since

$$\int_0^x \cos au \, du = \frac{\sin ax}{a}$$

is bounded on $(0, \infty)$, Theorem 12 implies that

$$F(s) = \int_0^\infty e^{-sx} \cos ax \, dx$$

converges and

$$F^{(n)}(s) = (-1)^n \int_0^\infty e^{-sx} x^n \cos ax \, dx, \quad s > 0. \tag{31}$$

(Note that this is also true if $a = 0$.) Elementary integration yields

$$F(s) = \frac{s}{s^2 + a^2}.$$

Hence, from (31),

$$\int_0^\infty e^{-sx} x^n \cos ax = (-1)^n \frac{d^n}{ds^n} \frac{s}{s^2 + a^2}, \quad n = 0, 1, \ldots.$$

# 9 Exercises

1. Suppose $g$ and $h$ are differentiable on $[a, b]$, with

$$a \leq g(y) \leq b \quad \text{and} \quad a \leq h(y) \leq b, \quad c \leq y \leq d.$$

   Let $f$ and $f_y$ be continuous on $[a, b] \times [c, d]$. Derive *Liebniz's rule*:

$$\frac{d}{dy} \int_{g(y)}^{h(y)} f(x, y)\, dx = f(h(y), y)h'(y) - f(g(y), y)g'(y)$$

$$+ \int_{g(y)}^{h(y)} f_y(x, y)\, dx.$$

   (Hint: Define $H(y, u, v) = \int_u^v f(x, y)\, dx$ and use the chain rule.)

2. Adapt the proof of Theorem 2 to prove Theorem 3.

3. Adapt the proof of Theorem 4 to prove Theorem 5.

4. Show that Definition 3 is independent of $c$; that is, if $\int_a^c f(x, y)\, dx$ and $\int_c^b f(x, y)\, dx$ both converge uniformly on $S$ for some $c \in (a, b)$, then they both converge uniformly on $S$ and every $c \in (a, b)$.

5. **(a)** Show that if $f$ is bounded on $[a, b] \times [c, d]$ and $\int_a^b f(x, y)\, dx$ exists as a proper integral for each $y \in [c, d]$, then it converges uniformly on $[c, d]$ according to all of Definition 1–3.
   **(b)** Give an example to show that the boundedness of $f$ is essential in **(a)**.

6. Working directly from Definition 1, discuss uniform convergence of the following integrals:

   **(a)** $\displaystyle\int_0^\infty \frac{dx}{1 + y^2 x^2}\, dx$  **(b)** $\displaystyle\int_0^\infty e^{-xy} x^2\, dx$

   **(c)** $\displaystyle\int_0^\infty x^{2n} e^{-yx^2}\, dx$  **(d)** $\displaystyle\int_0^\infty \sin x y^2\, dx$

   **(e)** $\displaystyle\int_0^\infty (3y^2 - 2xy)e^{-y^2 x}\, dx$  **(f)** $\displaystyle\int_0^\infty (2xy - y^2 x^2)e^{-xy}\, dx$

7. Adapt the proof of Theorem 6 to prove Theorem 7.

8. Use Weierstrass's test to show that the integral converges uniformly on $S$ :

   **(a)** $\displaystyle\int_0^\infty e^{-xy} \sin x\, dx, \quad S = [\rho, \infty), \quad \rho > 0$

   **(b)** $\displaystyle\int_0^\infty \frac{\sin x}{x^y}\, dx, \quad S = [c, d], \quad 1 < c < d < 2$

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(c) $\displaystyle\int_1^\infty e^{-px}\frac{\sin xy}{x}\,dx,\quad p>0,\quad S=(-\infty,\infty)$

(d) $\displaystyle\int_0^1 \frac{e^{xy}}{(1-x)^y}\,dx,\quad S=(-\infty,b),\quad b<1$

(e) $\displaystyle\int_{-\infty}^\infty \frac{\cos xy}{1+x^2y^2}\,dx,\quad S=(-\infty,-\rho]\cup[\rho,\infty),\quad \rho>0.$

(f) $\displaystyle\int_1^\infty e^{-x/y}\,dx,\quad S=[\rho,\infty),\quad \rho>0$

(g) $\displaystyle\int_{-\infty}^\infty e^{xy}e^{-x^2}\,dx,\quad S=[-\rho,\rho],\quad \rho>0$

(h) $\displaystyle\int_0^\infty \frac{\cos xy-\cos ax}{x^2}\,dx,\quad S=(-\infty,\infty)$

(i) $\displaystyle\int_0^\infty x^{2n}e^{-yx^2}\,dx,\quad S=[\rho,\infty),\quad \rho>0,\quad n=0,1,2,\dots$

9. **(a)** Show that
$$\Gamma(y)=\int_0^\infty x^{y-1}e^{-x}\,dx$$
converges if $y>0$, and uniformly on $[c,d]$ if $0<c<d<\infty$.

