More on Sturm-Liouville Theory

Ryan C. Daileda

Trinity University

Partial Differential Equations
April 19, 2012
Recall:

A **Sturm-Liouville (S-L) problem** consists of

- A Sturm-Liouville equation on an interval:

\[
(p(x)y')' + (q(x) + \lambda r(x))y = 0, \quad a < x < b,
\]

- **Boundary conditions**, i.e. specified behavior of \( y \) at \( x = a \) and \( x = b \).

Such a problem is called **regular** if:

- The boundary conditions are of the form

\[
\begin{align*}
    c_1 y(a) + c_2 y'(a) &= 0, \\
    d_1 y(b) + d_2 y'(b) &= 0,
\end{align*}
\]

- where \((c_1, c_2), (d_1, d_2) \neq (0, 0)\);

- \( p, q \) and \( r \) satisfy certain **regularity conditions** on \([a, b]\).
A **nonzero** function $y$ that solves an S-L problem is called an **eigenfunction**, and the corresponding value of $\lambda$ is called an **eigenvalue**. Eigenvalues and eigenfunctions of (regular) S-L problems have very nice properties.

**Theorem**

*The eigenvalues of a regular S-L problem form an increasing sequence of real numbers*

$$
\lambda_1 < \lambda_2 < \lambda_3 < \cdots
$$

with

$$
\lim_{n \to \infty} \lambda_n = \infty.
$$

Moreover, the eigenfunction $y_n$ corresponding to $\lambda_n$ is unique (up to a scalar multiple), and has exactly $n - 1$ zeros in the interval $a < x < b$. 

Daïdela  Sturm-Liouville Theory
Theorem

Suppose that $y_j$ and $y_k$ are eigenfunctions corresponding to distinct eigenvalues $\lambda_j$ and $\lambda_k$ of a (regular) S-L problem. Then $y_j$ and $y_k$ are orthogonal on $[a, b]$ with respect to the weight function $w(x) = r(x)$. That is

$$\langle y_j, y_k \rangle = \int_a^b y_j(x)y_k(x)r(x) \, dx = 0.$$  

- We have put the word “regular” in parentheses because this result actually holds for certain non-regular S-L problems, too.

- We will look at the proof of this result to see just where “regularity” is needed.
Proof of orthogonality

If \((y_j, \lambda_j), (y_k, \lambda_k)\) are eigenfunction/eigenvalue pairs then

\[
(py_j')' + (q + \lambda_j r)y_j = 0, \\
(py_k')' + (q + \lambda_k r)y_k = 0.
\]

Multiply the first by \(y_k\) and the second by \(y_j\), then subtract to get

\[
(py_j')'y_k - (py_k')'y_j + (\lambda_j - \lambda_k)y_jy_k r = 0.
\]

Moving the \(\lambda\)-terms to one side and “adding zero,” we get

\[
(\lambda_j - \lambda_k)y_jy_k r = (py_k')'y_j - (py_j')'y_k \\
= (py_k')'y_j + py_k'y_j' - py_j'y_k - (py_j')'y_k \\
= (py_k'y_j - py_j'y_k)'
= (p(y_k'y_j - y_j'y_k))'.
\]
If $\lambda_j \neq \lambda_k$, we can divide by $\lambda_j - \lambda_k$ and then integrate to get

$$\langle y_j, y_k \rangle = \int_a^b y_j(x)y_k(x)r(x)\,dx = \frac{p(x)\left(y'_k(x)y_j(x) - y'_j(x)y_k(x)\right)}{\lambda_j - \lambda_k} \bigg|_a^b.$$  

This proves the orthogonality of $y_j$ and $y_k$ whenever the RHS equals zero. This is guaranteed to happen if

$$p(a)\left(y'_k(a)y_j(a) - y'_j(a)y_k(a)\right) = p(b)\left(y'_k(b)y_j(b) - y'_j(b)y_k(b)\right) = 0.$$  

These equalities occur when:

- $y'_k(a)y_j(a) - y'_j(a)y_k(a) = 0$ or $p(a) = 0$;  
  \[A\] \[A'\]

- $y'_k(b)y_j(b) - y'_j(b)y_k(b) = 0$ or $p(b) = 0$.  
  \[B\] \[B'\]

While these conditions are sufficient for orthogonality, it should be pointed out that they are not necessary.
Orthogonality for regular S-L problems

