Flux Integrals: Stokes’ and Gauss’ Theorems

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Surfaces

A surface $S$ is a subset of $\mathbb{R}^3$ that is “locally planar,” i.e. when we zoom in on any point $P \in S$, $S$ looks like a piece of a plane.
Orientable surfaces

A surface \( S \) is *orientable* if it is “two sided.”

- Every surface shown above is orientable.

- The *Möbius band* is *not* orientable.

If \( S \) is an oriented surface, an *orientation* of \( S \) is a choice of a particular side of \( S \) as “positive.”
Planar flux

If $S$ is an oriented (finite) part of a plane and $\mathbf{F} = ai + bj + ck$ is a constant vector field, the flux of $\mathbf{F}$ through $S$ is defined to be

$$\text{comp}_n(\mathbf{F}) A(S)$$

where:

- $\mathbf{n}$ is the normal vector to $S$, in the “positive” direction;
- $A(S)$ is the area of $S$.

If $F$ represents the “flow” of some quantity, then the flux is the amount of “stuff” that passes through $S$ in one unit of time.
General flux

Suppose $S$ is a more general oriented surface, and $\mathbf{F} = \mathbf{F}(x, y, z)$ is a possibly nonconstant vector field.

- Subdivide $S$ into (approximately) planar pieces with (inherited) normal vectors $\mathbf{n}_j$ and areas $\Delta S_j$.
- Choose a point $P_j$ in the $j$th subdivision, and assume that $\mathbf{F} \approx \mathbf{F}(P_j)$ throughout this subdivision.
- Compute the “local planar flux” on each subdivision and add to get the total approximate flux:

$$\sum_j \text{comp}_{\mathbf{n}_j}(\mathbf{F}(P_j)) \Delta S_j.$$  

- Take the limit as the areas of the subdivisions tend to zero to get the flux of $\mathbf{F}$ through $S$:

$$\int \int_S \mathbf{F} \cdot d\mathbf{S} = \lim_{\Delta S \to 0} \sum_j \text{comp}_{\mathbf{n}_j}(\mathbf{F}(P_j)) \Delta S_j.$$
Remarks

- If we assume that $\mathbf{n}_j$ is a unit vector, then

$$\text{comp}_{\mathbf{n}_j}(\mathbf{F}(P_j)) \Delta S_j = (\mathbf{F}(P_j) \cdot \mathbf{n}_j) \Delta S_j$$

$$= \mathbf{F}(P_j) \cdot (\Delta S_j \mathbf{n}_j)$$

$$= \mathbf{F}(P_j) \cdot \Delta S_j,$$

where $\Delta S_j$ is a normal vector with area $\Delta S_j$.

The $d\mathbf{S}$ is thus meant to represent an “infinitesimal area normal vector” to $S$.

- As with planar flux, if $\mathbf{F}$ represents the “flow” of some quantity, then $\iint_S \mathbf{F} \cdot d\mathbf{S}$ represents the amount of “stuff” that passes through $S$ in one unit of time.
Computing flux integrals

In order to compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$, one must first parametrize $S$ via a two-variable vector function:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}, \quad (u, v) \in D \subset \mathbb{R}^2.$$  

If we define

$$\mathbf{T}_u = \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j} + \frac{\partial z}{\partial u} \mathbf{k},$$

$$\mathbf{T}_v = \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j} + \frac{\partial z}{\partial v} \mathbf{k}$$

then $\mathbf{T}_u \times \mathbf{T}_v$ is normal to $S$ at every point. If the direction of $\mathbf{n}$ agrees with the orientation of $S$, a Riemann sum argument shows

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v) \, du \, dv.$$
Example

Find the flux of the vector field

\[ \mathbf{F} = x \mathbf{i} - z \mathbf{j} + y \mathbf{k} \]

through the portion of the sphere \( x^2 + y^2 + z^2 = 4 \) in the first octant, oriented toward the origin.

The portion of the sphere in question can be parametrized as

\[ \mathbf{r}(u, v) = 2 \sin u \cos v \mathbf{i} + 2 \sin u \sin v \mathbf{j} + 2 \cos u \mathbf{k}, \]

\[ 0 \leq u \leq \pi/2, \quad 0 \leq v \leq \pi/2. \]

The tangent vectors are

\[ \mathbf{T}_u = 2 \cos u \cos v \mathbf{i} + 2 \cos u \sin v \mathbf{j} - 2 \sin u \mathbf{k}, \]

\[ \mathbf{T}_v = -2 \sin u \sin v \mathbf{i} + 2 \sin u \cos v \mathbf{j}. \]
We have

\[ \mathbf{T}_u \times \mathbf{T}_v = 4 \sin^2 u \cos v \mathbf{i} + 4 \sin^2 u \sin v \mathbf{j} + 4 \sin u \cos u \mathbf{k} \]

and

\[ \mathbf{F}(\mathbf{r}(u, v)) = 2 \sin u \cos v \mathbf{i} - 2 \cos u \mathbf{j} + 2 \sin u \sin v \mathbf{k}, \]

so that

\[ \mathbf{F} \cdot (\mathbf{T}_u \times \mathbf{T}_v) = 8 \sin^3 u \cos^2 v. \]