**(b)** Use integration by parts to show that
$$\Gamma(y)=\frac{\Gamma(y+1)}{y},\quad y\ge 0,$$
and then show by induction that
$$\Gamma(y)=\frac{\Gamma(y+n)}{y(y+1)\cdots(y+n-1)},\quad y>0,\quad n=1,2,3,\dots.$$
How can this be used to define $\Gamma(y)$ in a natural way for all $y\neq 0,-1,$ $-2,\dots$? (This function is called the *gamma function*.)

**(c)** Show that $\Gamma(n+1)=n!$ if $n$ is a positive integer.

**(d)** Show that
$$\int_0^\infty e^{-st}t^\alpha\,dt=s^{-\alpha-1}\Gamma(\alpha+1),\quad \alpha>-1,\quad s>0.$$

10. Show that Theorem 8 remains valid with assumption **(c)** replaced by the assumption that $|g_x(x,y)|$ is monotonic with respect to $x$ for all $y\in S$.

11. Adapt the proof of Theorem 8 to prove Theorem 9.

12. Use Dirichlet's test to show that the following integrals converge uniformly on $S=[\rho,\infty)$ if $\rho>0$:

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**(a)** $\displaystyle\int_1^\infty \frac{\sin xy}{x^y}\,dx$    **(b)** $\displaystyle\int_2^\infty \frac{\sin xy}{\log x}\,dx$

**(c)** $\displaystyle\int_0^\infty \frac{\cos xy}{x+y^2}\,dx$    **(d)** $\displaystyle\int_1^\infty \frac{\sin xy}{1+xy}\,dx$

**13.** Suppose $g$, $g_x$ and $h$ are continuous on $[a,b) \times S$, and denote $H(x,y) = \int_a^x h(u,y)\,du$, $a \le x < b$. Suppose also that

$$\lim_{x \to b-}\left\{\sup_{y \in S}|g(x,y)H(x,y)|\right\} = 0 \quad\text{and}\quad \int_a^b g_x(x,y)H(x,y)\,dx$$

converges uniformly on $S$. Show that $\int_a^b g(x,y)h(x,y)\,dx$ converges uniformly on $S$.

**14.** Prove Theorem 10 for the case where $f = f(x,y)$ is continuous on $(a,b] \times [c,d]$.

**15.** Prove Theorem 11 for the case where $f = f(x,y)$ is continuous on $(a,b] \times [c,d]$.

**16.** Show that

$$C(y) = \int_{-\infty}^\infty f(x)\cos xy\,dx \quad\text{and}\quad S(y) = \int_{-\infty}^\infty f(x)\sin xy\,dx$$

are continuous on $(-\infty,\infty)$ if

$$\int_{-\infty}^\infty |f(x)|\,dx < \infty.$$

**17.** Suppose $f$ is continuously differentiable on $[a,\infty)$, $\lim_{x\to\infty} f(x) = 0$, and

$$\int_a^\infty |f'(x)|\,dx < \infty.$$

Show that the functions

$$C(y) = \int_a^\infty f(x)\cos xy\,dx \quad\text{and}\quad S(y) = \int_a^\infty f(x)\sin xy\,dx$$

are continuous for all $y \ne 0$. Give an example showing that they need not be continuous at $y = 0$.