If our S-L problem is regular then at $x = a$ we have

$$c_1y_j(a) + c_2y_j'(a) = 0,$$
$$c_1y_k(a) + c_2y_k'(a) = 0,$$

or in matrix form

$$\begin{pmatrix} y_j(a) & y_j'(a) \\ y_k(a) & y_k'(a) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $(c_1, c_2) \neq (0, 0)$ the determinant must be zero, that is

$$y_j(a)y_k'(a) - y_k(a)y_j'(a) = 0,$$

which is condition $A$. Likewise, the boundary condition at $x = b$ gives condition $B$, which verifies orthogonality.
Examples

Example

Use the preceding results to verify orthogonality of the eigenfunctions of

\[ y'' + \lambda y = 0, \quad 0 < x < L, \]
\[ y(0) = y(L) = 0. \]

This is a regular S-L problem with eigenfunctions

\[ y_n = \sin\left(\frac{n\pi x}{L}\right). \]

Since \( r(x) = 1 \), we immediately deduce that

\[ \int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \, dx = 0 \]

for \( m \neq n \).
Example

If \( m \geq 0 \), use the preceding results to verify orthogonality of the eigenfunctions of

\[
x^2 y'' + xy' + (\lambda^2 x^2 - m^2)y = 0, \quad 0 < x < a,
\]

\( y(0) \) is finite, \( y(a) = 0 \).

This is a singular S-L problem with eigenfunctions

\[
y_n = J_m(\alpha_{mn}x/a).
\]

Since \( p(x) = x \), \( p(0) = 0 \). This gives condition \( A' \). Since the boundary condition \( y(a) = 0 \) is regular, we get also get condition \( B \). With \( r(x) = x \), we immediately deduce that

\[
\int_0^a J_m \left( \frac{\alpha_{mk}x}{a} \right) J_m \left( \frac{\alpha_{ml}x}{a} \right) x \, dx = 0
\]

for \( k \neq \ell \).
Example

Use the preceding results to verify orthogonality of the eigenfunctions of

\[ y'' + \lambda y = 0, \quad -p < x < p, \]

\[ y(-p) = y(p), \]

\[ y'(-p) = y'(p). \]

This is an S-L problem with 2p-periodic boundary conditions. It is left as an exercise to verify that the eigenvalues are

\[ \lambda_n = \frac{n^2 \pi^2}{p^2} \]

for \( n = 0, 1, 2, 3, \ldots \) with eigenfunctions

\[ y_n = \cos \frac{n\pi x}{p} \text{ or } \sin \frac{n\pi x}{p}. \]
Although $A$, $A'$, $B$ and $B'$ may not hold, the periodic boundary conditions imply that

\[
(y'_k(p)y_j(p) - y'_j(p)y_k(p)) - (y'_k(-p)y_j(-p) - y'_j(-p)y_k(-p)) = 0.
\]

Since $r(x) = 1$, this \textit{immediately} implies the orthogonality relations

\[
\int_{-p}^{p} \sin \frac{m \pi x}{p} \sin \frac{n \pi x}{p} \, dx = 0,
\]

\[
\int_{-p}^{p} \cos \frac{m \pi x}{p} \cos \frac{n \pi x}{p} \, dx = 0,
\]

\[
\int_{-p}^{p} \sin \frac{m \pi x}{p} \cos \frac{n \pi x}{p} \, dx = 0,
\]

for $m \neq n$. 
“Fourier convergence” for S-L problems

The eigenfunctions of an S-L problem provide a family of orthogonal functions. As with sine and cosine, we can use these to give series expansions for “sufficiently nice” functions.

**Theorem**

Let \( y_1, y_2, y_3, \ldots \) be the eigenfunctions of a regular S-L problem on \([a, b]\). If \( f \) is piecewise smooth on \([a, b]\), then

\[
\frac{f(x^+) + f(x^-)}{2} = \sum_{n=1}^{\infty} A_n y_n(x),
\]

where

\[
A_n = \frac{\langle f, y_n \rangle}{\langle y_n, y_n \rangle} = \frac{\int_a^b f(x)y_n(x)r(x) \, dx}{\int_a^b y_n^2(x)r(x) \, dx}.
\]
The series $\sum_{n=1}^{\infty} A_n y_n$ is called the **eigenfunction expansion** of $f$.

Recall that $f(x) = \frac{f(x+) + f(x-)}{2}$ anywhere $f$ is continuous. So the eigenfunction expansion is equal to $f$ at most points.

Although we have only stated this result for regular S-L problems, it frequently holds for singular problems as well.