Since \( \mathbf{T}_u \times \mathbf{T}_v \) is oriented outward we have

\[
\int \int_S \mathbf{F} \cdot d\mathbf{S} = - \int_0^{\pi/2} \int_0^{\pi/2} 8 \sin^3 u \cos^2 v \, du \, dv = -\frac{4\pi}{3}
\]
A relationship between surface and line integrals

**Stokes’ Theorem**

Let $S$ be an oriented surface bounded by a closed curve $\partial S$. If $\mathbf{F}$ is a $C^1$ vector field and $\partial S$ is oriented positively relative to $S$, then

$$\int\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$
Stokes’ Theorem is another generalization of FTOC. It relates the integral of “the derivative” of $\mathbf{F}$ on $S$ to the integral of $\mathbf{F}$ itself on the boundary of $S$.

If $D \subset \mathbb{R}^2$ is a 2D region (oriented upward) and $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a 2D vector field, one can show that

$$\iint_D \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA.$$

That is, Stokes’ Theorem includes Green’s Theorem as a special case.
Interpreting the curl

Let $\mathbf{F}$ be a vector field. Fix a point $P \in \mathbb{R}^3$ and a unit vector $\mathbf{n}$ based at $P$. Let $C_a$ denote a circle of radius $a$, centered at $P_0$, in the plane normal to $\mathbf{n}$, oriented using the right hand rule.

The tendency of $\mathbf{F}$ to “circulate” about $\mathbf{n}$ (in the positive sense) can be measured by

$$\lim_{a \to 0^+} \frac{1}{\pi a^2} \int_{C_a} \mathbf{F} \cdot d\mathbf{r}.$$
If \( D_a \) is the disk bounded by \( C_a \), Stokes’ Theorem implies

\[
\lim_{a \to 0^+} \frac{1}{\pi a^2} \int_{C_a} \mathbf{F} \cdot d\mathbf{r} = \lim_{a \to 0^+} \frac{1}{\pi a^2} \iint_{D_a} \nabla \times \mathbf{F} \cdot d\mathbf{S}
= \lim_{a \to 0^+} \frac{1}{\pi a^2} (\nabla \times \mathbf{F})(P_0) \cdot \mathbf{n} A(D_a)
= (\nabla \times \mathbf{F})(P_0) \cdot \mathbf{n}.
\]

Thus, the circulation at \( P \) about \( \mathbf{n} \) is \textit{maximized when} \( \mathbf{n} \) \textit{points in the same direction as} \( \nabla \times \mathbf{F} \).
A relationship between surface and triple integrals

Gauss’ Theorem (a.k.a. The Divergence Theorem)

Let $E \subset \mathbb{R}^3$ be a solid region bounded by a surface $\partial E$. If $\mathbf{F}$ is a $C^1$ vector field and $\partial E$ is oriented outward relative to $E$, then

$$\iiint_E \nabla \cdot \mathbf{F} \, dV = \iint_{\partial E} \mathbf{F} \cdot d\mathbf{S}.$$
Remarks

- This can be viewed as yet another generalization of FTOC.
- Gauss’ Theorem reduces computing the flux of a vector field through a \textit{closed surface} to integrating its divergence over the region contained by that surface.
- As above, this can be used to derive a physical interpretation of $\nabla \cdot \mathbf{F}$:

\[
(\nabla \cdot \mathbf{F})(P) = \lim_{a \to 0^+} \frac{1}{V(B_a)} \int \int \int_{B_a} \nabla \cdot \mathbf{F} \, dV = \lim_{a \to 0^+} \frac{1}{V(B_a)} \int \int_{S_a} \mathbf{F} \cdot d\mathbf{S},
\]

where $P \in \mathbb{R}^3$, $B_a$ and $S_a$ are the solid ball and sphere (respectively) of radius $a$ centered at $P$. 

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Stokes' & Gauss' Theorems
A vast generalization

- We have studied various types of differentiation and integration in 2 and 3 dimensions.
- These can be generalized to arbitrary dimension $n$ using the notions of “manifold” and “differential form.”
- The following theorem unifies and extends much of our integration theory in one statement.

**Generalized Stokes Theorem**

*If $M$ is an $n$-dimensional “manifold with boundary,” and $\omega$ is a “differential $(n-1)$-form,” then*

$$
\int_M d\omega = \int_{\partial M} \omega.
$$