**18.** Evaluate $F(y)$ and use Theorem 11 to evaluate $I$:

**(a)** $\displaystyle F(y) = \int_0^\infty \frac{dx}{1+y^2x^2}$, $y \ne 0$;    $\displaystyle I = \int_0^\infty \frac{\tan^{-1}ax - \tan^{-1}bx}{x}\,dx$,
$a,b > 0$

**(b)** $F(y) = \int_0^\infty x^y \, dx, \ y > -1; \quad I = \int_0^\infty \dfrac{x^a - x^b}{\log x} \, dx, \quad a, b > -1$

**(c)** $F(y) = \int_0^\infty e^{-xy} \cos x \, dx, \quad y > 0$

$I = \int_0^\infty \dfrac{e^{-ax} - e^{-bx}}{x} \cos x \, dx, \quad a, b > 0$

**(d)** $F(y) = \int_0^\infty e^{-xy} \sin x \, dx, \quad y > 0$

$I = \int_0^\infty \dfrac{e^{-ax} - e^{-bx}}{x} \sin x \, dx, \quad a, b > 0$

**(e)** $F(y) = \int_0^\infty e^{-x} \sin xy \, dx; \ I = \int_0^\infty e^{-x} \dfrac{1 - \cos ax}{x} \, dx$

**(f)** $F(y) = \int_0^\infty e^{-x} \cos xy \, dx; \ I = \int_0^\infty e^{-x} \dfrac{\sin ax}{x} \, dx$

**19.** Use Theorem 11 to evaluate:

**(a)** $\displaystyle\int_0^1 (\log x)^n x^y \, dx, \quad y > -1, \quad n = 0, 1, 2, \dots$

**(b)** $\displaystyle\int_0^\infty \dfrac{dx}{(x^2 + y)^{n+1}} \, dx, \quad y > 0, \quad n = 0, 1, 2, \dots$

**(c)** $\displaystyle\int_0^\infty x^{2n+1} e^{-yx^2} \, dx, \quad y > 0, \quad n = 0, 1, 2, \dots$

**(d)** $\displaystyle\int_0^\infty xy^x \, dx, \quad 0 < y < 1.$

**20.** **(a)** Use Theorem 11 and integration by parts to show that

$$F(y) = \int_0^\infty e^{-x^2} \cos 2xy \, dx$$

satisfies

$$F' + 2yF = 0.$$

**(b)** Use part **(a)** to show that

$$F(y) = \dfrac{\sqrt{\pi}}{2} e^{-y^2}.$$

**21.** Show that

$$\int_0^\infty e^{-x^2} \sin 2xy \, dx = e^{-y^2} \int_0^y e^{u^2} \, du.$$

(Hint: See Exercise 20.)

**22.** State a condition implying that

$$C(y) = \int_a^\infty f(x) \cos xy \, dx \quad \text{and} \quad S(y) = \int_a^\infty f(x) \sin xy \, dx$$

are $n$ times differentiable on for all $y \neq 0$. (Your condition should imply the hypotheses of Exercise 16.)

**23.** Suppose $f$ is continuously differentiable on $[a, \infty)$,

$$\int_a^\infty |(x^k f(x))'| \, dx < \infty, \quad 0 \le k \le n,$$

and $\lim_{x \to \infty} x^n f(x) = 0$. Show that if

$$C(y) = \int_a^\infty f(x) \cos xy \, dx \quad \text{and} \quad S(y) = \int_a^\infty f(x) \sin xy \, dx,$$

then

$$C^{(k)}(y) = \int_a^\infty x^k f(x) \cos xy \, dx \quad \text{and} \quad S^{(k)}(y) = \int_a^\infty x^k f(x) \sin xy \, dx,$$

$0 \le k \le n$.

**24.** Differentiating

$$F(y) = \int_1^\infty \cos \frac{y}{x} \, dx$$

under the integral sign yields

$$-\int_1^\infty \frac{1}{x} \sin \frac{y}{x} \, dx,$$

which converges uniformly on any finite interval. (Why?) Does this imply that $F$ is differentiable for all $y$?

**25.** Show that Theorem 11 and induction imply Eq. (30).

**26.** Prove Theorem 12.

**27.** Show that if $F(s) = \int_0^\infty e^{-sx} f(x) \, dx$ converges for $s = s_0$, then it converges uniformly on $[s_0, \infty)$. (What's the difference between this and Theorem 13?)

**28.** Prove: If $f$ is continuous on $[0, \infty)$ and $\int_0^\infty e^{-s_0 x} f(x) \, dx$ converges, then

$$\lim_{s \to s_0+} \int_0^\infty e^{-sx} f(x) \, dx = \int_0^\infty e^{-s_0 x} f(x) \, dx.$$

(Hint: See the proof of Theorem 4.5.12, p. 273.)