The “original” Fourier convergence theorem provides an example of this phenomenon (the S-L problem involved in this case is non-regular).
The hanging chain

Consider a chain (or heavy rope, cable, etc.) of length $L$ hanging from a fixed point, subject to only to downward gravitational force.

We place the chain along the (vertical) $x$-axis, displace the chain from rest, and let

$$u(x, t) = \text{Horizontal deflection of chain from equilibrium at height } x \text{ and time } t.$$
Under ideal assumptions (e.g. planar motion, small deflection, no energy loss due to friction or air resistance, etc.) we obtain the boundary value problem

\[ u_{tt} = g \left( xu_{xx} + u_x \right), \quad 0 < x < L, \quad t > 0, \]

\[ u(L, t) = 0, \quad t > 0, \]

\[ u(x, 0) = f(x), \]

\[ u_t(x, 0) = v(x), \]

where

- \( f(x) \) is the initial shape of the chain,
- \( v(x) \) is the initial (horizontal) velocity of the chain,
- \( g \) is the acceleration due to gravity.
Writing \( u(x, t) = X(x) T(t) \), separation of variables (and physical considerations) yields

\[
T'' + \lambda^2 g T = 0, \quad t > 0, \\
xX'' + X' + \lambda^2 X = 0, \quad 0 < x < L, \\
X(0) \text{ finite, } X(L) = 0.
\]

The general solution for \( T \) is

\[
T(t) = A \cos (\sqrt{g} \lambda t) + B \sin (\sqrt{g} \lambda t).
\]

The ODE for \( X \) can be rewritten as

\[
(xX')' + \lambda^2 X = 0,
\]

yielding a **singular** S-L problem (with \( p(x) = x \), \( q(x) = 0 \), \( r(x) = 1 \), and parameter \( \lambda^2 \)).
To find the eigenfunctions, we substitute $s = 2\sqrt{x}$. This yields the parametric Bessel equation of order 0:

$$s^2 \frac{d^2 X}{ds^2} + s \frac{dX}{ds} + \lambda^2 s^2 X = 0, \quad 0 < s < 2\sqrt{L},$$

$X(0)$ finite, $X(2\sqrt{L}) = 0$.

As we have seen, this means

$$\lambda = \lambda_n = \frac{\alpha_n}{2\sqrt{L}},$$

$$X(s) = X_n(s) = J_0 \left( \frac{\alpha_n s}{2\sqrt{L}} \right),$$

where $\alpha_n$ is the $n$th positive zero of $J_0$. Back-substitution then gives

$$X(x) = X_n(x) = J_0 \left( \frac{\alpha_n \sqrt{x}}{L} \right).$$
From this we find that

\[ T(t) = T_n(t) = A_n \cos (\sqrt{g} \lambda_n t) + B_n \sin (\sqrt{g} \lambda_n t) \]

\[ = A_n \cos \left( \sqrt{\frac{g}{L}} \alpha_n t \right) + B_n \sin \left( \sqrt{\frac{g}{L}} \alpha_n t \right), \]

and superposition gives the **general solution**

\[
\begin{align*}
 u(x, t) &= \sum_{n=1}^{\infty} X_n(x) T_n(t) \\
 &= \sum_{n=1}^{\infty} J_0 \left( \alpha_n \sqrt{\frac{x}{L}} \right) \left( A_n \cos \left( \sqrt{\frac{g}{L}} \frac{\alpha_n t}{2} \right) + B_n \sin \left( \sqrt{\frac{g}{L}} \frac{\alpha_n t}{2} \right) \right).
\end{align*}
\]
The initial shape condition requires that

\[ f(x) = u(x, 0) = \sum_{n=1}^{\infty} A_n J_0 \left( \alpha_n \sqrt{\frac{x}{L}} \right) X_n(x). \]

According to S-L theory, this means that

\[ A_n = \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle} = \frac{\int_0^L f(x) J_0 \left( \alpha_n \sqrt{\frac{x}{L}} \right) \, dx}{\int_0^L J_0^2 \left( \alpha_n \sqrt{\frac{x}{L}} \right) \, dx} = \frac{1}{L J_1^2(\alpha_n)} \int_0^L f(x) J_0 \left( \alpha_n \sqrt{\frac{x}{L}} \right) \, dx. \]

Setting \( u_t(x, 0) = v(x) \) and using similar reasoning yields

\[ B_n = \frac{2}{\alpha_n J_1^2(\alpha_n) \sqrt{gL}} \int_0^L v(x) J_0 \left( \alpha_n \sqrt{\frac{x}{L}} \right) \, dx. \]