**29.** Under the assumptions of Exercise 28, show that

$$\lim_{s \to s_0+} \int_r^\infty e^{-sx} f(x) \, dx = \int_r^\infty e^{-s_0 x} f(x) \, dx, \quad r > 0.$$

**30.** Suppose $f$ is continuous on $[0, \infty)$ and

$$F(s) = \int_0^\infty e^{-sx} f(x)\, dx$$

converges for $s = s_0$. Show that $\lim_{s \to \infty} F(s) = 0$. (Hint: Integrate by parts.)

**31.** **(a)** Starting from the result of Exercise 18(d), let $b \to \infty$ and invoke Exercise 30 to evaluate

$$\int_0^\infty e^{-ax} \frac{\sin x}{x}\, dx, \quad a > 0.$$

**(b)** Use **(a)** and Exercise 28 to show that

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

**32.** **(a)** Suppose $f$ is continuously differentiable on $[0, \infty)$ and

$$|f(x)| \le M e^{s_0 x}, \quad 0 \le x \le \infty.$$

Show that

$$G(s) = \int_0^\infty e^{-sx} f'(x)\, dx$$

converges uniformly on $[s_1, \infty)$ if $s_1 > s_0$. (Hint: Integrate by parts.)

**(b)** Show from part **(a)** that

$$G(s) = \int_0^\infty e^{-sx} x e^{x^2} \sin e^{x^2}\, dx$$

converges uniformly on $[\rho, \infty)$ if $\rho > 0$. (Notice that this does not follow from Theorem 6 or 8.)

**33.** Suppose $f$ is continuous on $[0, \infty)$,

$$\lim_{x \to 0+} \frac{f(x)}{x}$$

exists, and

$$F(s) = \int_0^\infty e^{-sx} f(x)\, dx$$

converges for $s = s_0$. Show that

$$\int_{s_0}^\infty F(u)\, du = \int_0^\infty e^{-s_0 x} \frac{f(x)}{x}\, dx.$$

# 10 Answers to selected exercises

**5. (b)** If $f(x, y) = 1/y$ for $y \neq 0$ and $f(x, 0) = 1$, then $\int_a^b f(x, y)\, dx$ does not converge uniformly on $[0, d]$ for any $d > 0$.

**6. (a)**, **(d)**, and **(e)** converge uniformly on $(-\infty, \rho] \cup [\rho, \infty)$ if $\rho > 0$; **(b)**, **(c)**, and **(f)** converge uniformly on $[\rho, \infty)$ if $\rho > 0$.

**17.** Let $C(y) = \int_1^\infty \dfrac{\cos xy}{x}\, dx$ and $S(y) = \int_1^\infty \dfrac{\sin xy}{x}\, dx$. Then $C(0) = \infty$ and $S(0) = 0$, while $S(y) = \pi/2$ if $y \neq 0$.

**18. (a)** $F(y) = \dfrac{\pi}{2|y|};\quad I = \dfrac{\pi}{2}\log\dfrac{a}{b}$  **(b)** $F(y) = \dfrac{1}{y+1};\quad I = \log\dfrac{a+1}{b+1}$

**(c)** $F(y) = \dfrac{y}{y^2+1};\quad I = \dfrac{1}{2}\dfrac{b^2+1}{a^2+1}$

**(d)** $F(y) = \dfrac{1}{y^2+1};\quad I = \tan^{-1} b - \tan^{-1} a$

**(e)** $F(y) = \dfrac{y}{y^2+1};\quad I = \dfrac{1}{2}\log(1+a^2)$

**(f)** $F(y) = \dfrac{1}{y^2+1};\quad I = \tan^{-1} a$

**19. (a)** $(-1)^n n!(y+1)^{-n-1}$   **(b)** $\pi 2^{-2n-1}\dbinom{2n}{n} y^{-n-1/2}$

**(c)** $\dfrac{n!}{2y^{n+1}} (\log y)^{-2}$ **(d)** $\dfrac{1}{(\log x)^2}$

**22.** $\displaystyle\int_{-\infty}^\infty |x^n f(x)|\, dx < \infty$

**24.** No; the integral defining $F$ diverges for all $y$.

**31. (a)** $\dfrac{\pi}{2} - \tan^{-1} a$